Engineering Electromagnetics

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Chapter 1:

Vector Analysis

1.1 Scalars and Vectors

- Scalar : magnitude or quantity
- Complex scalar (or phasor): a set of scalar

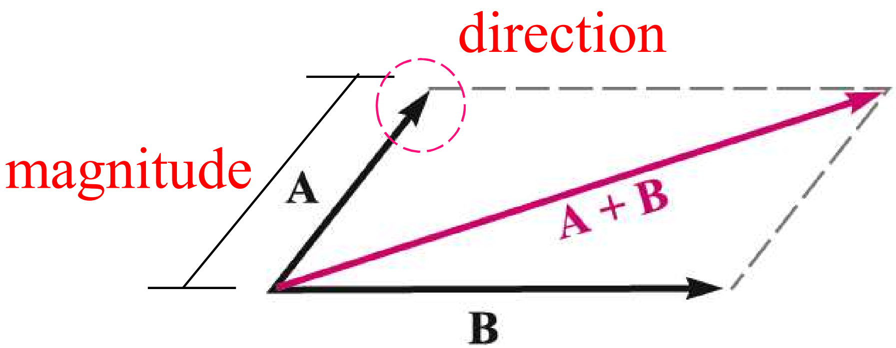
  Ex.] $\boldsymbol{M\angle\theta}$, $\boldsymbol{R + jX}$

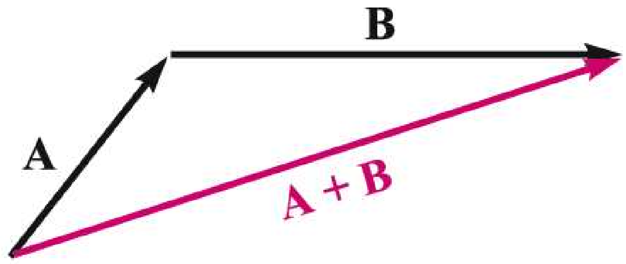
- Vector : identity having magnitudes and directions in the ***n*-dimensional spaces**

1.2 Vector Algebra

1.2.1 Addition and Subtraction

- The addition of vectors follows the parallelogram law.





Communicative Law: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

Associative Law: $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

Subtraction: $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

Inverse of direction

1.2.2 Multiplication and Division

- Multiplication of vectors by scalar

  Distributive Law: $(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B})$

  $$= r\vec{\boldsymbol{A}} + r\vec{\boldsymbol{B}} + s\vec{\boldsymbol{A}} + s\vec{\boldsymbol{B}}$$

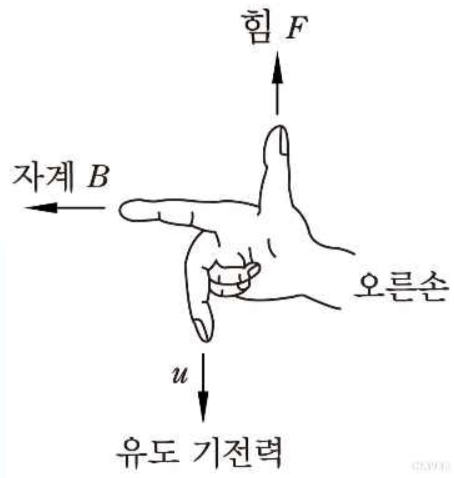
- Division of vector by a scalar

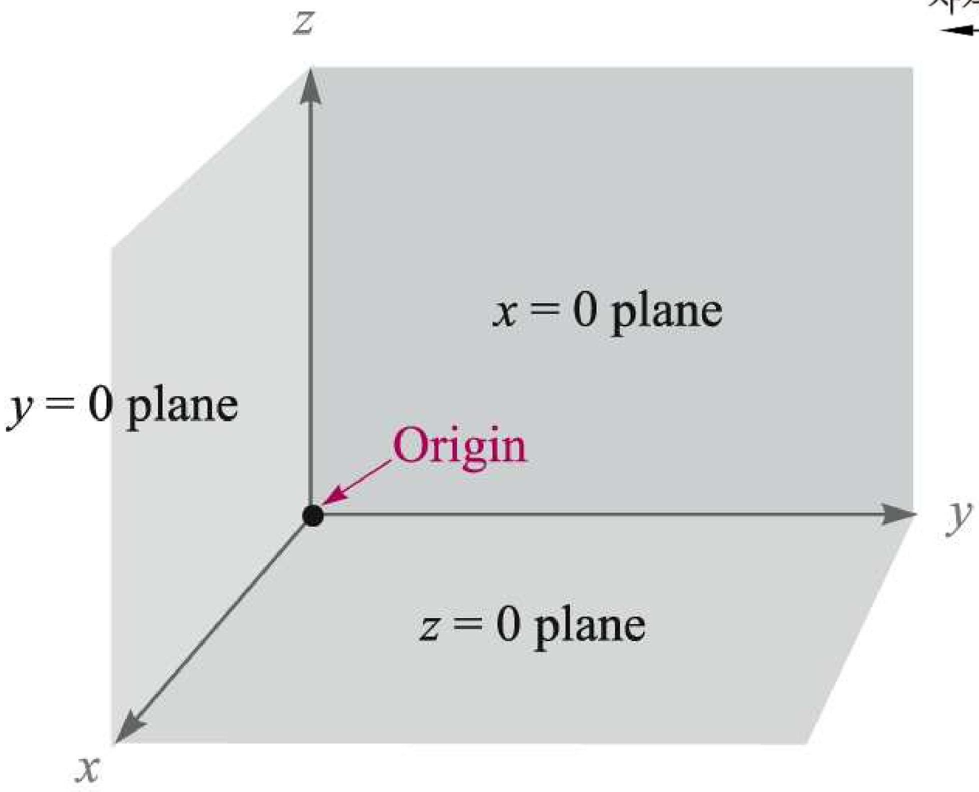
  $$\frac{\vec{\boldsymbol{A}}}{\boldsymbol{r}} = \frac{1}{r}\vec{\boldsymbol{A}}$$

- Equal: $\vec{\boldsymbol{A}} = \vec{\boldsymbol{B}}$ or $\vec{\boldsymbol{A}} - \vec{\boldsymbol{B}} = \mathbf{0}$

1.3 Rectangular Coordinate System

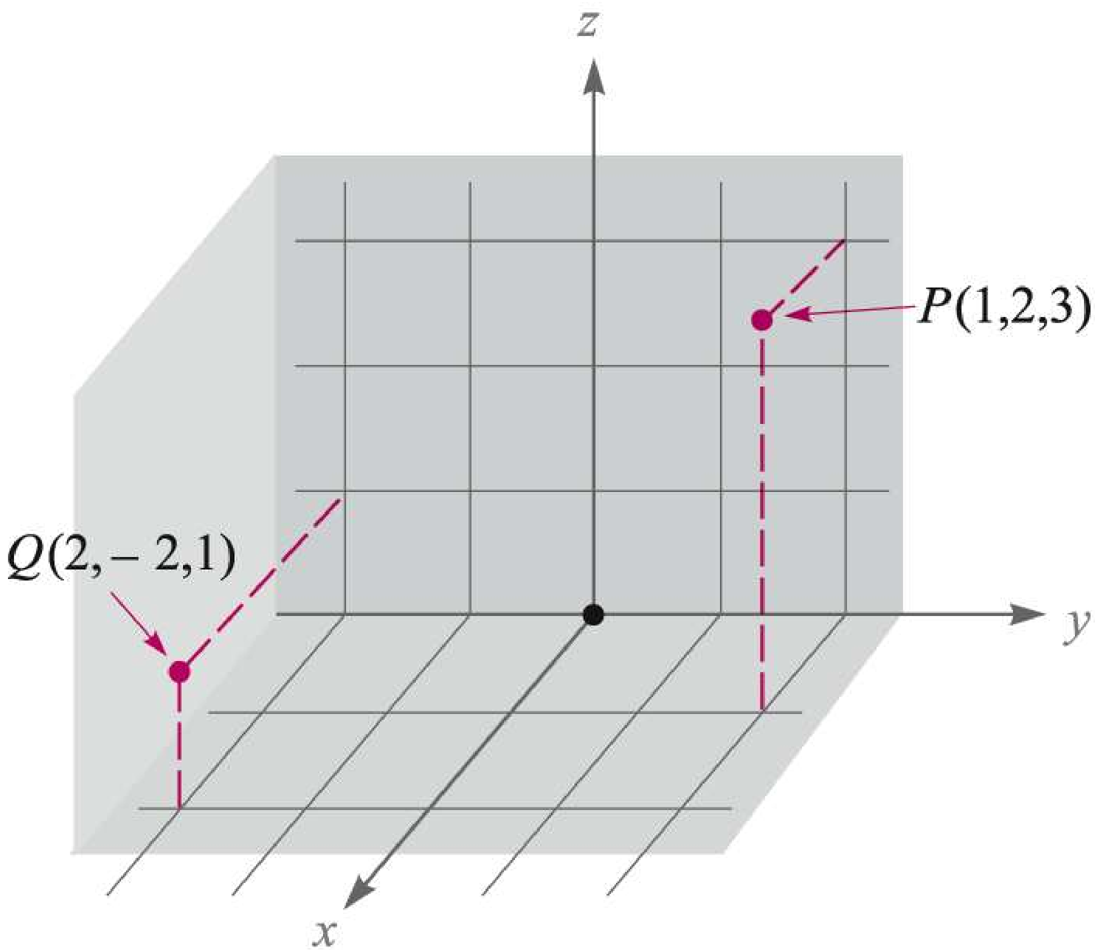
- Right-handed coordinate system: Three coordinate axes are located mutually at right angle to each other.
  $(x \rightarrow y \rightarrow z \rightarrow x \rightarrow \cdots)$
  (Ex.] Fleming's right hand rule)





