Engineering Electromagnetics

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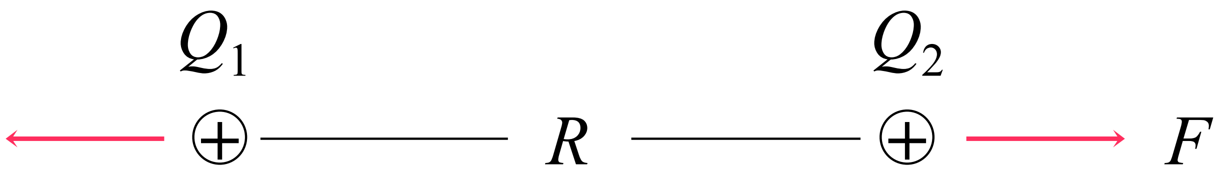
Chapter 2

Coulomb's Law and Electric Field Intensity

2.1 Coulomb's Experimental Law

- Assumption: "Static (or time-invariant) electric field"

- Force of repulsion, *F,* occurs when charges have the same sign. Charges attract when of opposite sign.



$$F = k\frac{Q_1 Q_2}{R^2} \quad \text{where} \quad k = \frac{1}{4\pi\epsilon_0}$$

- **Free space** permittivity

$$\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi} 10^{-9} \text{ F/m}$$

with which the Coulomb force becomes:

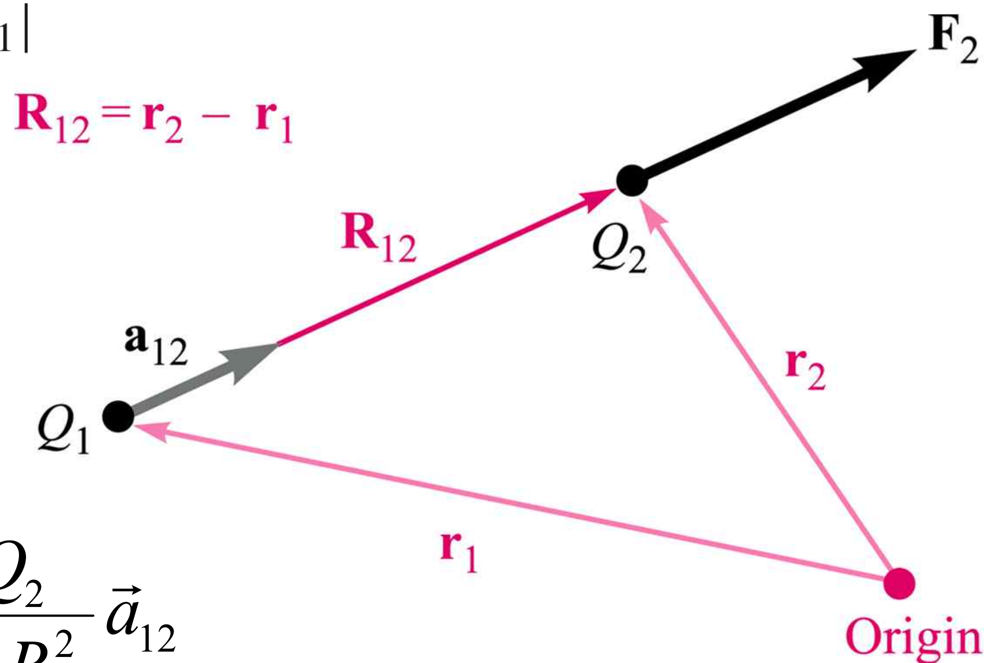
$$F = \frac{Q_1 Q_2}{4\pi \epsilon_0 R^2}$$

➔ Scalar,
Not vector

- $\overrightarrow{\boldsymbol{F_2}}$: Force on Q_2 in case Q_1 and Q_2 have the same sign.

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^3}\overrightarrow{R_{12}}$$

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$



$$\vec{F}_1 = -\vec{F}_2 = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\vec{a}_{21} = -\frac{Q_1 Q_2}{4\pi\varepsilon_0 R_{12}^2}\vec{a}_{12}$$

Ex.] $Q_1 = 3\times10^{-4}\text{C}$ @$(1, 2, 3)$, $Q_2 = -10^{-4}\text{C}$ @$(2, 0, 5)$

$$\vec{R}_{12} = \vec{r}_2 - \vec{r}_1 = (2-1)\vec{a}_x + (0-2)\vec{a}_y + (5-3)\vec{a}_z = \vec{a}_x - 2\vec{a}_y + 2\vec{a}_z$$

$$\vec{a}_{12} = \frac{\vec{a}_x - 2\vec{a}_y + 2\vec{a}_z}{\sqrt{1+4+4}} = \frac{1}{3}(\vec{a}_x - 2\vec{a}_y + 2\vec{a}_z)$$

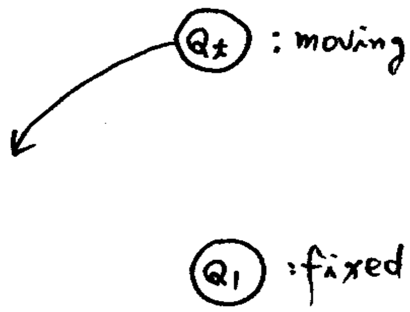
$$\vec{F}_2 = \frac{\left(3\times10^{-4}\right)\cdot\left(-10^{-4}\right)}{4\pi\times\frac{1}{36\pi}\times10^{-9}\times9}\frac{\vec{a}_x - 2\vec{a}_y + 2\vec{a}_z}{3} = -30\left(\frac{\vec{a}_x - 2\vec{a}_y + 2\vec{a}_z}{3}\right)N$$

$$= -10\vec{a}_x + 20\vec{a}_y - 20\vec{a}_z \ \ [\text{N}]$$

- The force on a charge in the presence of several other charges is the sum of the forces on a charge due to each of the other charges acting alone. (**Superposition Theorem**)

2.2 Electric Field Intensity

- Consider the force acting on a test charge, Q_t , arising from charge Q_1:



