Engineering Electromagnetics

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Chapter 3:

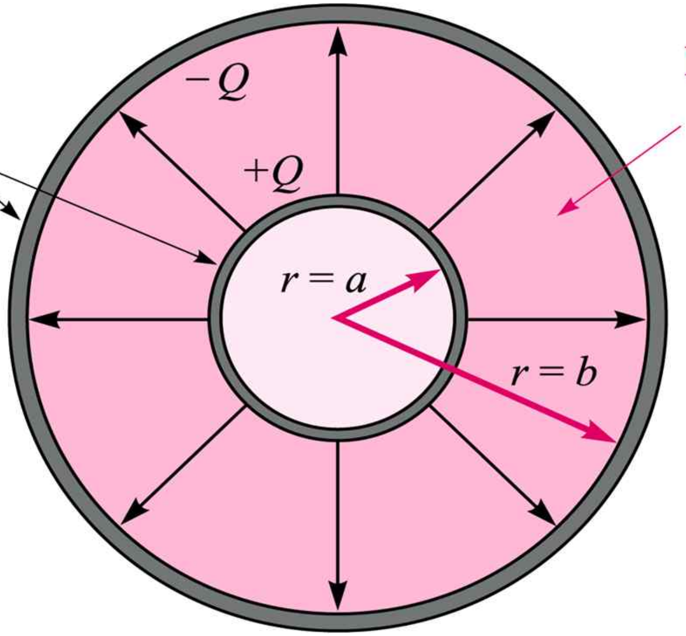
Electric Flux Density, Gauss' Law,

and Divergence

3.1 Electric Flux Density
3.1.1 Faraday's Experiments on Electric Displacement

- Experimental steps

1. With the equipment dismantled, the inner sphere was given a known positive charge.
2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.
3. The outer sphere was discharged by connecting it momentarily to ground.
4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.



5. The inner charge, Q, induces an equal and opposite charge, $-Q$, on the inside surface of the outer sphere. This phenomenon is maintained for intermediate materials.

Faraday conclusion: There was some sort of "displacement" from the inner sphere to the outer which was independent of the medium ➔ Displacement flux or Electric flux: Ψ [psi]