Point Locations in Rectangular Coordinates

Ex.] $(x_1, y_1, z_1) \rightarrow x = x_1, y = y_1, z = z_1$



Differential Elements

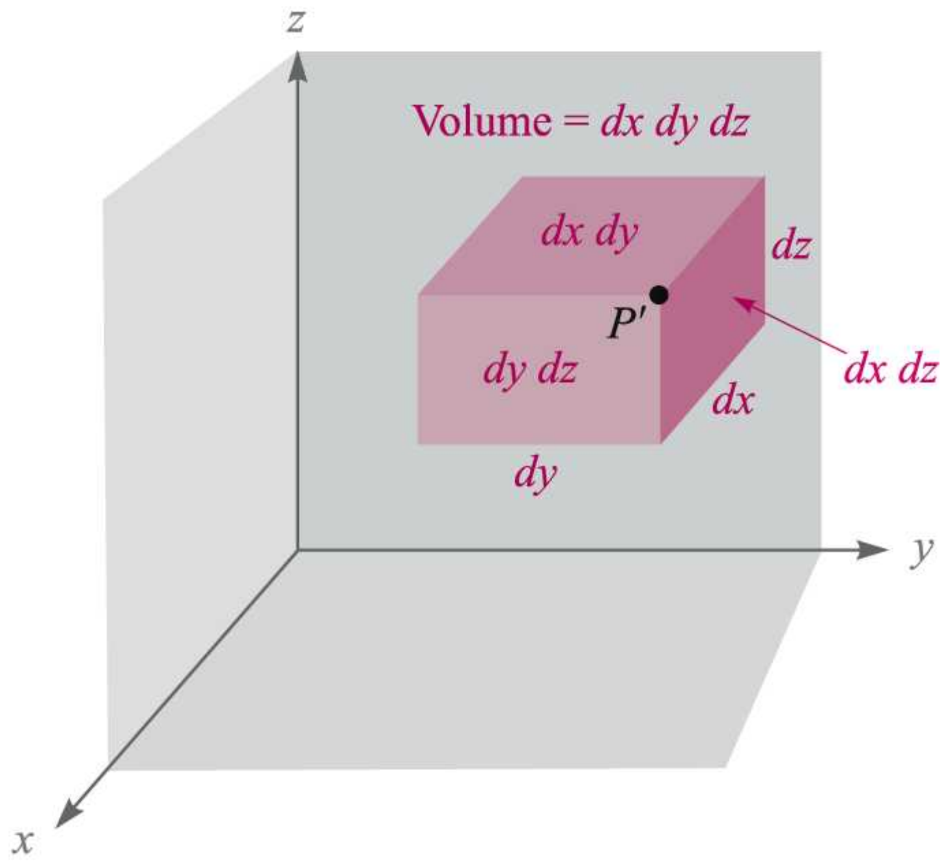
- Differential lengths: dx, dy, dz

$$dl_{pp'} = \sqrt{(dx)^2+(dy)^2+(dz)^2}$$

- Differential areas: $dS = dxdy,\ dydz,\ dzdx$

- Differential volume:

$$dv = dxdydz$$

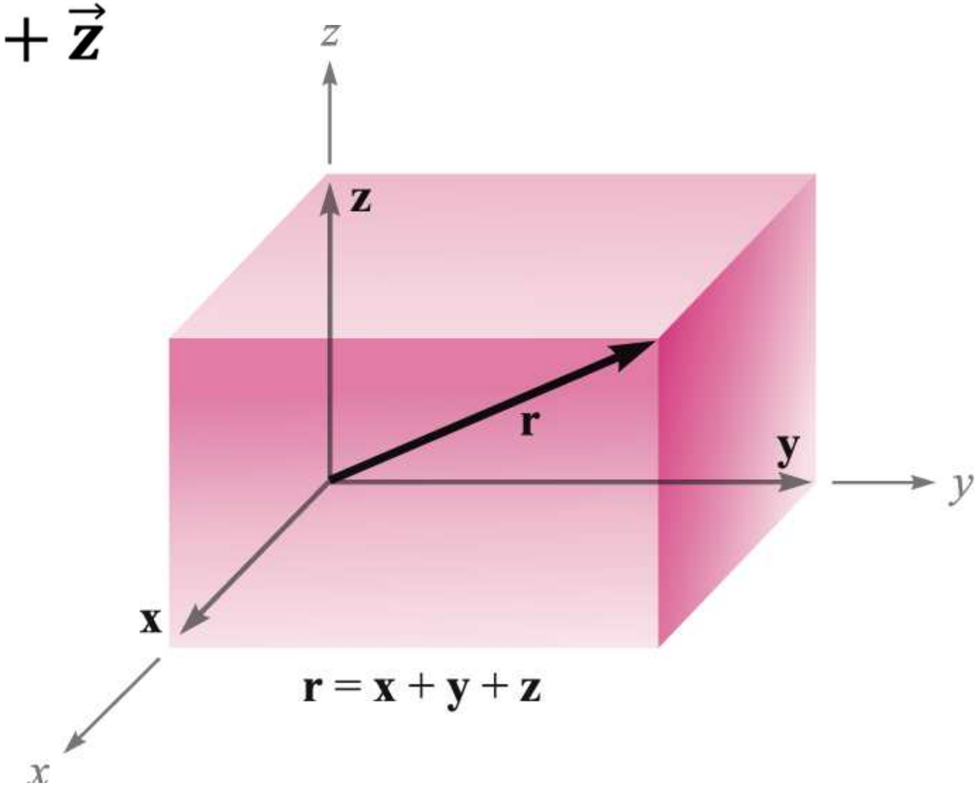


1.4 Vector Components and Unit Vectors

- Orthogonal Vector Components

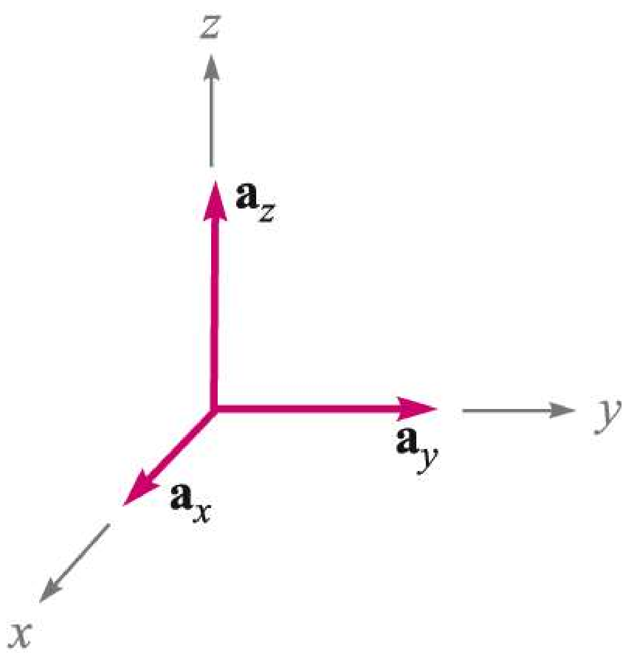
$$\vec{x} \perp \vec{y}, \qquad \vec{y} \perp \vec{z}, \qquad \vec{z} \perp \vec{x}$$

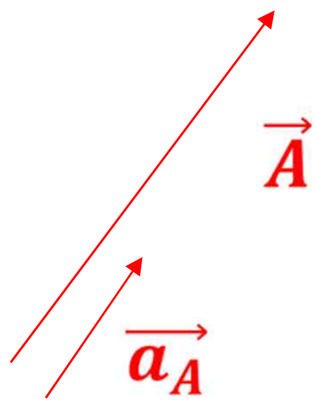
$$\vec{r} = \vec{x} + \vec{y} + \vec{z}$$