$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

where $\mathbf{a}_{1t}$: unit vector directed from Q_1 to Q_t

- **Electric field intensity** : Force per unit test charge

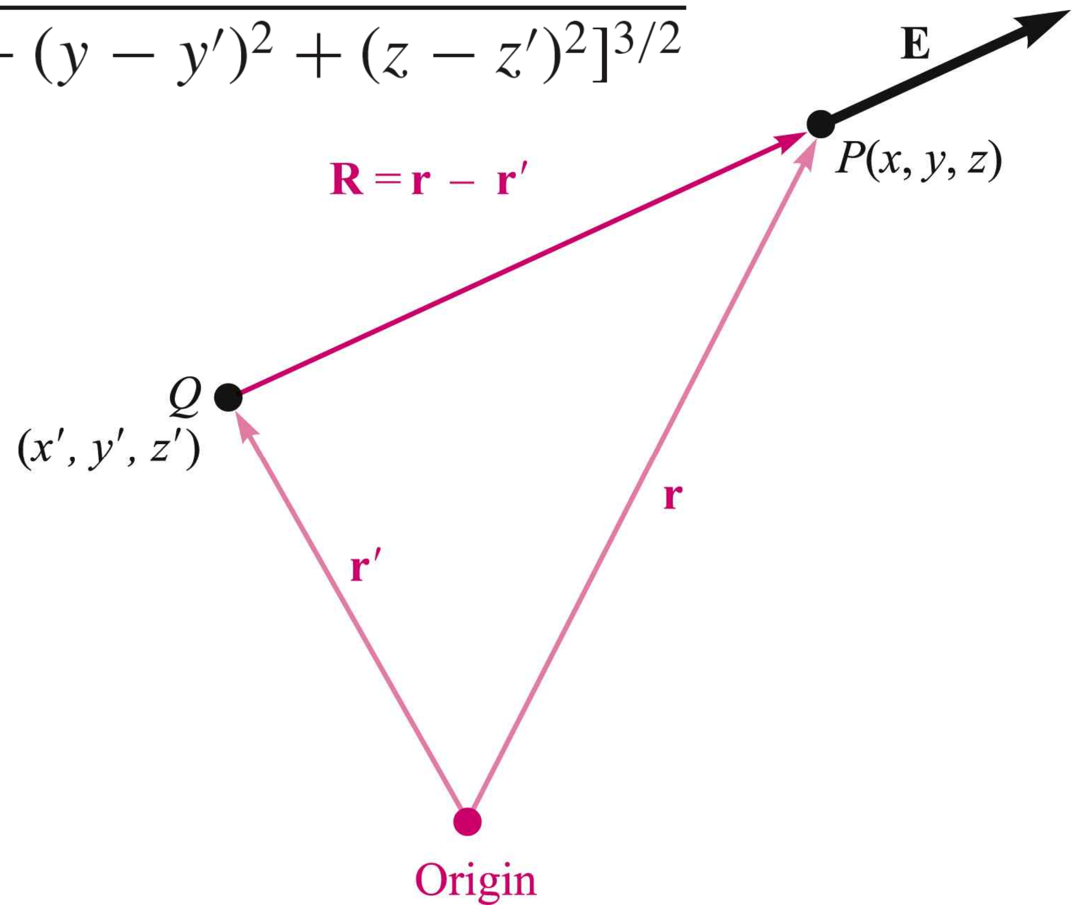
$$\boxed{\overrightarrow{E_t} = \frac{\overrightarrow{F_t}}{Q_t}} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\overrightarrow{a_{1t}} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^3}\overrightarrow{R_{1t}}$$

- A more convenient unit for electric field is V/m, as will be shown.

Electric Field of a Charge Off-Origin

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$

$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2 + (y-y')^2 + (z-z')^2]^{3/2}}$$

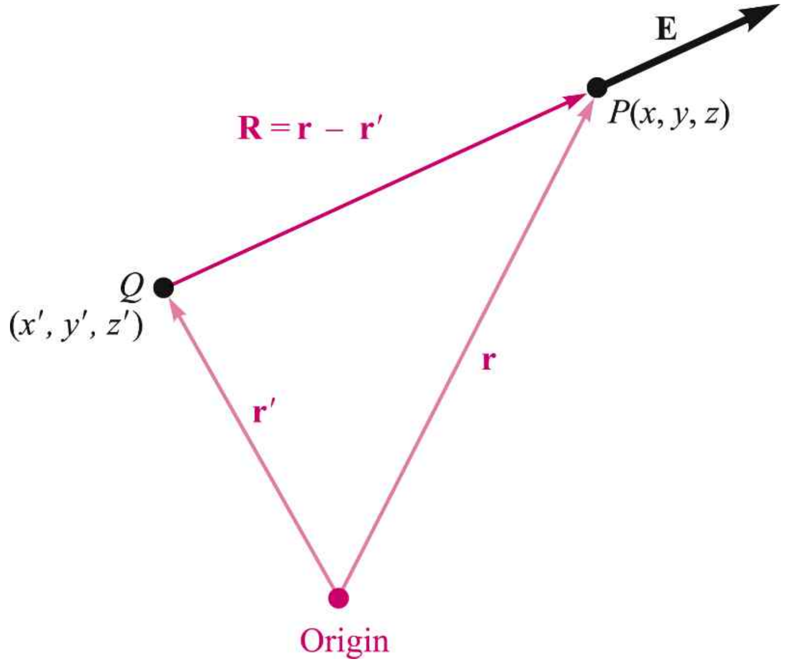


Electric Field of a Charge at Origin

- Q is at the origin and a test charge (1 C) is at (x, y, z).

$$\vec{R} = \vec{r} = x\vec{a}_x + y\vec{a}_y + z\vec{a}_z$$

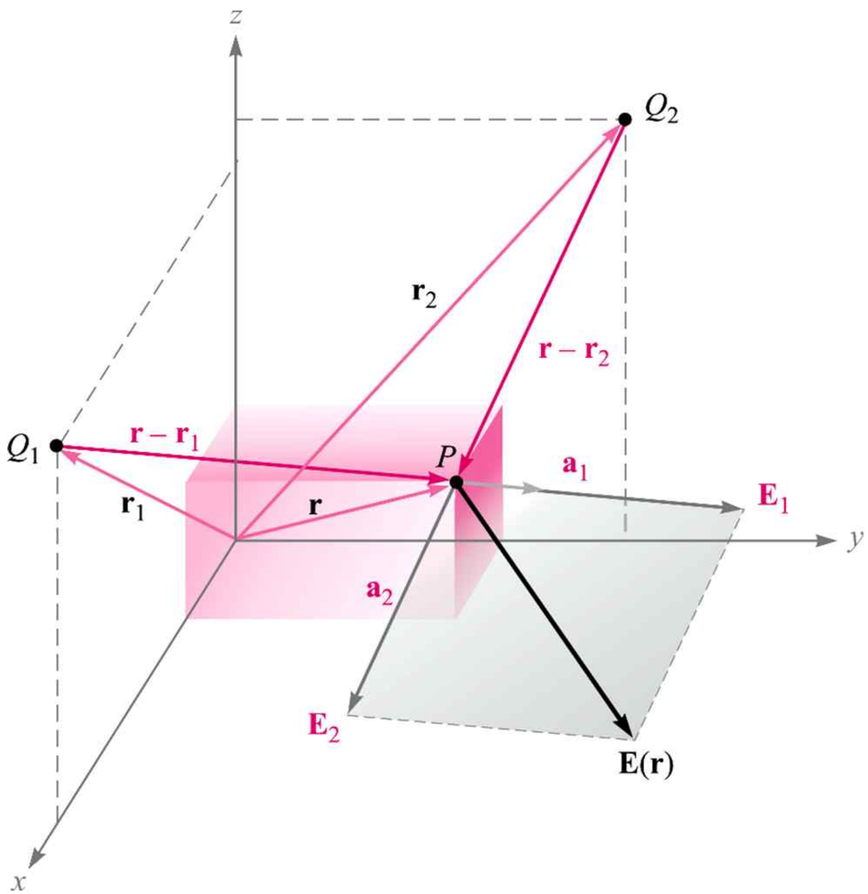
$$\vec{a}_R = \vec{a}_r = \frac{x\vec{a}_x + y\vec{a}_y + z\vec{a}_z}{\sqrt{x^2 + y^2 + z^2}}$$



$$\vec{E} = \frac{Q}{4\pi\varepsilon_o(x^2 + y^2 + z^2)}\left(\frac{x}{\sqrt{x^2 + y^2 + z^2}}\vec{a}_x + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\vec{a}_y + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\vec{a}_z\right)$$

Superposition of Fields from Two Point Charges

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$



For *n* charges:

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$

* Assumption : Individual charge must be **independent to each other**.