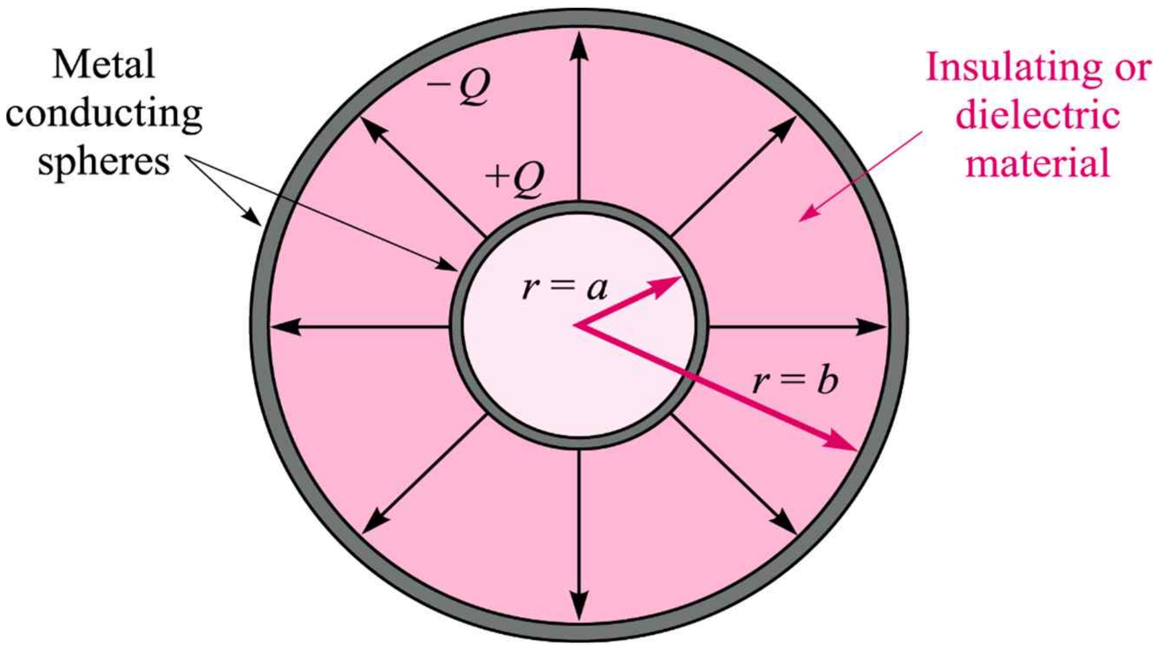
$$\therefore \ \Psi = Q$$

3.1.2 Electric Flux Density

- At the surface of the inner sphere, Ψ coulombs of electric flux are produced by the charge Q (= Ψ) coulombs distributed uniformly over a surface having an area of $4\pi a^2$ [m^2].

- **Electric Flux Density** ($\vec{D}$) : density of flux at the specific surface

$$\boxed{D(r=a) = \frac{\Psi}{4\pi a^2} = \frac{Q}{4\pi a^2}}$$



Radially-Dependent Electric Flux Density

- Electric flux densities:

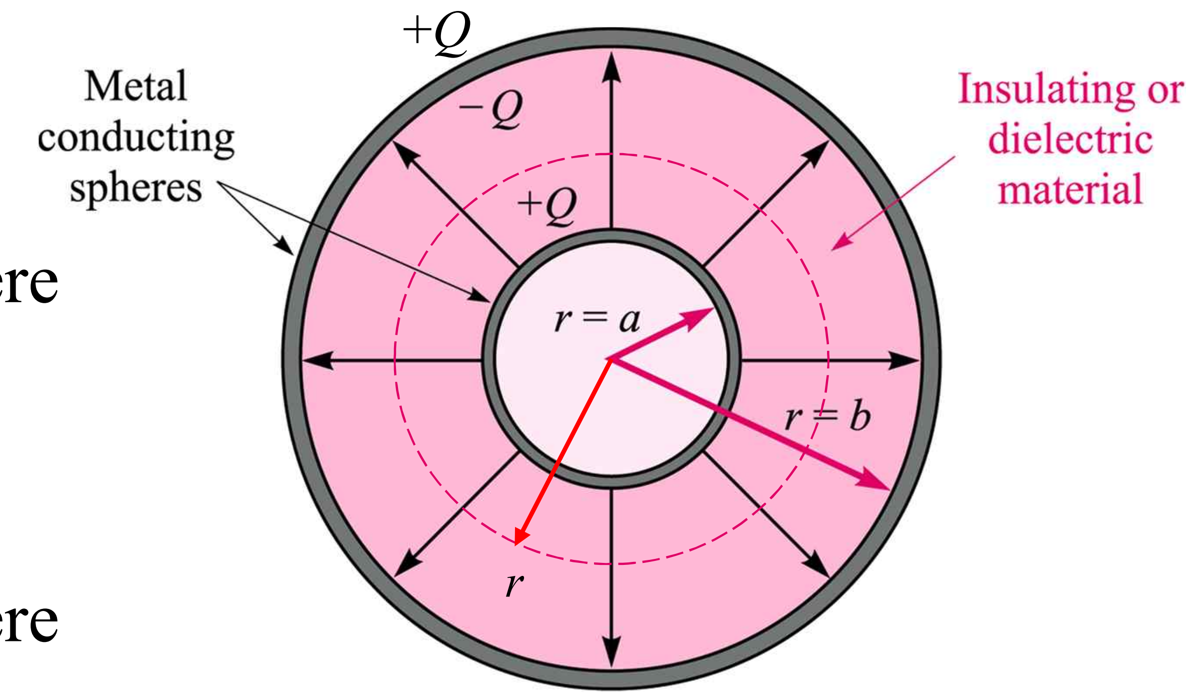
$$\vec{D}\Big|_{r=a} = \frac{Q}{4\pi a^2}\vec{a}_r$$

@ surface of inner sphere

$$\vec{D}\Big|_{r=b} = \frac{Q}{4\pi b^2}\vec{a}_r$$

@ surface of outer sphere

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a}_r \quad @\, a \le r \le b$$



Point Charge Fields

- Let the inner sphere make smaller and smaller, still retaining a charge of Q, it becomes a point charge.