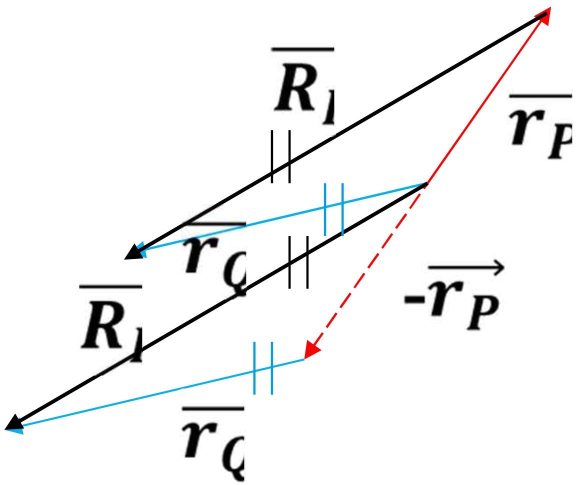
- Unit vector : A vector having unit magnitude directed along the direction of the arbitrary vector

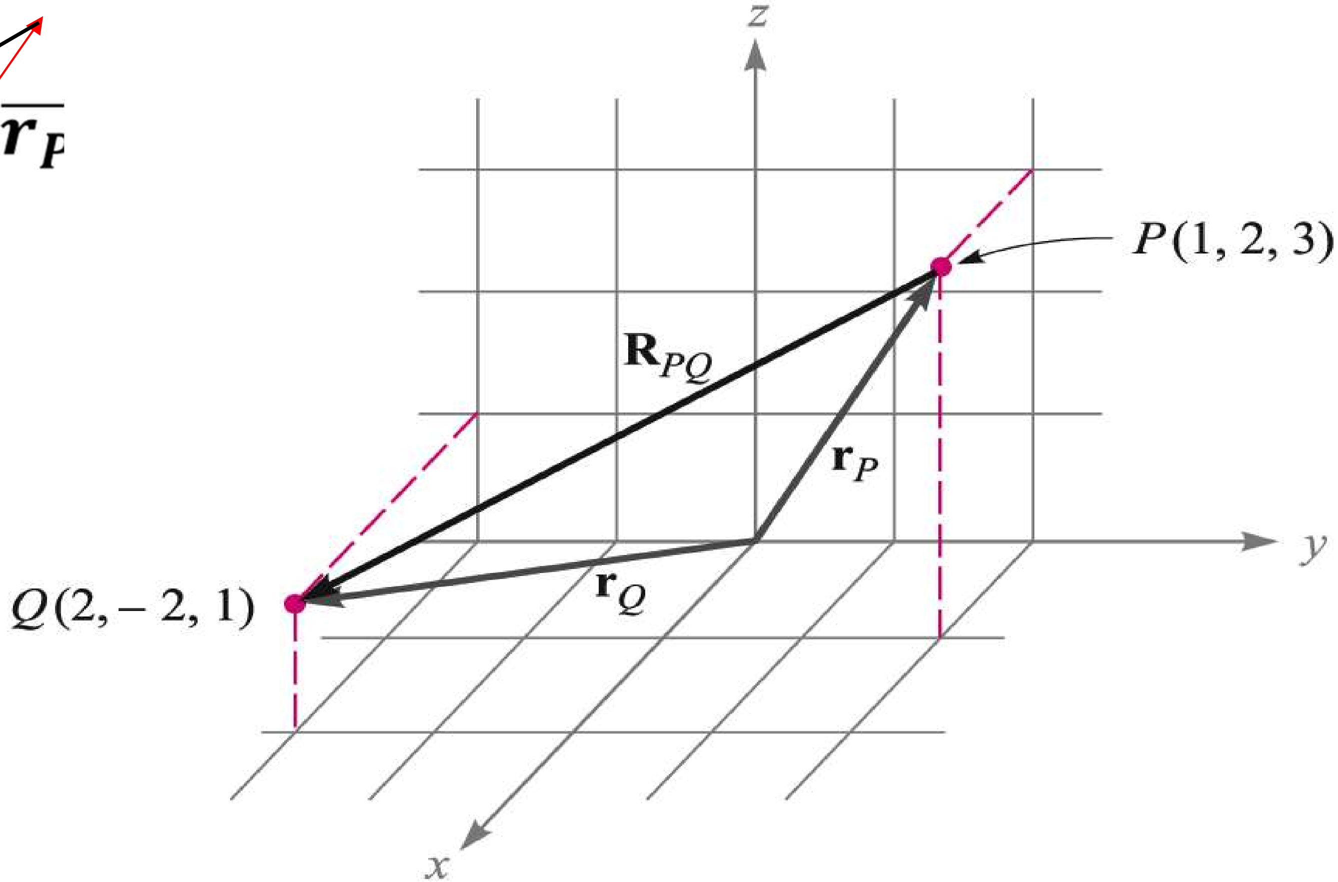
Ex.] $\overrightarrow{\boldsymbol{a}_x}, \overrightarrow{\boldsymbol{a}_y}, \overrightarrow{\boldsymbol{a}_z}, \overrightarrow{\boldsymbol{a}_A}$, etc, .





- Vector representation in terms of orthogonal rectangular components





source

$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$

$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

target

- Vector expressions in rectangular coordinates

General vector, **B**:

$$\mathbf{B} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$$

Magnitude of **B**:

$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Unit vector in the direction of **B:**

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

Example

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \frac{2}{3}\mathbf{a}_x - \frac{2}{3}\mathbf{a}_y - \frac{1}{3}\mathbf{a}_z = \underline{0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z}$$

1.5 Vector Field

We are accustomed to thinking of a specific vector:

$$\mathbf{v} = v_x \mathbf{a}_x + v_y \mathbf{a}_y + v_z \mathbf{a}_z$$

A vector field is **a *function* defined in space** that has magnitude and direction at all points:

$$\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$$

where $\mathbf{r} = (x, y, z)$

1.6 The Dot Product (or Scalar Product)

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta_{AB}$$

$$= |\vec{A}||\vec{B}|cos(-\theta_{BA}) = |\vec{B}||\vec{A}|cos(\theta_{BA}) = \vec{B} \cdot \vec{A}$$

Commutative Law: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

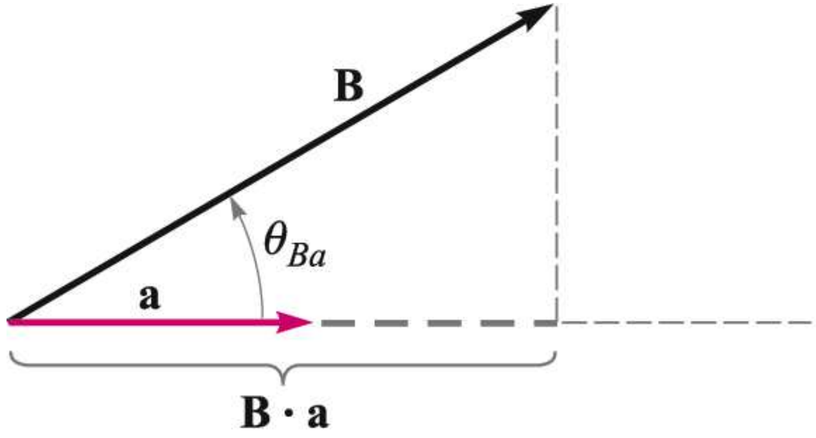
$$\vec{A} \cdot \vec{A} = |\vec{A}||\vec{A}|cos(0) = |\vec{A}|^2$$

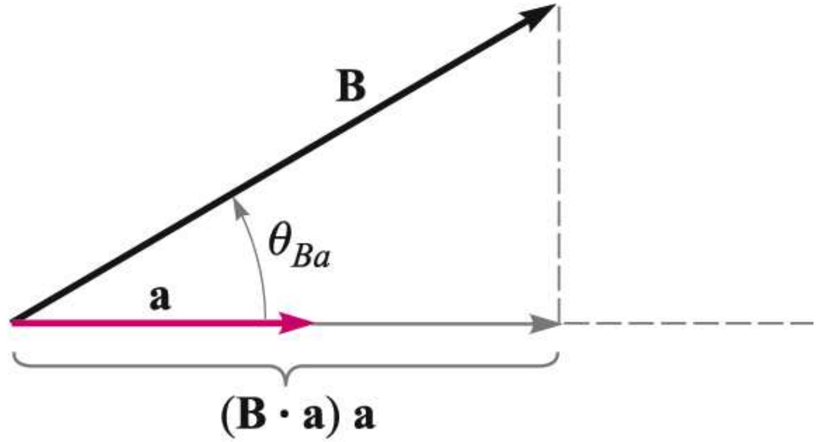
$$\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}|cos(0) = 1$$

$$\vec{A} \cdot \vec{a} = |\vec{A}||\vec{a}|cos(\theta_{Aa}) = |\vec{A}|cos(\theta_{Aa})$$

$$(\vec{A} \cdot \vec{a})\vec{a} = \{|\vec{A}||\vec{a}|cos(\theta_{Aa})\}\vec{a} = |\vec{A}|cos(\theta_{Aa})\,\vec{a}$$

- (Vector) **Projection** using the dot product : Finding the magnitude component of a vector in given direction