➔ Superposition theorem

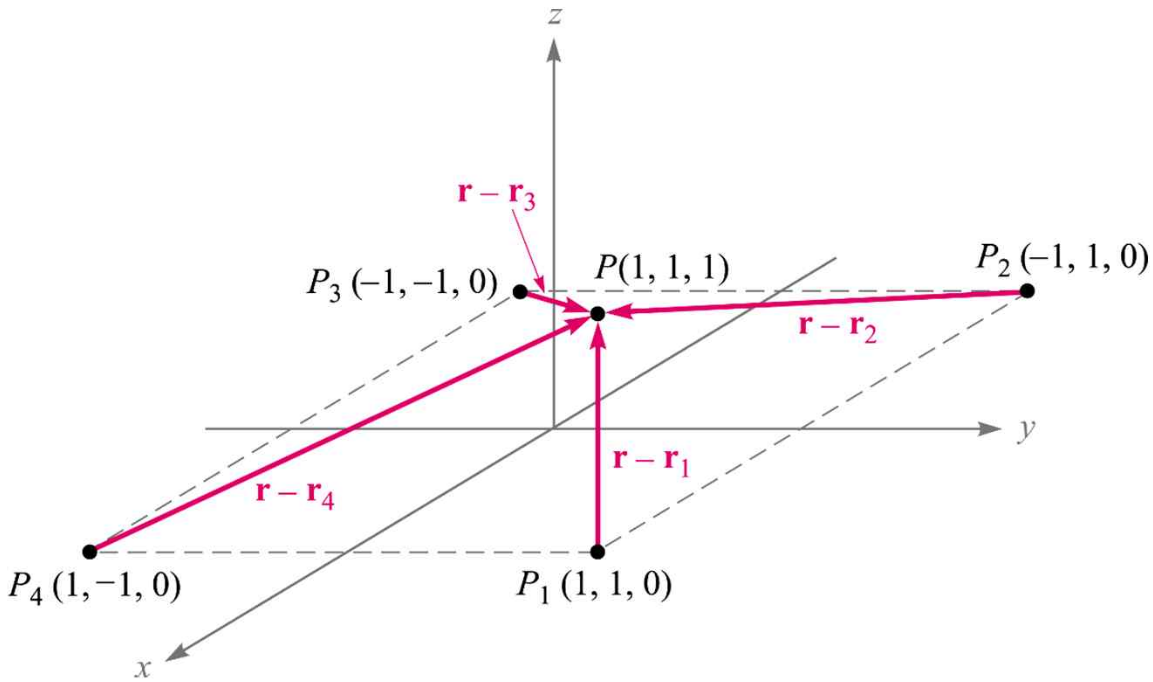
Example

- 3 nC @ $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, $P_4(1, -1, 0)$

  $\vec{E} = ?$ @$P(1, 1, 1)$

Find **E** at *P*, using

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$



First, find the vectors:

$$\vec{R}_1 = \vec{r} - \vec{r}_1 = \vec{a}_z \qquad \vec{R}_2 = \vec{r} - \vec{r}_2 = 2\vec{a}_x + \vec{a}_z$$

$$\vec{R}_3 = \vec{r} - \vec{r}_3 = 2\vec{a}_x + 2\vec{a}_y + \vec{a}_z \qquad \vec{R}_4 = \vec{r} - \vec{r}_4 = 2\vec{a}_y + \vec{a}_z$$

Example (continued)

$$Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96[\mathrm{V} \cdot \mathrm{m}]$$

$$\therefore \vec{E} = \sum_{m=1}^{4} \frac{Q}{4\pi\varepsilon_0} \frac{1}{R_m^3} \vec{R}_m = \frac{Q}{4\pi\varepsilon_0} \sum_{m=1}^{4} \frac{1}{R_m^3} \vec{R}_m$$

$$\mathbf{E} = 26.96 \left[\frac{\mathbf{a}_z}{1} \frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3} \frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} \right]$$

$$= \underline{6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \ \mathrm{V/m}}$$

2.3 Field arising from a Continuous Volume Charge Distribution

2.3.1 Volume Charge Density Definition

- Volume charge density (ρ_v [C/m^3]): Distribution of very small particles with a smooth continuous distribution
- Small amount of charge ΔQ within a small volume Δv :

$$\Delta Q = \rho_v \Delta v \quad \Rightarrow \quad \rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v} = \frac{dQ}{dv}$$

- Total charge : $Q = \int_{\text{vol}} \rho_v \, dv$

$$dv = dx\,dy\,dz, \quad \rho\,d\rho\,d\phi\,dz, \quad r^2 \sin\theta\,dr\,d\theta\,d\phi$$

Ex. 2-3] Find the charge contained within a 2-cm length of the electron beam shown below.

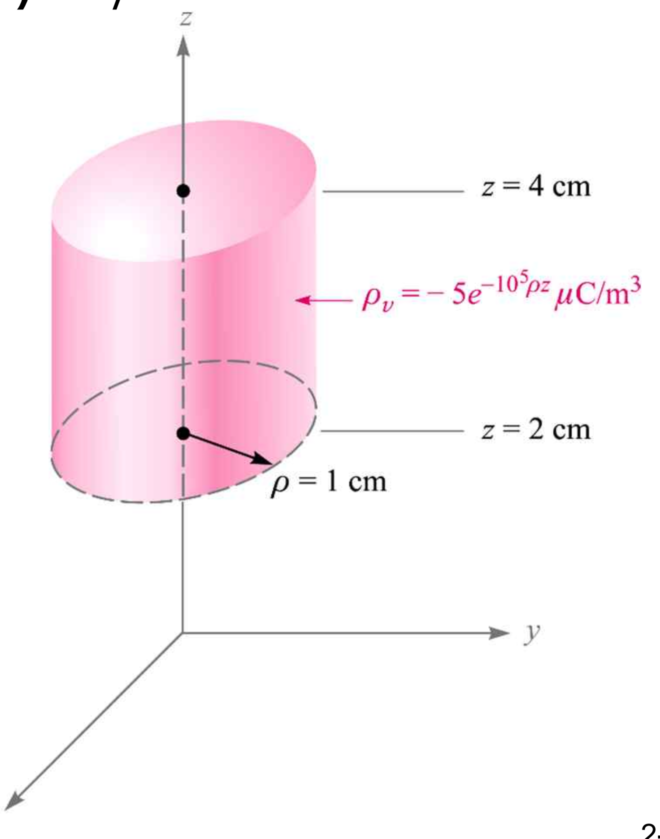
- Charge density : $\rho_v = -5e^{-10^5 \rho z}\left[\mu\text{C}/\text{m}^3\right]$

$$= -5\times 10^{-6}\times e^{-10^5 \rho z}\left[\text{C}/\text{m}^3\right]$$

$$Q = \underbrace{\int_{0.02}^{0.04}}_{\Delta z} \underbrace{\int_{0}^{2\pi}}_{\Delta\phi} \underbrace{\int_{0}^{0.01}}_{\Delta\rho} \left(-5\times 10^{-6}\times e^{-10^5 \rho z}\right)\rho\, d\rho\, d\phi\, dz$$

$$= \int_{0.02}^{0.04}\int_{0}^{0.01}\left(-10^{-5}\pi\times e^{-10^5 \rho z}\right)\rho\, d\rho\, dz$$

$$= \int_{0}^{0.01}\left[\frac{-10^{-5}\pi}{-10^{5}\rho} e^{-10^5 \rho z}\rho\right]_{z=0.02}^{z=0.04} d\rho$$



$$= \int_0^{0.01} (10^{-10}\pi)(e^{-4000\rho} - e^{-2000\rho})d\rho = \int_0^{0.01} (-10^{-10}\pi)(e^{-2000\rho} - e^{-4000\rho})d\rho$$

$$= \left(-10^{-10}\pi\right)\left[\frac{e^{-2000\rho}}{(-2000)} - \frac{e^{-4000\rho}}{(-4000)}\right]_0^{0.01}$$