- Electric flux density for a point charge

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r} \quad [\mathrm{C/m^2}] \quad (0 < r < \infty)$$

: symmetrically directed outward from the point and pass through an imaginary spherical surface of area $4\pi r^2$.

- Compare with $$\boxed{\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\mathbf{a}_r} \quad [\mathrm{V/m}] \quad (@\, 0 < r < \infty),$$

then $\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}}$ (free space only)

Finding E and D from Charge Distributions

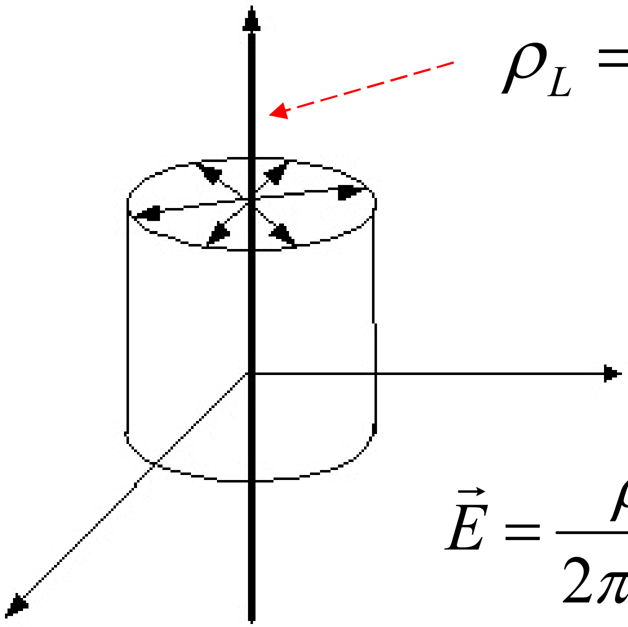
- In chapter 2,

$$\boxed{\mathbf{E} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon_0 R^2}\mathbf{a}_R} \qquad \text{(free space only)}$$

- As similar manner,

$$\boxed{\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2}\mathbf{a}_R}$$

Ex.] (임의의 가상 원통면을 관통하여 밖으로 나가는) $\vec{D} = ?$



$\rho_L = 8$ [nC/m]

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a}_\rho = \frac{8\times10^{-9}}{2\pi\times8.854\times10^{-12}\rho}\vec{a}_\rho = \frac{143.8}{\rho}\vec{a}_\rho \ [\text{V/m}]$$

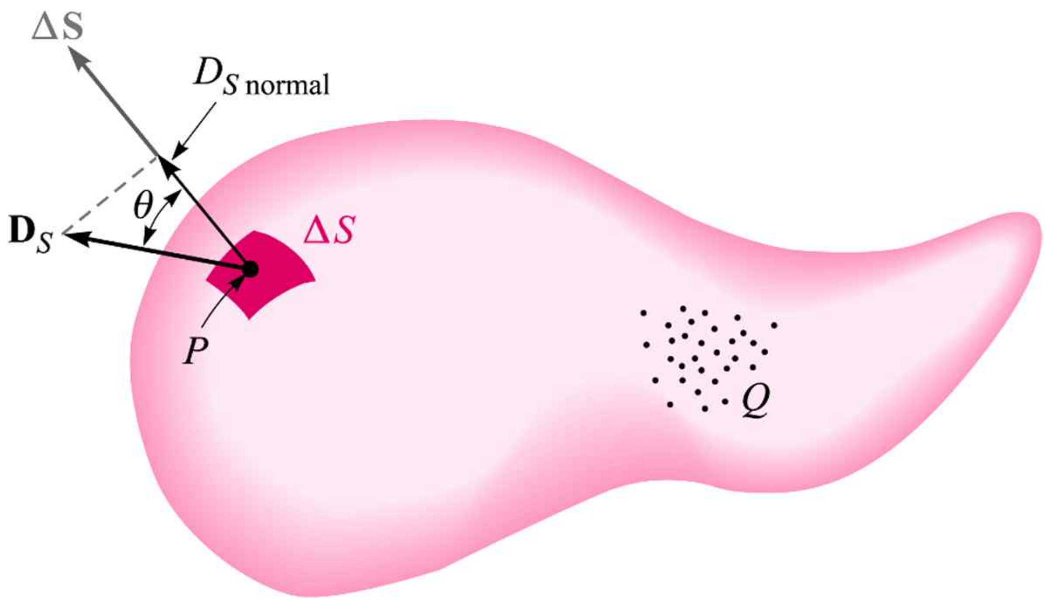
$$@\ \rho = 3\ \text{m},\quad \vec{E} = 47.9\vec{a}_\rho\ [\text{V/m}]$$