B • a gives the component of **B** in the horizontal direction

Magnitude of $\vec{B}$ in direction of $\vec{a}$

(B • a) a gives the *vector* component of **B** in the horizontal direction

$\vec{a}$

- Operational Use of the Dot Product

Given
$$\begin{cases} \mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z \\ \mathbf{B} = B_x\mathbf{a}_x + B_y\mathbf{a}_y + B_z\mathbf{a}_z \end{cases}$$

Find
$$\begin{aligned} \vec{A}\cdot\vec{B} &= \left(A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z}\right)\cdot\left(B_x\overrightarrow{a_x} + B_y\overrightarrow{a_y} + B_z\overrightarrow{a_z}\right) \\ &= A_xB_x\overrightarrow{a_x}\cdot\overrightarrow{a_x} + A_xB_y\overrightarrow{a_x}\cdot\overrightarrow{a_y} + A_xB_z\overrightarrow{a_x}\cdot\overrightarrow{a_z} \\ &\quad + A_yB_x\overrightarrow{a_y}\cdot\overrightarrow{a_x} + A_yB_y\overrightarrow{a_y}\cdot\overrightarrow{a_y} + A_yB_z\overrightarrow{a_y}\cdot\overrightarrow{a_z} \\ &\quad + A_zB_x\overrightarrow{a_z}\cdot\overrightarrow{a_x} + A_zB_y\overrightarrow{a_z}\cdot\overrightarrow{a_y} + A_zB_z\overrightarrow{a_z}\cdot\overrightarrow{a_z} \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned}$$

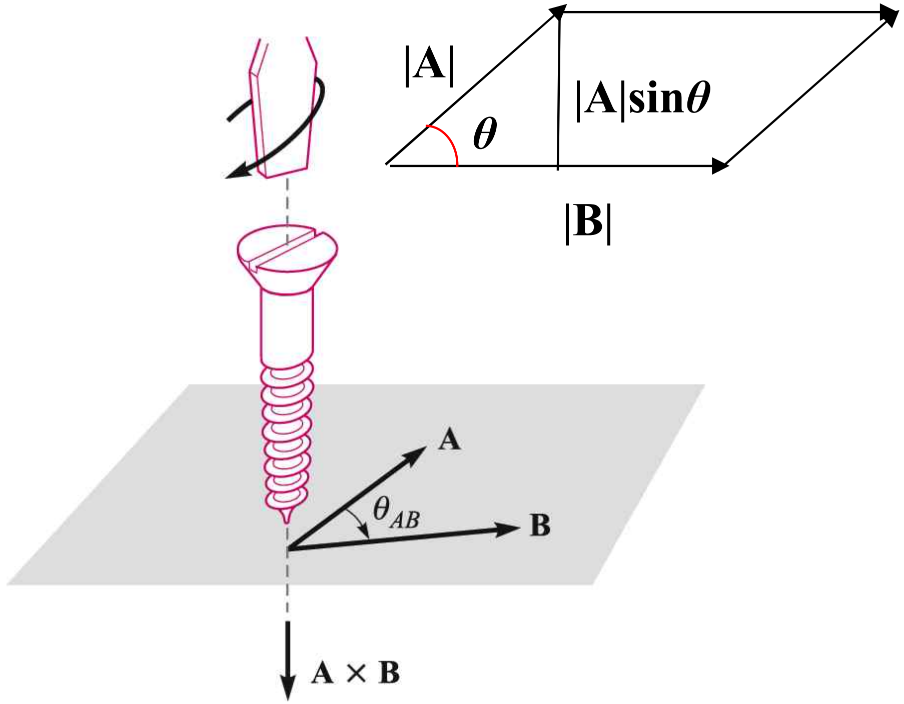
where
$$\begin{cases} \mathbf{a}_x\cdot\mathbf{a}_y = \mathbf{a}_y\cdot\mathbf{a}_z = \mathbf{a}_x\cdot\mathbf{a}_z = 0 \\ \mathbf{a}_x\cdot\mathbf{a}_x = \mathbf{a}_y\cdot\mathbf{a}_y = \mathbf{a}_z\cdot\mathbf{a}_z = 1 \end{cases}$$

Note also:
$$\boxed{\mathbf{A}\cdot\mathbf{A} = A^2 = |\mathbf{A}|^2}$$

1.7 Cross Product (or Vector Product)

$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\,|\mathbf{B}| \sin\theta_{AB}$$

right-handed screw direction



Since $\sin\boldsymbol{\theta}_{AB} = -\sin\boldsymbol{\theta}_{BA}$,

$$\vec{A} \times \vec{B} = -|\vec{A}||\vec{B}| \sin\theta_{BA}\, \overrightarrow{a_N}$$
$$= -|\vec{B}||\vec{A}| \sin\theta_{BA}\, \overrightarrow{a_N}$$
$$= -(\vec{B} \times \vec{A})$$

$$\vec{A} \times \vec{B} = |\vec{A}| sin\theta_{AB} |\vec{B}| \overrightarrow{a_N}$$

← Vector 크기: 평행사변형 면적
방향: 벡터 $\vec{A}$ 와 $\vec{B}$ 의
오른나사 진행방향

1.7.2 Operational Definition of the Cross Product

$$\vec{A} \times \vec{B} = (A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}) \times (B_x\vec{a_x} + B_y\vec{a_y} + B_z\vec{a_z})$$
$$= A_xB_x\vec{a_x} \times \vec{a_x} + A_xB_y\vec{a_x} \times \vec{a_y} + A_xB_z\vec{a_x} \times \vec{a_z}$$

$$A_xB_x\mathbf{a}_x \times \mathbf{a}_x + A_xB_y\mathbf{a}_x \times \mathbf{a}_y + A_xB_z\mathbf{a}_x \times \mathbf{a}_z$$
$$+ A_yB_x\mathbf{a}_y \times \mathbf{a}_x + A_yB_y\mathbf{a}_y \times \mathbf{a}_y + A_yB_z\mathbf{a}_y \times \mathbf{a}_z$$
$$+ A_zB_x\mathbf{a}_z \times \mathbf{a}_x + A_zB_y\mathbf{a}_z \times \mathbf{a}_y + A_zB_z\mathbf{a}_z \times \mathbf{a}_z$$

where $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z, \ \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x, \ \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$

Therefore:

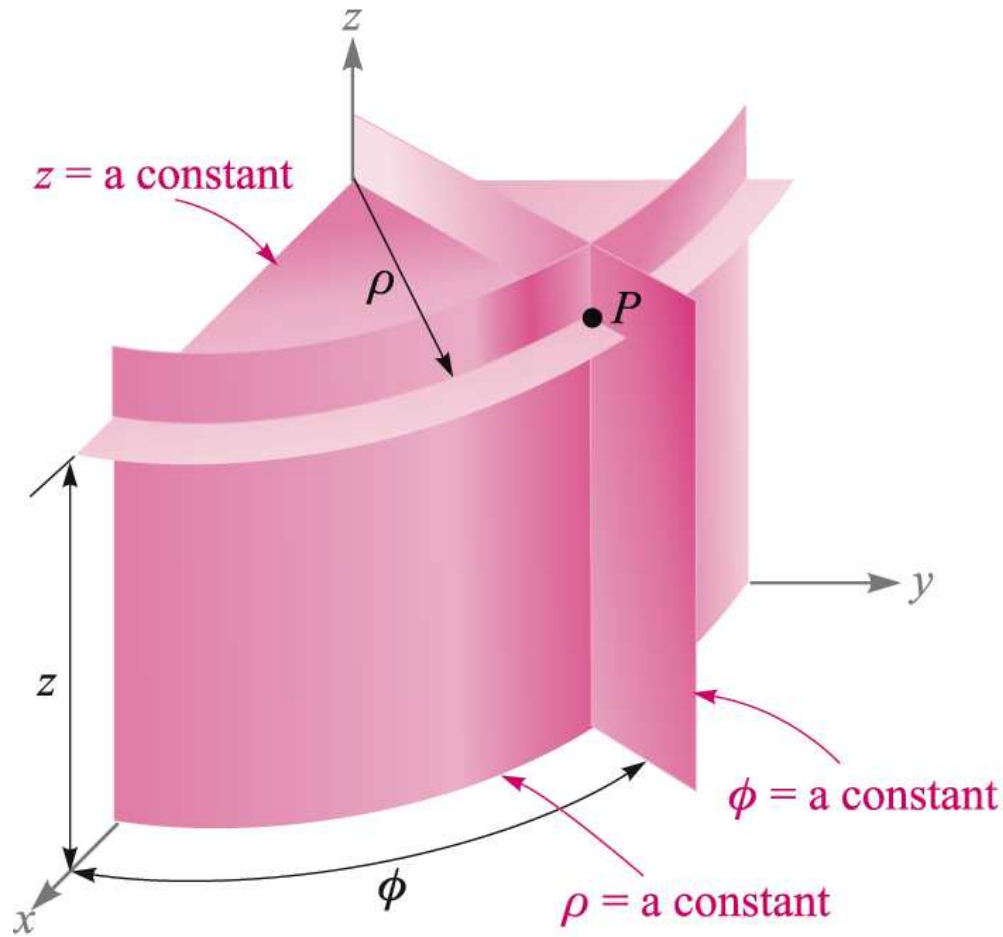
$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z$$