무시 가능

$$= \left(-10^{-10}\pi\right)\left[\frac{e^{-20}}{(-2000)} - \frac{1}{-2000} - \frac{e^{-40}}{(-4000)} + \frac{1}{(-4000)}\right] \leftarrow e^{-1} \approx 0.368$$

$$\approx \left(-10^{-10}\pi\right)\left(\frac{1}{2000} - \frac{1}{4000}\right) = \left(-10^{-12}\pi\right)\left(\frac{1}{20} - \frac{1}{40}\right) = \left(-\frac{\pi}{40}\right) \times 10^{-12} \text{ [C]}$$

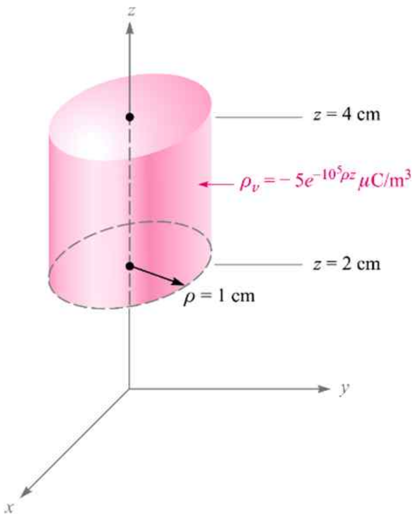
$$= -\frac{\pi}{40} \text{ [pC]}$$

2.3.2 Electric Field from Volume Charge Distributions

- The incremental contribution to the electric field intensity at $\vec{r}$ produced by an incremental charge ΔQ at $\vec{r}\,'$:

$$\Delta\vec{E}(r) = \frac{\Delta Q}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^2} \cdot \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|} = \frac{\rho_v \Delta v}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')$$

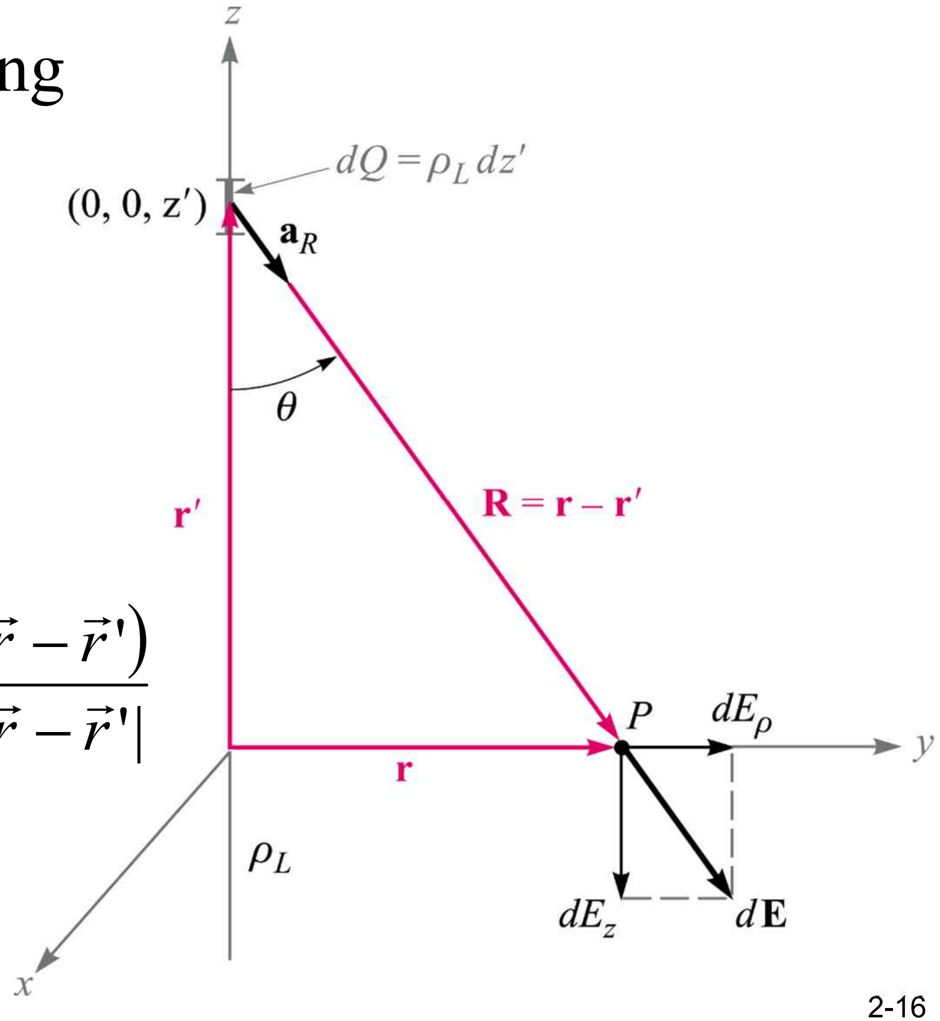
$$\therefore \vec{E}(r) = \int_{vol} \frac{\rho_v(r')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^3}(\vec{r}-\vec{r}\,')dv'$$



🡐 Sum all contributions throughout a volume and take the limit as Δv approaches zero

2.4 Line Charge Electric Field

- A filamentlike distribution of volume charge density such as a very fine and sharp beam ➔ Line charge density ρ_L [C/m]
- Straight-line charge extending along the z-axis
- Move around the line charge, varying ϕ while keeping ρ and z constantly.
- Unit charge: $dQ = \rho_L dz'$
- Unit electric field intensity:



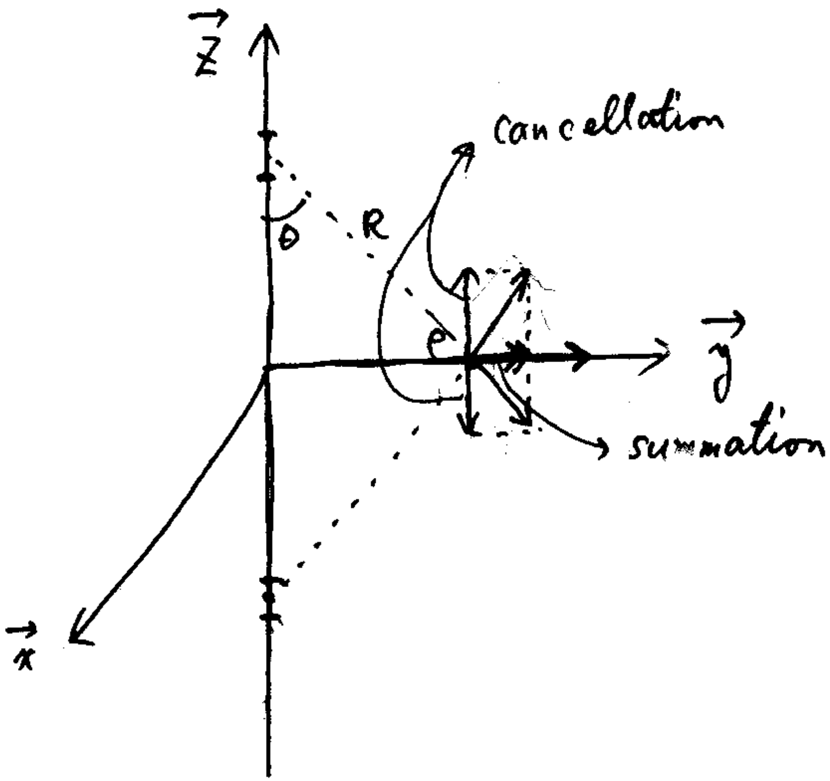
$$d\vec{E} = \frac{\rho_L dz'(\vec{r}-\vec{r}\,')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^3} = \frac{\rho_L dz'}{4\pi\varepsilon_0 |\vec{r}-\vec{r}\,'|^2} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}$$