$$\vec{D} = \frac{\rho_L}{2\pi\rho}\vec{a}_\rho = \frac{8\times10^{-9}}{2\pi\rho}\vec{a}_\rho = \frac{1.273\times10^{-9}}{\rho}\vec{a}_\rho [\text{C/m}^2]$$

$$@\ \rho = 3\ \text{m},\quad \vec{D} = 0.424\vec{a}_\rho\ [\text{nC/m}^2]$$

3.2 Gauss' Law

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface



- ΔS : incremental (surface) element of surface at P

  $\Delta\vec{S} = \Delta S\vec{a}_N$

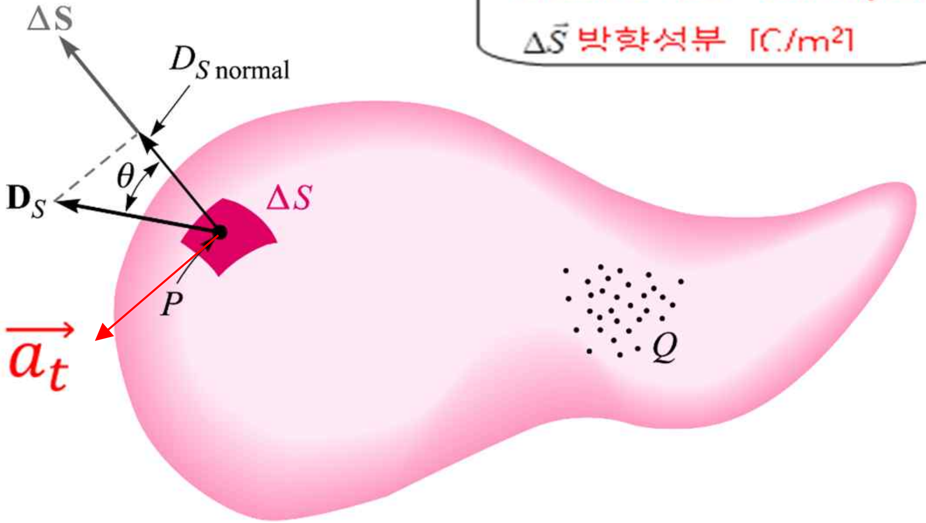
  $\vec{D}_S$: angled about θ with $\Delta\vec{S}$

Development of Gauss' Law

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}}\Delta S = D_S \cos\theta \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

Electric flux density의 $\Delta\vec{S}$ 방향성분 [C/m²]

ΔS 의 수직방향성분 고려



** Tangential 방향 성분의 vector 들의 합은 "0"
➔ 무시 가능)

- Total flux passing through the closed surface:

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S} \quad \text{where } dS = dxdy, \rho d\phi dz, r^2 \sin\theta d\theta d\phi, \cdots$$