Or…

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

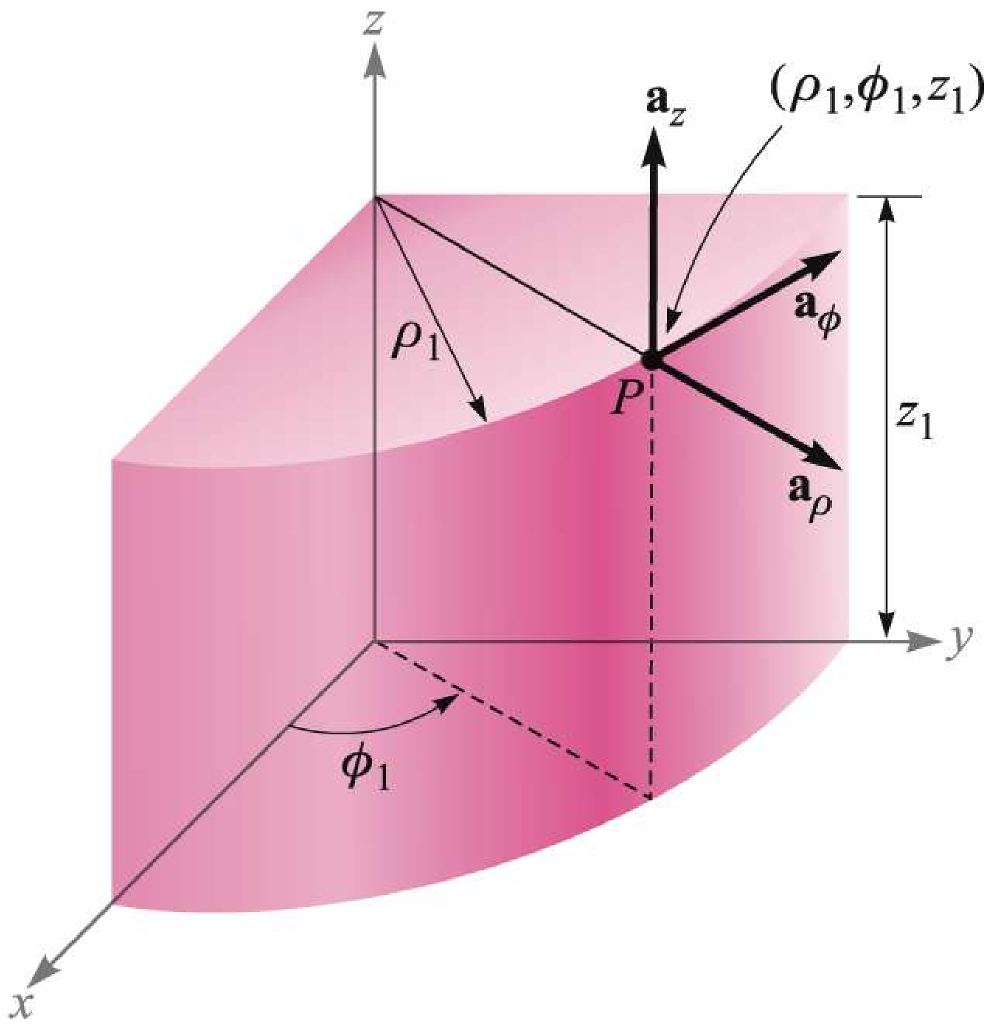
1.8 (Circular) Cylindrical Coordinates

- Point *P* has coordinately specified by $P(\rho, \phi, z)$.
- Right-handed coordinate
- $\rho \rightarrow \phi \rightarrow z \rightarrow \rho \rightarrow \cdots$



- Orthogonal unit vectors in cylindrical coordinates

$$\overrightarrow{a_\rho} \times \overrightarrow{a_\phi} = \overrightarrow{a_z},\quad \overrightarrow{a_\phi} \times \overrightarrow{a_z} = \overrightarrow{a_\rho},\quad \overrightarrow{a_z} \times \overrightarrow{a_\rho} = \overrightarrow{a_\phi}$$



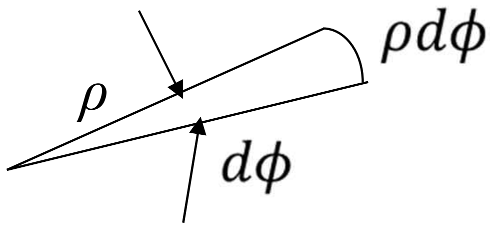
Differential elements in Cylindrical Coordinates

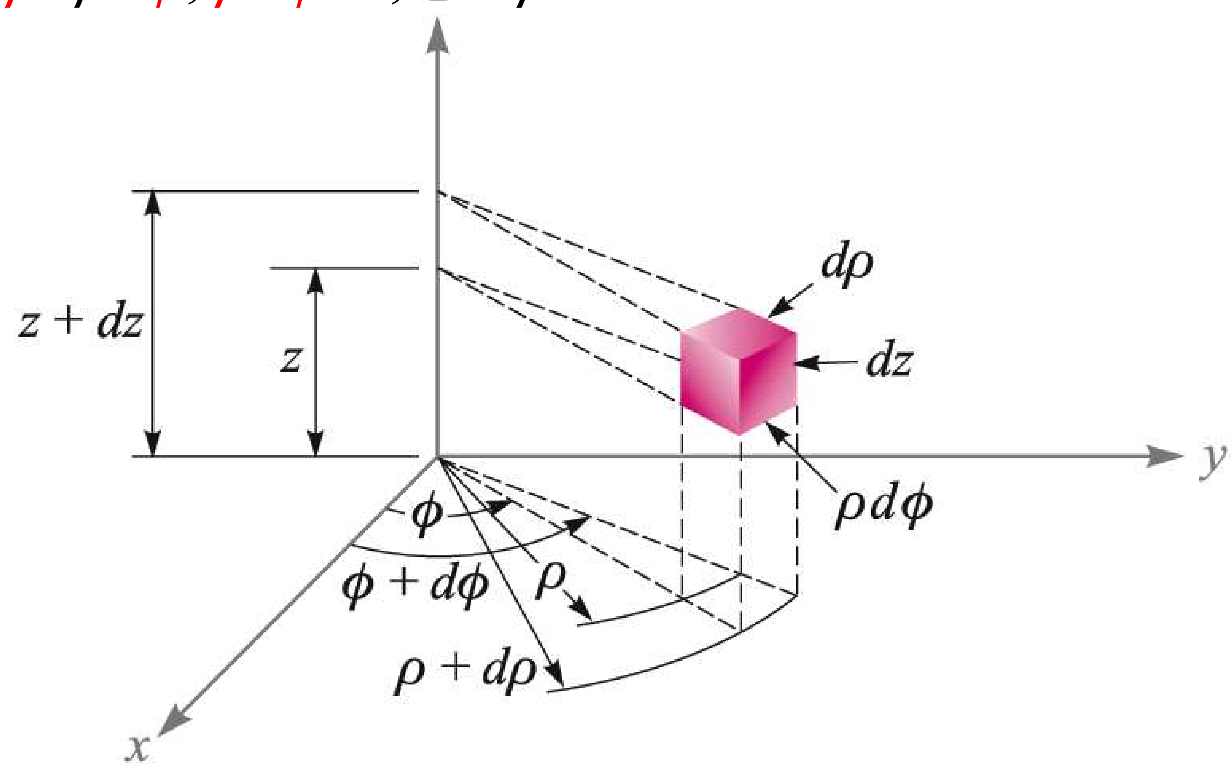
- Differential lengths: $d\rho$, $\rho d\phi$, dz