where $\vec{r} = y\vec{a}_y = \rho\vec{a}_\rho \qquad \vec{r}\,' = z'\vec{a}_z$

$\vec{r}-\vec{r}\,' = \rho\vec{a}_\rho - z'\vec{a}_z$

$$\therefore d\vec{E} = \frac{\rho_L dz'}{4\pi\varepsilon_0\left(\rho^2 + z'^2\right)} \frac{\rho\vec{a}_\rho - z'\vec{a}_z}{\sqrt{\rho^2 + z'^2}}$$

$$= \frac{\rho_L dz'}{4\pi\varepsilon_0} \frac{\left(\rho\vec{a}_\rho - z'\vec{a}_z\right)}{\left(\rho^2 + z'^2\right)^{3/2}}$$



- By dE_z cancellation,

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

$$E_\rho = \int_{-\infty}^{\infty} \frac{\rho_L \rho dz'}{4\pi\epsilon_0(\rho^2 + z'^2)^{3/2}}$$

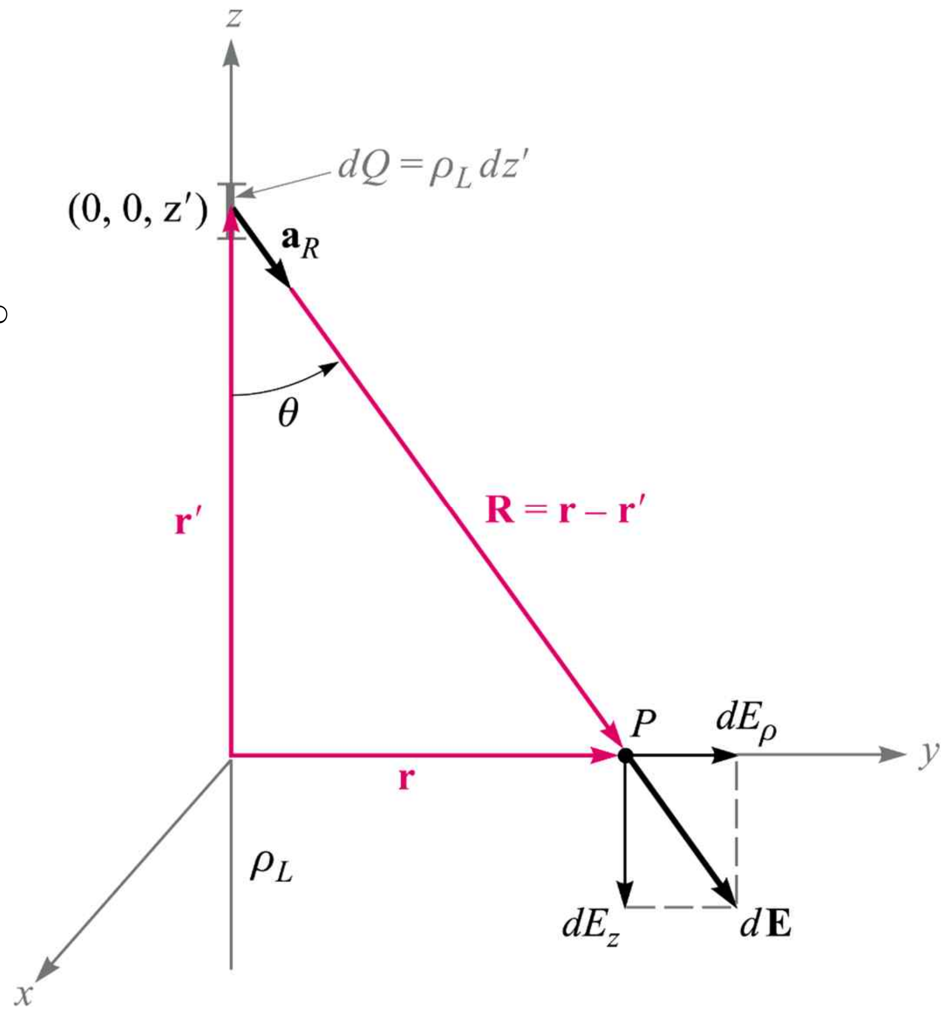
- Appling $\int \frac{dx}{\left(a^2+x^2\right)^{3/2}} = \frac{x}{a^2\sqrt{a^2+x^2}}$,

$$E_\rho = \frac{\rho_L}{4\pi\varepsilon_0}\cdot\rho\left(\frac{1}{\rho^2}\frac{z'}{\sqrt{\rho^2+z'^2}}\right)_{-\infty}^{\infty}$$

$$= \frac{\rho_L\cdot\rho}{4\pi\varepsilon_0}\frac{1}{\rho^2}\cdot\left(1-(-1)\right)$$

$$= \frac{\rho_L}{2\pi\varepsilon_0\rho}$$

$$\vec{E} = E_\rho\vec{a}_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a}_\rho$$