내부로부터 밖(radial 방향)으로 향하는 flux

실제로 이루어진 closed surfac의 단위 면적 및 그 normal 벡터 성분

Mathematical Statement of Gauss' Law

$$\boxed{\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q}$$

Several point charges: $Q = \sum Q_m$

Line charge: $Q = \int \rho_L \, dL$

(Open) Surface charge: $Q = \int_S \rho_S dS$

Volume charge: $Q = \int_{\text{vol}} \rho_v \, dv$

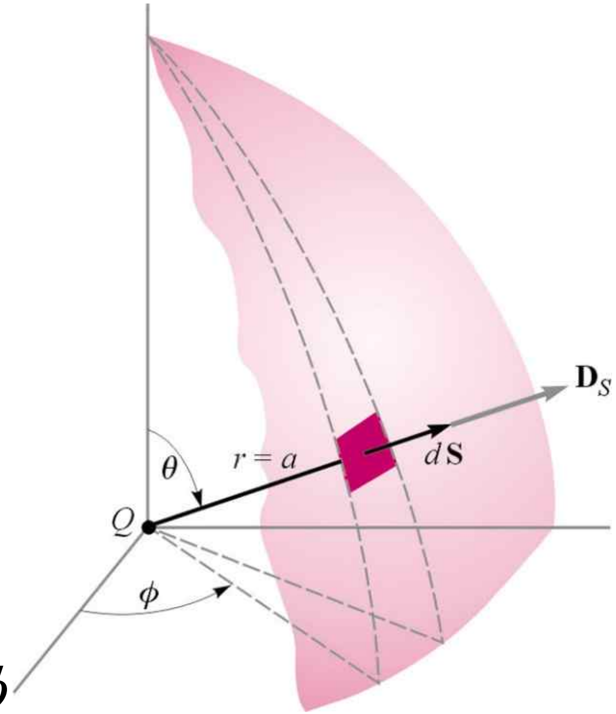
$$\therefore \quad \boxed{\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv}$$

(폐곡면을 수직으로 빠져나가는 electric flux density의 합 = 폐곡면 내부의 전하량 합)

Ex. 3.1] Check the results of Faraday's experimental

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$

$$\vec{D} = \varepsilon_0\vec{E} = \frac{Q}{4\pi r^2}\vec{a}_r$$



At the surface of the sphere,

$$\vec{D}_S = \frac{Q}{4\pi a^2}\vec{a}_r$$

$$dS = r^2 \sin\theta d\theta d\phi = a^2 \sin\theta d\theta d\phi$$

$$\vec{D}_S \cdot d\vec{S} = \left(\frac{Q}{4\pi a^2}\vec{a}_r\right)\cdot\left(a^2 \sin\theta d\theta d\phi \vec{a}_r\right) = \frac{Q}{4\pi}\sin\theta d\theta d\phi$$

$$\therefore \text{Total charge} = \oint_S \vec{D}_S \cdot d\vec{S} = \int_{\phi=0}^{\phi=2\pi}\int_{\theta=0}^{\theta=\pi}\frac{Q}{4\pi}\sin\theta d\theta d\phi$$

$$= \int_{\phi=0}^{\phi=2\pi}\frac{Q}{4\pi}[-\cos\theta]_0^{\pi} d\phi = \int_{\phi=0}^{\phi=2\pi}\frac{Q}{2\pi}d\phi$$

$$= \frac{Q}{2\pi}\int_{\phi=0}^{\phi=2\pi} d\phi = \left[\frac{Q}{2\pi}\phi\right]_0^{2\pi} = Q$$

3.3 Application of Gauss Law: Some Symmetrical Charge Distributions

- Gauss' Law $$\boxed{Q = \oint_S \mathbf{D}_S \cdot d\mathbf{S}}$$
- The solution can be obtained easily if

1. $\mathbf{D}_S$ is everywhere either normal or tangential to the closed surface, so that $\mathbf{D}_S \cdot d\mathbf{S}$ becomes either $D_S dS$ or zero, respectively.

2. On that portion of the closed surface for which $\mathbf{D}_S \cdot d\mathbf{S}$ is not zero, $D_S =$ constant.

$$\oint_S \mathbf{D}_s \cdot d\mathbf{S} = \underbrace{\oint_S D_s \, dS}_{\text{Condition 1}} = \underbrace{D_s \oint_S dS}_{\text{Condition 2}} = Q$$