- Differential areas: $dS = \rho d\rho d\phi$, $\rho d\phi dz$, $dz d\rho$

- Differential volume:

$$dv = \rho d\rho d\phi dz$$

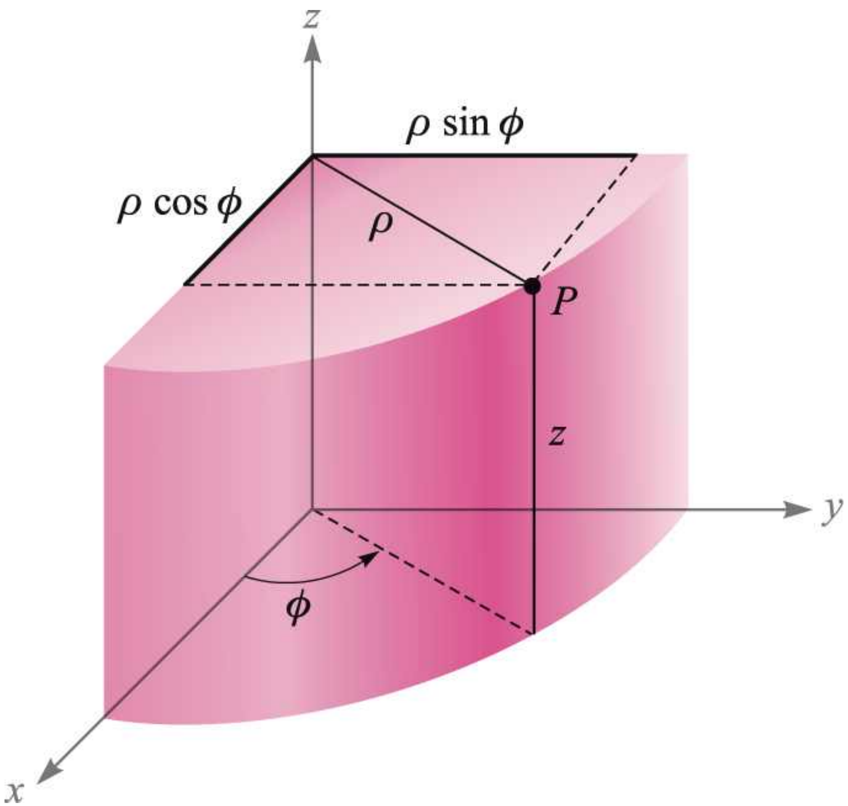




Relation between Cartesian and cylindrical Coordinates

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$
$$\phi = \tan^{-1} \frac{y}{x}$$
$$z = z$$

$$\frac{\boldsymbol{\rho sin\phi}}{\boldsymbol{\rho cos\phi}} = \boldsymbol{tan\phi} = \frac{\boldsymbol{y}}{\boldsymbol{x}}$$



$$x = \rho \cos \phi$$
$$y = \rho \sin \phi$$
$$z = z$$

Ex.] x = -3, y = 4 → 2nd quadrature plane $\rho = 5, \phi = 180^o - 53.1^o = 126.9^o$

x = 3, y = -4 → 4th quadrature plane $\rho = 5, \phi = -53.1^o$

Relation between Cartesian and cylindrical Coordinates

$$\vec{A} = A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z} = A_\rho\overrightarrow{a_\rho} + A_\phi\overrightarrow{a_\phi} + A_z\overrightarrow{a_z}$$

where

$$A_\rho = \vec{A}\cdot\overrightarrow{a_\rho} = (A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z})\cdot\overrightarrow{a_\rho}$$
$$= A_x\overrightarrow{a_x}\cdot\overrightarrow{a_\rho} + A_y\overrightarrow{a_y}\cdot\overrightarrow{a_\rho}$$

$$A_\phi = \vec{A}\cdot\overrightarrow{a_\phi} = A_x\overrightarrow{a_x}\cdot\overrightarrow{a_\phi} + A_y\overrightarrow{a_y}\cdot\overrightarrow{a_\phi}$$

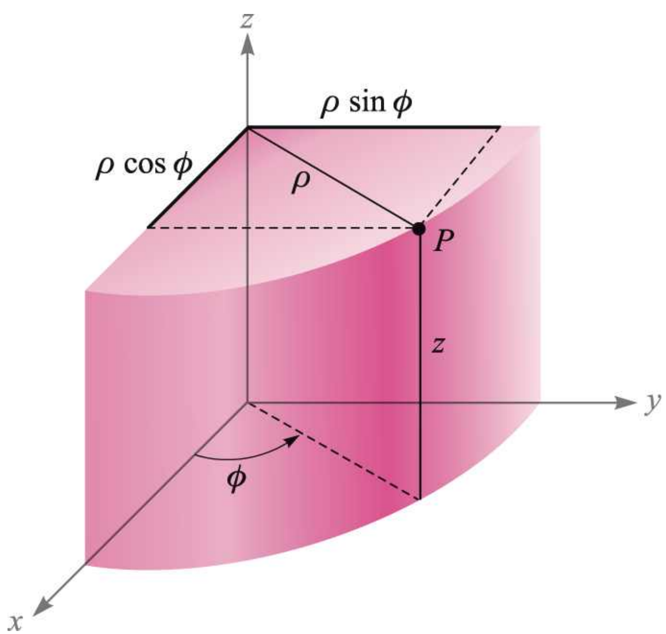
$$A_z = \vec{A}\cdot\overrightarrow{a_z} = \left(A_x\overrightarrow{a_x} + A_y\overrightarrow{a_y} + A_z\overrightarrow{a_z}\right)\cdot\overrightarrow{a_z} = A_z$$

$$\overrightarrow{\boldsymbol{a_\rho}} \cdot \overrightarrow{\boldsymbol{a_x}} = \cos \boldsymbol{\phi}$$

$$\overrightarrow{\boldsymbol{a_\rho}} \cdot \overrightarrow{\boldsymbol{a_y}} = \cos(90^{o} - \phi) = \sin \phi$$

$$\overrightarrow{\boldsymbol{a_\phi}} \cdot \overrightarrow{\boldsymbol{a_x}} = -\cos(90^{o} - \phi) = -\sin \phi$$

$$\overrightarrow{\boldsymbol{a_\phi}} \cdot \overrightarrow{\boldsymbol{a_y}} = \sin(90^{o} - \phi) = \cos\phi$$



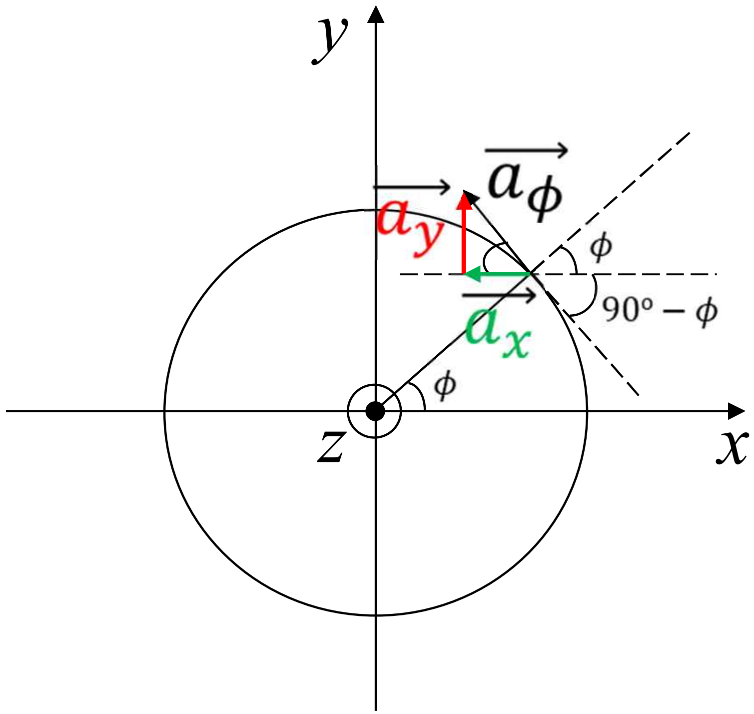


Table 1.1 Dot Products of Unit Vectors in Cylindrical and Rectangular Coordinate Systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	**1**

Example

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Start with:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Then:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y \cos\phi - x \sin\phi = \rho \sin\phi \cos\phi - \rho \cos\phi \sin\phi = 0$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

$$= -y \sin\phi - x \cos\phi = -\rho \sin^2\phi - \rho \cos^2\phi = -\rho$$

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Finally:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

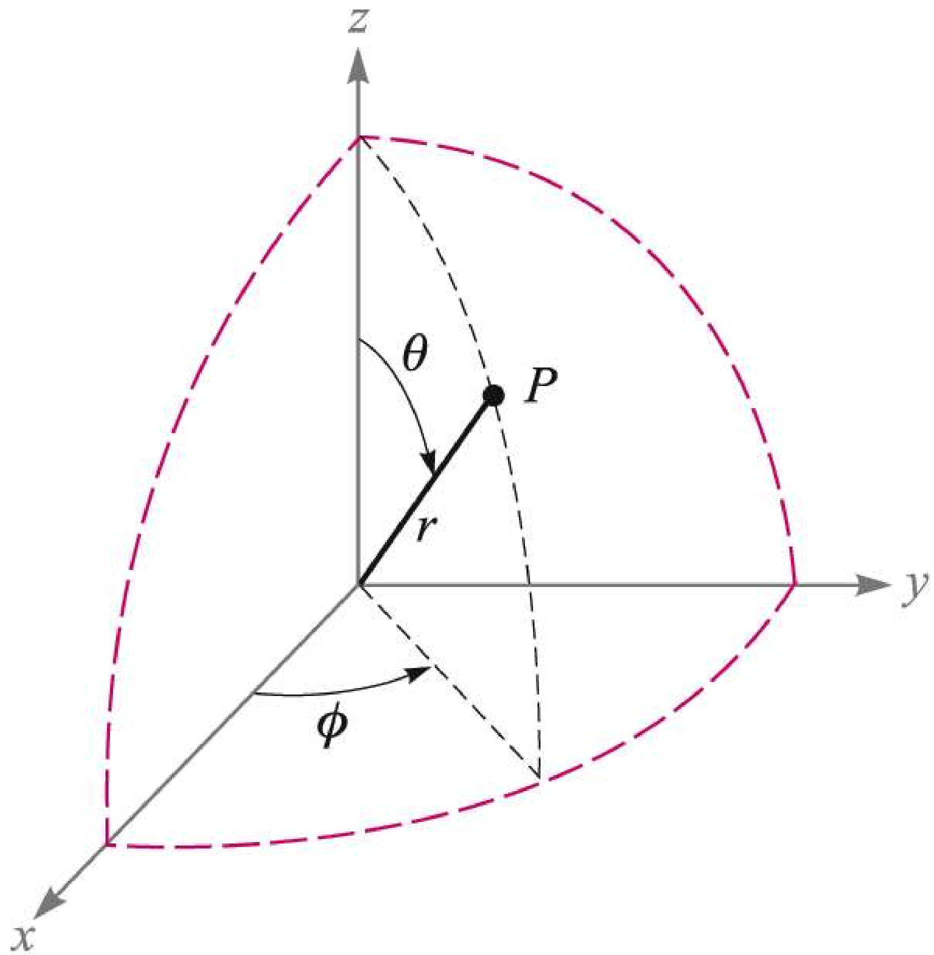
$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