- Another method

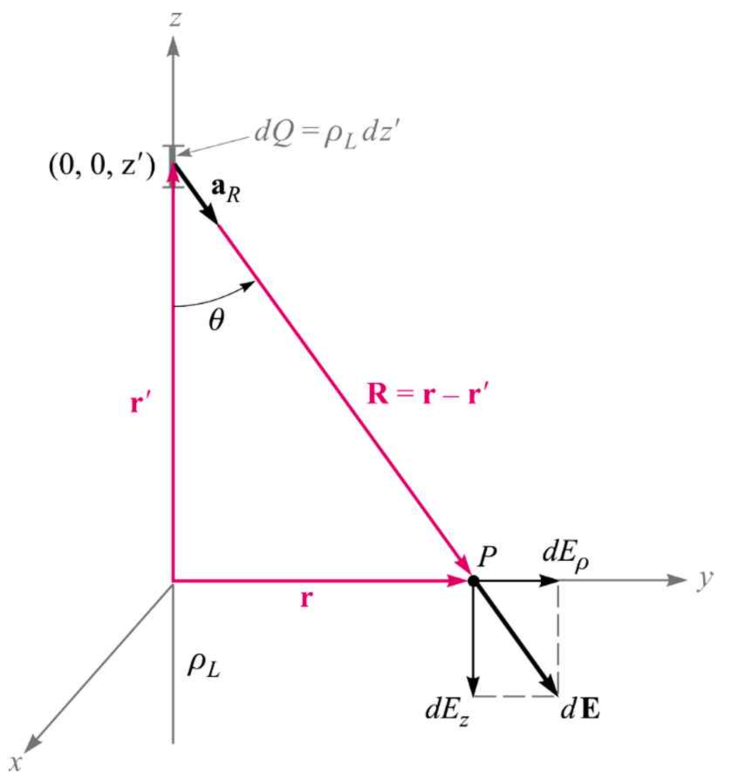
$$z' = \rho \cot\theta$$

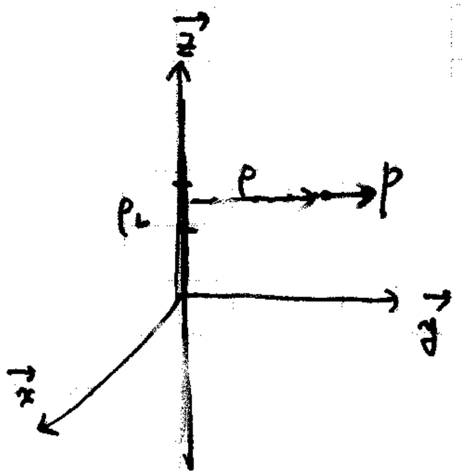
$$\frac{dz'}{d\theta} = -\rho \csc^2\theta \qquad \therefore dz' = -\rho\csc^2\theta d\theta$$

$$R = \rho\csc\theta$$

$$dE_\rho = \frac{\rho_L \rho dz'}{4\pi\varepsilon_0 R^3} = \frac{\rho_L}{4\pi\varepsilon_0}\frac{\rho}{\rho^3\csc^3\theta}(-\rho)\csc^2\theta d\theta$$

$$= -\frac{\rho_L}{4\pi\varepsilon_0\rho\csc\theta}d\theta = -\frac{\rho_L\sin\theta}{4\pi\varepsilon_0\rho}d\theta$$





$$E_\rho = -\frac{\rho_L}{4\pi\varepsilon_0\rho}\int_\pi^0 \sin\theta d\theta = \frac{\rho_L}{4\pi\varepsilon_0\rho}[\cos\theta]_\pi^0$$

$$= \frac{\rho_L}{2\pi\varepsilon_0\rho}$$

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a}_\rho$$

선전하가 있을 때 임의의 위치에서의 field는 측정 점에서 제일 가까운 위치에 있는 선전하 밀도로부터 거리에 반비례하고 선전하 밀도에 비례

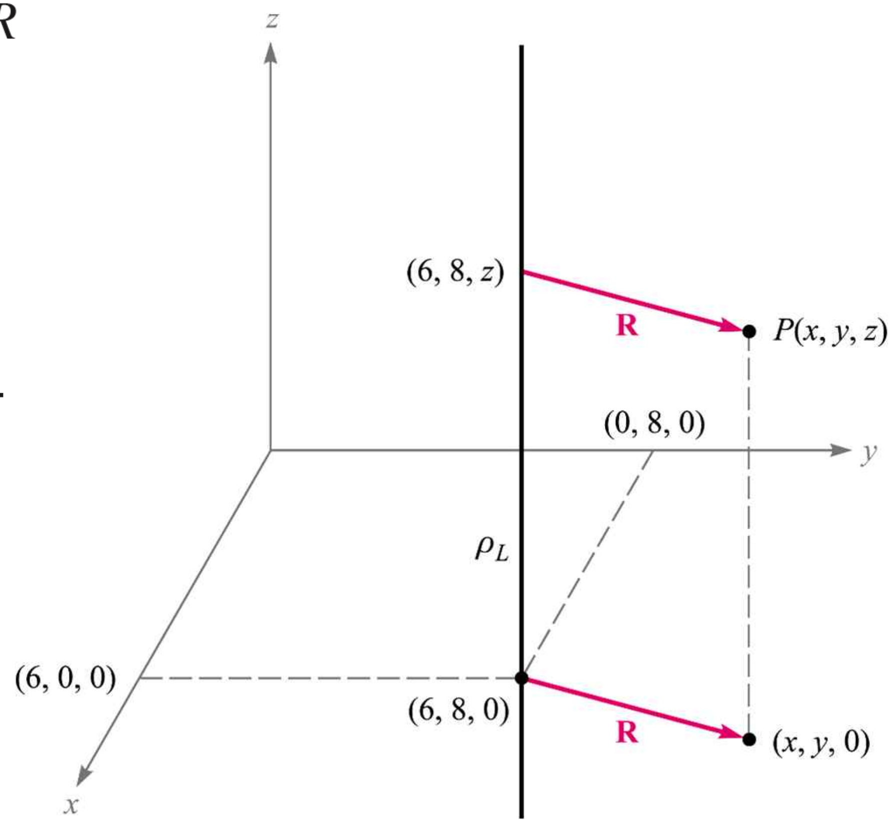
2.4.2 Field of an Off-Axis Line Charge

- Electric field in case of line displaced to (6,8)

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0\sqrt{(x-6)^2+(y-8)^2}}\mathbf{a}_R$$

where

$$\mathbf{a}_R = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{(x-6)\mathbf{a}_x+(y-8)\mathbf{a}_y}{\sqrt{(x-6)^2+(y-8)^2}}$$