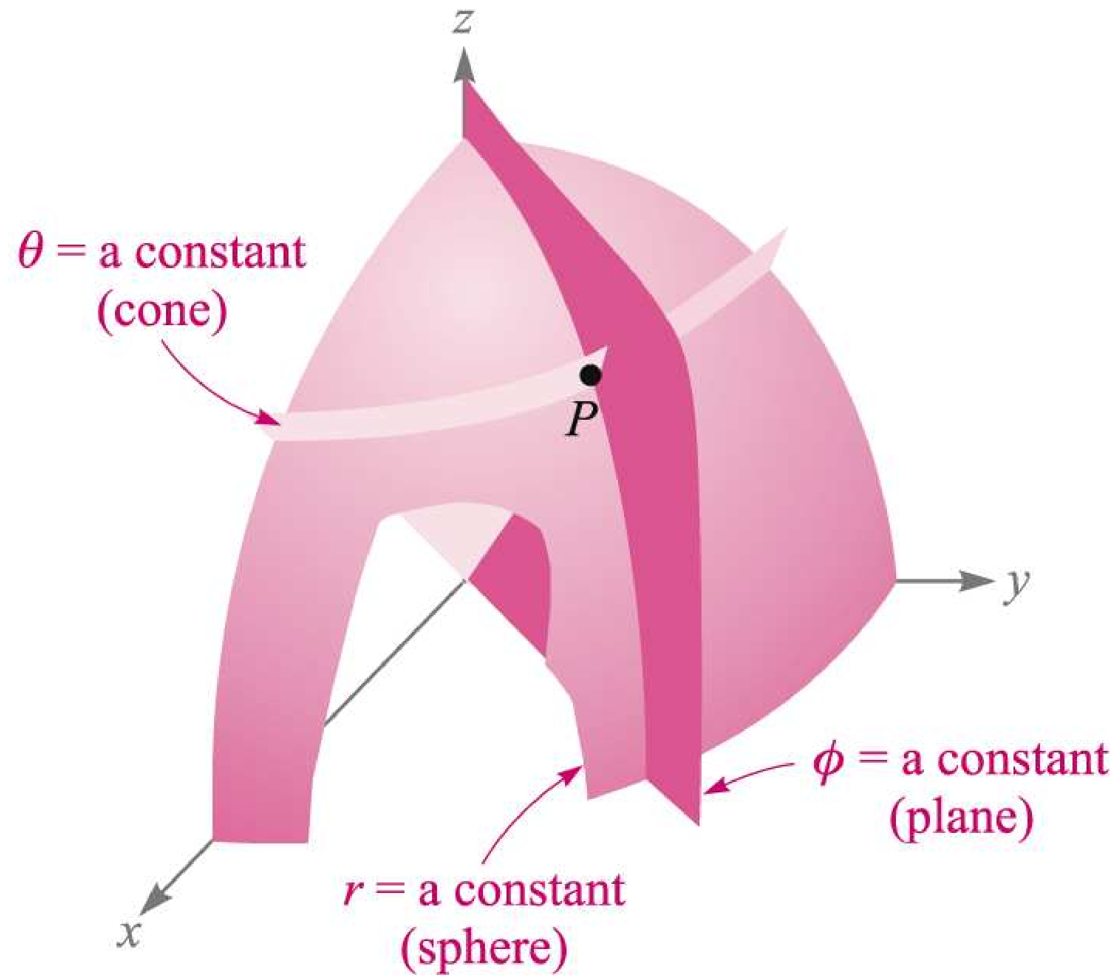
$$\boxed{\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z}$$

1.9 Spherical Coordinate System

- Point P has coordinately specified by $P(r, \theta, \phi)$.
- Right-handed coordinate $r \rightarrow \theta \rightarrow \phi \rightarrow r \rightarrow \cdots$

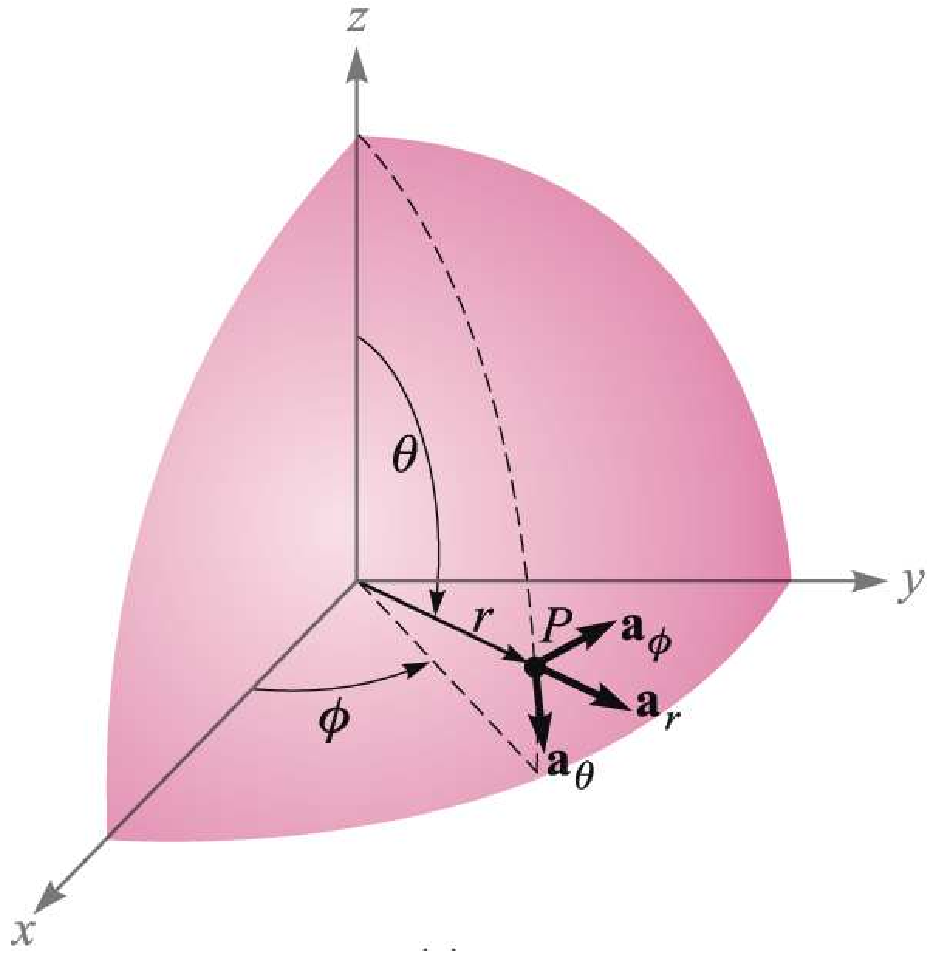


Constant Coordinate Surfaces in Spherical Coordinates



Unit Vector Components in Spherical Coordinates

$$\overrightarrow{a_r} \times \overrightarrow{a_\theta} = \overrightarrow{a_\phi}, \quad \overrightarrow{a_\theta} \times \overrightarrow{a_\phi} = \overrightarrow{a_r}, \quad \overrightarrow{a_\phi} \times \overrightarrow{a_r} = \overrightarrow{a_\theta}$$