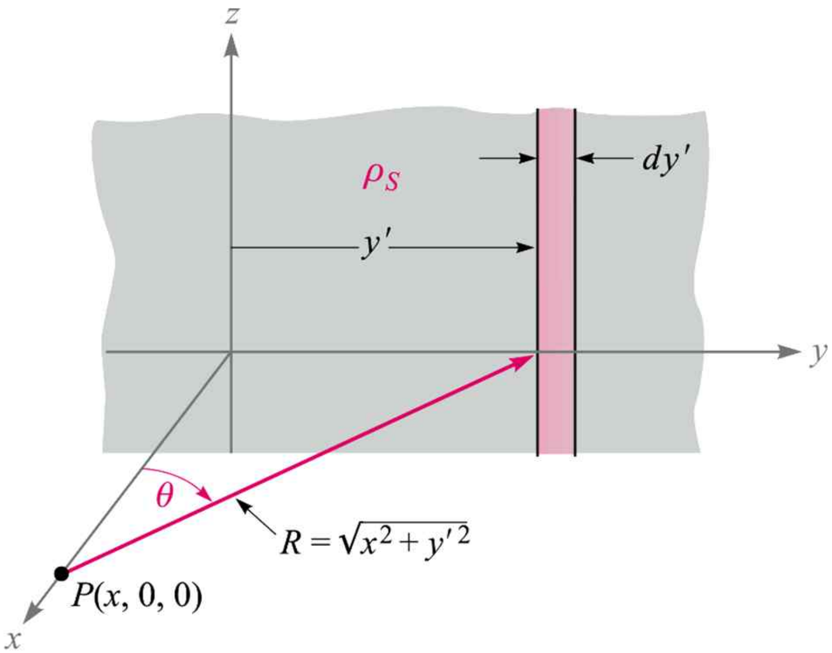


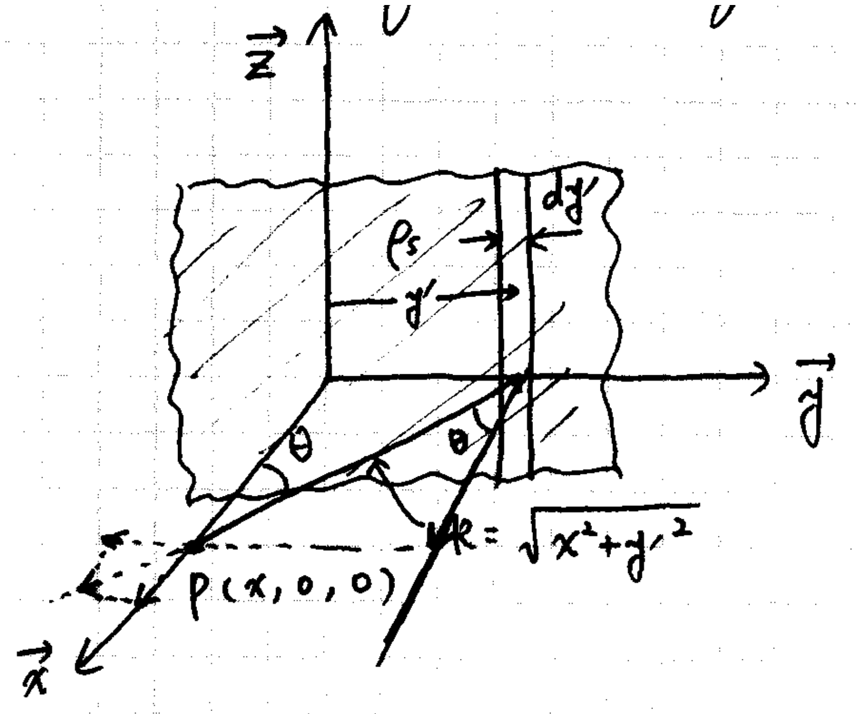
- Finally:

$$\mathbf{E} = \frac{\rho_L}{2\pi\epsilon_0}\frac{(x-6)\mathbf{a}_x+(y-8)\mathbf{a}_y}{(x-6)^2+(y-8)^2}$$

2.5 Field of a Sheet Charge

- Surface charge density ρ_S[C/m^2]: infinite sheet of charge having a uniform density
- Consider the field of the infinite line charge by dividing the infinite sheet into differential-width (*dy'*) strips

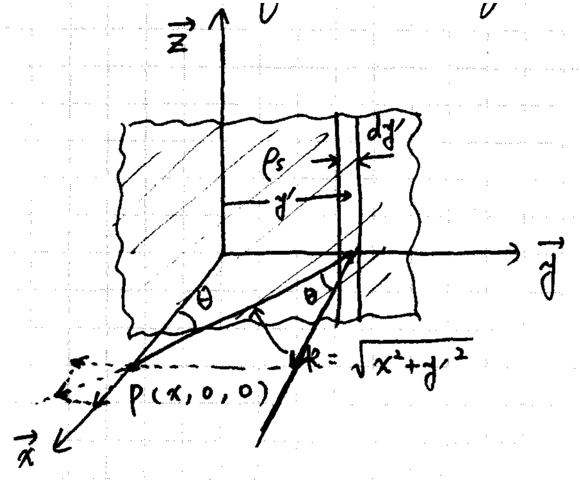




$$\rho_L = \rho_s dy'$$

$$R = \sqrt{x^2 + y'^2}$$

대칭성에 의해 *dEy*
성분은 Canceling-out 됨.



$$\therefore dE_x = \frac{\rho_s dy'}{2\pi\varepsilon_0 \sqrt{x^2 + y'^2}} \cos\theta = \frac{\rho_s x}{2\pi\varepsilon_0 \left(x^2 + y'^2\right)} dy'$$

$$E_x = \int_{-\infty}^{\infty} \frac{\rho_s x}{2\pi\varepsilon_0 \left(x^2 + y'^2\right)} dy' \qquad \Leftarrow y' = x\tan\theta \qquad dy' = x\sec^2\theta d\theta$$

$$= \frac{\rho_s}{2\pi\varepsilon_0} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \sec^2\theta}{x^2 \sec^2\theta} d\theta = \frac{\rho_s}{2\pi\varepsilon_0} \left[\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{\rho_s}{2\varepsilon_0}$$

$$\vec{E} = \frac{\rho_s}{2\varepsilon_0} \vec{a}_x$$

$$\therefore \quad \vec{E} = \frac{\rho_s}{2\varepsilon_0} \vec{a}_N \quad \text{(in general)}$$

1) charge 평면에 **n**ormal 한 방향.
2) 거리에 무관.

2.5.3 Capacitor Model

- $\vec{E} = ?$ in conditions of ρ_S @ $x = 0$ and $-\rho_S$ @ $x = a$

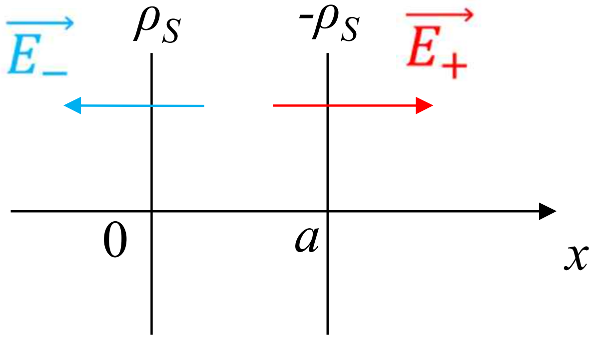
1) In the region of $x > a$, $\vec{E}_+ = \frac{\rho_s}{2\varepsilon_0}\vec{a}_x \qquad \vec{E}_- = \frac{-\rho_s}{2\varepsilon_0}\vec{a}_x$

$$\therefore \vec{E} = \vec{E}_+ + \vec{E}_- = 0$$

2) In the region of $x < 0$, $\vec{E}_+ = \frac{\rho_s}{2\varepsilon_0}(-\vec{a}_x) \qquad \vec{E}_- = \frac{-\rho_s}{2\varepsilon_0}(-\vec{a}_x)$

$$\therefore \vec{E} = \vec{E}_+ + \vec{E}_- = 0$$

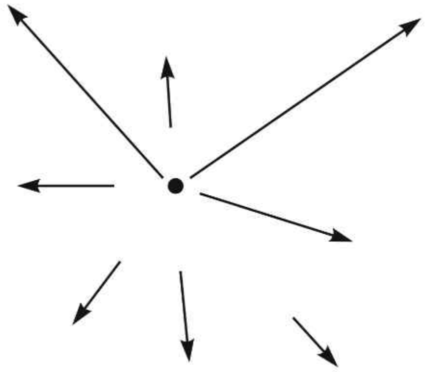
3) In the region of $0 < x < a$, $\vec{E}_+ = \frac{\rho_s}{2\varepsilon_0}\vec{a}_x \qquad \vec{E}_- = \frac{-\rho_s}{2\varepsilon_0}(-\vec{a}_x)$



$$\therefore \vec{E} = \vec{E}_+ + \vec{E}_- = \frac{\rho_s}{\varepsilon_o}\vec{a}_x$$

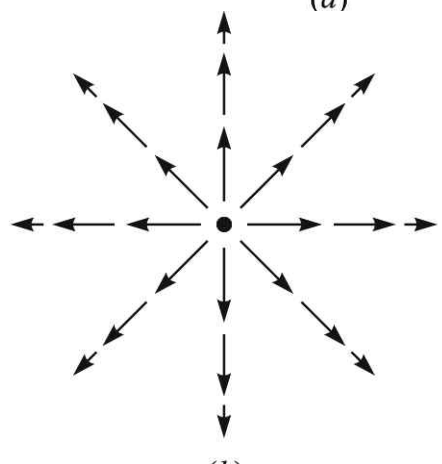
2.6 Streamline and Sketches of Fields:

- One picture is worth about a thousand words if we just knew what picture to draw. (百聞不如一見)
- Electric field due to line charge: $\vec{E} = \dfrac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a}_\rho$



(a)

a) $\vec{\phi}$ Symmetric property is not explained.

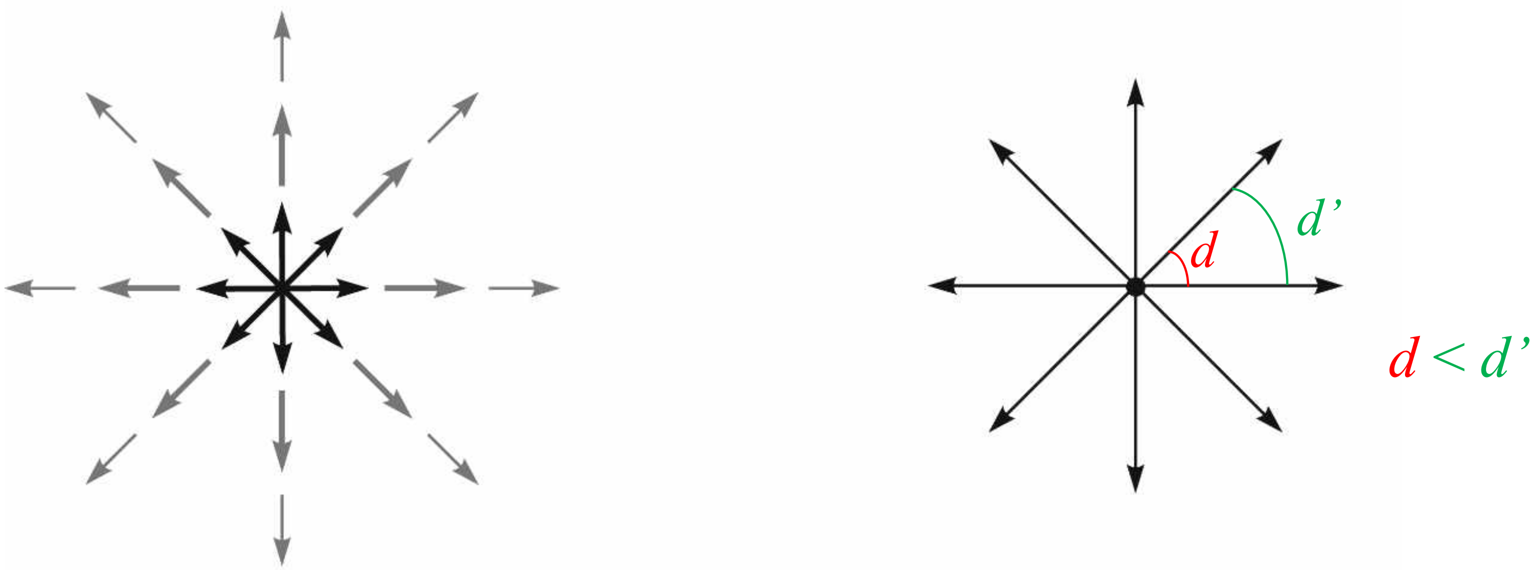


(b)

b) Although $\vec{\phi}$ symmetry property is explained, the longest lines must be drawn in the most crowded region.

c) Although $\vec{\phi}$ symmetry property is explained, the stronger field must be explained with the thicker line, especially at origin.

d) The spacing of the lines is inversely proportional to the strength of the field.



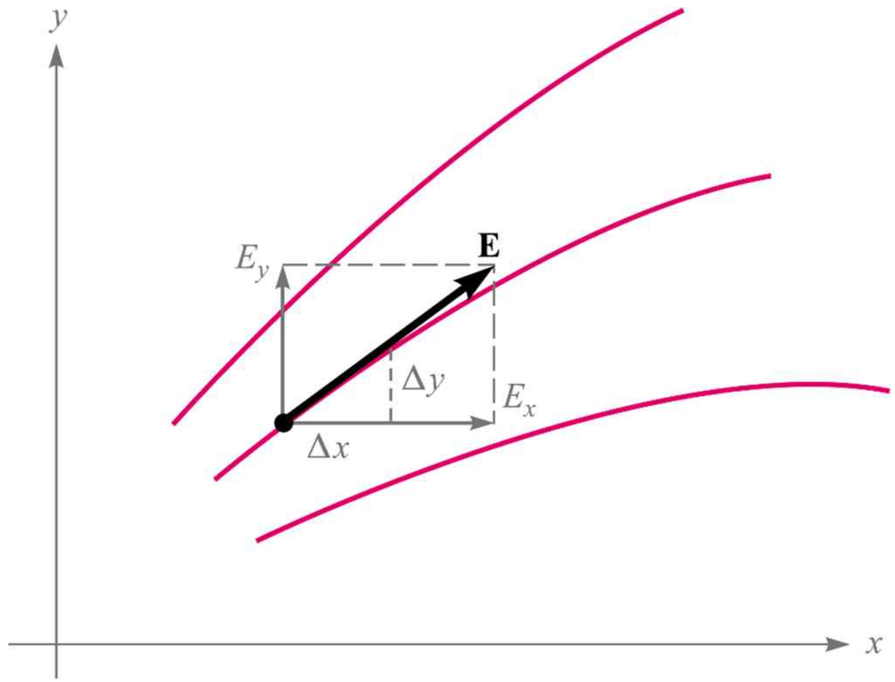
(c) (d)

Methodology of Streamline Construction

- Sketch the field of the point charge in 2-dimension (*xy*-plane).

$$\frac{E_y}{E_x} = \frac{dy}{dx} \qquad @\ E_z = 0$$

➔ This equation will enable us to obtain the equations of the streamlines.



Ex.] $\vec{E} = \frac{\rho_L}{2\pi\varepsilon\rho}\vec{a}_\rho \qquad \Leftarrow \rho_L = 2\pi\varepsilon$

$= \frac{1}{\rho}\vec{a}_\rho$ in cylindrical coordinate

$$\vec{E} = \frac{1}{\rho}\left[(\vec{a}_\rho \cdot \vec{a}_x)\vec{a}_x + (\vec{a}_\rho \cdot \vec{a}_y)\vec{a}_y + (\vec{a}_\rho \cdot \vec{a}_z)\vec{a}_z\right] = \frac{1}{\rho}\left[\cos\phi\vec{a}_x + \sin\phi\vec{a}_y\right]$$

$$= \frac{1}{\sqrt{x^2+y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}\vec{a}_x + \frac{y}{\sqrt{x^2+y^2}}\vec{a}_y\right)$$

$= \frac{x}{x^2+y^2}\vec{a}_x + \frac{y}{x^2+y^2}\vec{a}_y = E_x\vec{a}_x + E_y\vec{a}_y$: in Catesian coordinate

$\frac{dy}{dx} = \frac{E_y}{E_x} = \frac{y/\left(x^2+y^2\right)}{x/\left(x^2+y^2\right)} = \frac{y}{x}$ or $\frac{dy}{y} = \frac{dx}{x}$

$\therefore \ln y = \ln x + C$ or $\ln y = \ln x + \ln C = \ln Cx.$

$y = Cx$

If this stream line pass through P(-2, 7, 10), C = -3.5