Differential Elements in Spherical Coordinates

- Differential lengths: dr, $rd\theta$, $r\sin\theta d\phi$

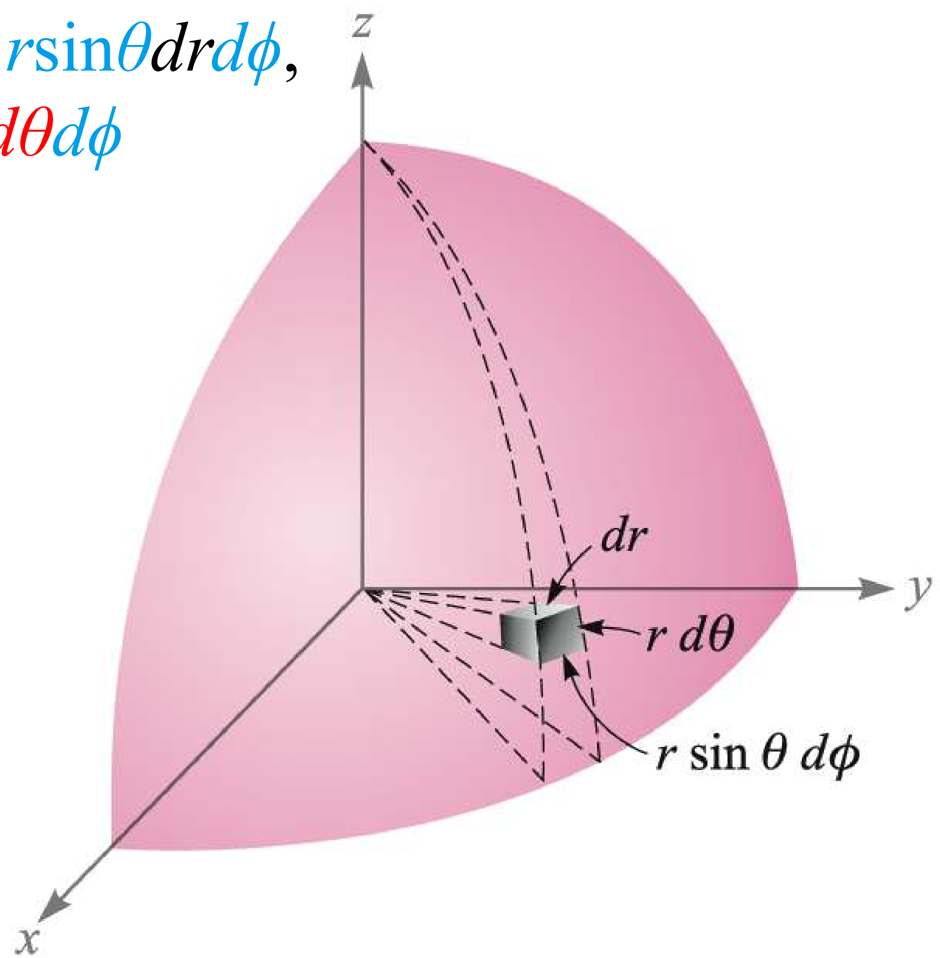
  $\leftarrow (r + dr)d\theta \approx rd\theta$, $(r+dr)\sin\theta d\phi \approx r\sin\theta d\phi$

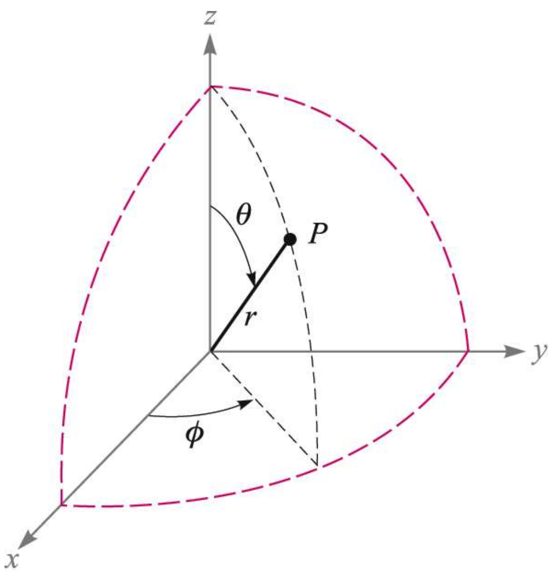
- Differential areas: $dS = rdrd\theta$, $r\sin\theta drd\phi$,

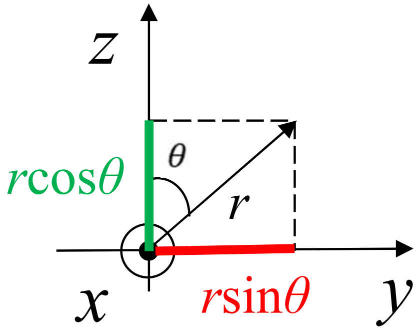
  $r^2\sin\theta d\theta d\phi$

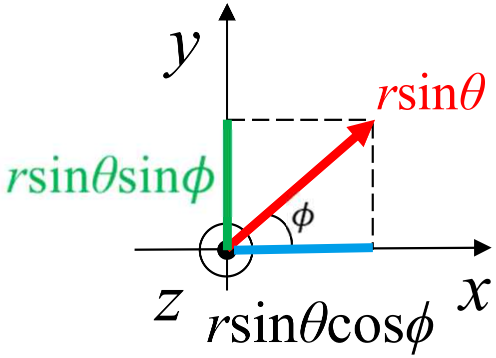
- Differential volume:

  $dv = r^2\sin\theta drd\theta\, d\phi$









$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

$$\theta = \cos^{-1}\frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$

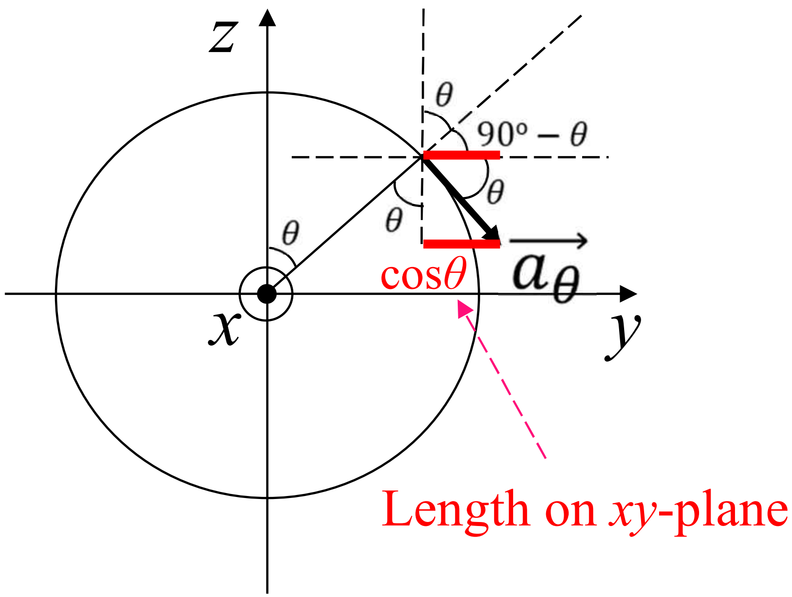
$$\phi = \tan^{-1}\frac{y}{x}$$

$$\vec{A} = A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z} = A_r\vec{a_r} + A_\theta\vec{a_\theta} + A_\phi\vec{a_\phi}$$

$$\vec{a_r} \cdot \vec{a_x} = \sin\theta \left(\cos\phi\,\vec{a_x} + \sin\phi\,\vec{a_y}\right) \cdot \vec{a_x} = \sin\theta\cos\phi$$

$$\vec{a_r} \cdot \vec{a_y} = \sin\theta \left(\cos\phi\,\vec{a_x} + \sin\phi\,\vec{a_y}\right) \cdot \vec{a_y} = \sin\theta\sin\phi$$

$$\vec{a_r} \cdot \vec{a_z} = \cos\theta\vec{a_z} \cdot \vec{a_z} = \cos\theta$$



$$\vec{a_\theta} \cdot \vec{a_x} = \cos\theta\cos\phi$$

$$\vec{a_\theta} \cdot \vec{a_y} = \cos\theta\sin\phi$$

$$\vec{a_\theta} \cdot \vec{a_z} = -\cos(90^\circ - \theta) = -\sin\theta$$

$\vec{a_\phi} \cdot \vec{a_x}, \vec{a_\phi} \cdot \vec{a_y}, \vec{a_\phi} \cdot \vec{a_z}$: Table 1.1

Table 1.2 Dot Products of Unit Vectors in the Spherical and Rectangular Coordinate Systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Example: Vector Component Transformation

Transform the field, $\mathbf{G} = (xz/y)\mathbf{a}_x$, into spherical coordinates and components

$$G_r = \mathbf{G}\cdot\mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi = \frac{r\sin\theta\cos\phi \times r\cos\theta}{r\sin\theta\sin\phi}\times\sin\theta\cos\phi$$

$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G_\theta = \mathbf{G}\cdot\mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi = \frac{r\sin\theta\cos\phi \times r\cos\theta}{r\sin\theta\sin\phi}\times\cos\theta\cos\phi$$

$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G\phi = \mathbf{G}\cdot\mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x\cdot\mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi) = \frac{r\sin\theta\cos\phi \times r\cos\theta}{r\sin\theta\sin\phi}\times(-\sin\phi)$$

$$= -r\cos\theta\cos\phi$$

$$\mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$