So that: $$\boxed{D_s = \frac{Q}{\oint_S dS}}$$

3.3.1 Point Charge Field

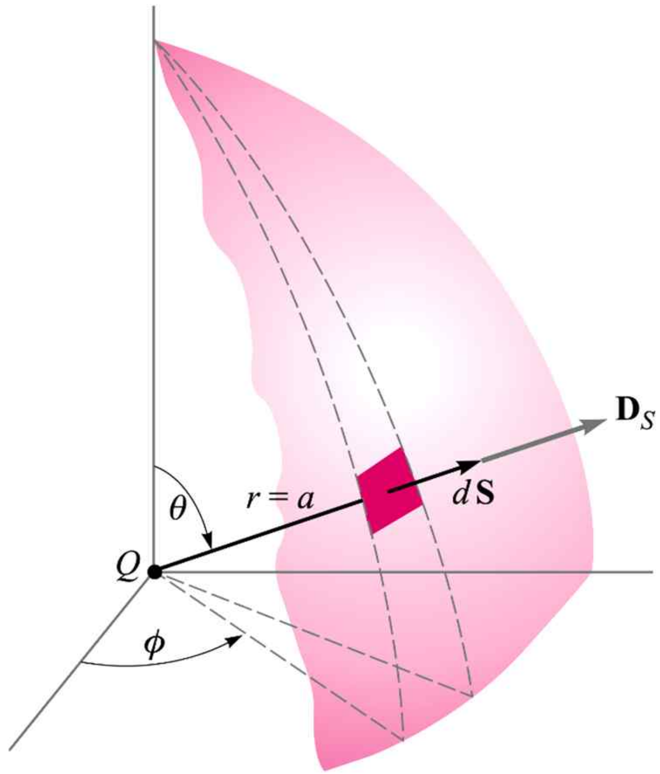
- $\vec{D}_S$ is everywhere normal to the surface and its magnitude is constant.

$$Q = \oint_S \vec{D}_S \cdot d\vec{S} = \oint_{sphere} D_S dS = D_S \oint dS$$

$$= D_S \int_0^{2\pi} \int_0^{\pi} r^2 \sin\theta d\theta d\phi = D_S \int_0^{2\pi} 2r^2 d\phi$$

$$= 4\pi r^2 D_S$$

$$\therefore D_S = \frac{Q}{4\pi r^2}$$



- Since r may have any value and $\vec{D}_S$ is radially outwarded,

$$\vec{D} = \frac{Q}{4\pi r^2}\vec{a}_r \qquad \vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a}_r$$

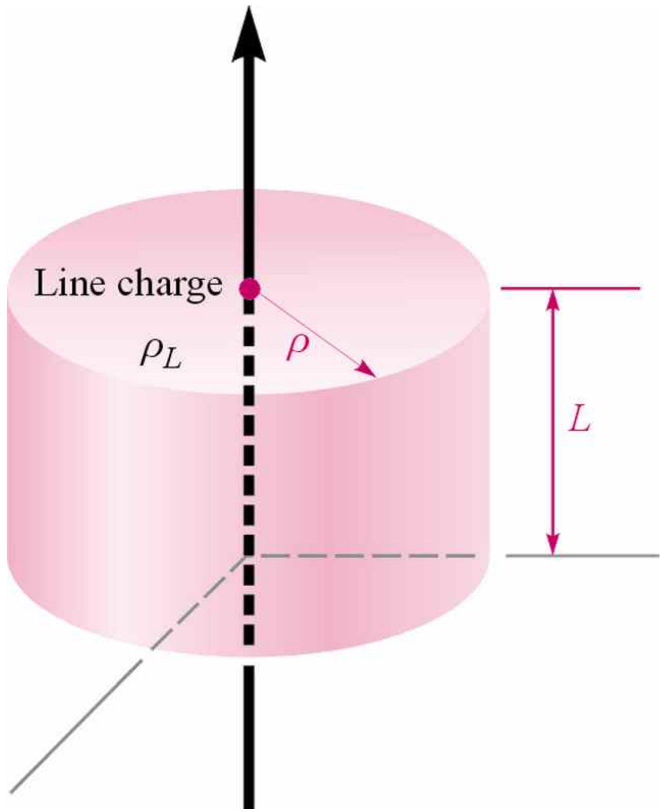
3.3.2 Line Charge Field

$$\vec{D} = D_\rho \vec{a}_\rho (\because \vec{a}_\rho \text{ directional } \text{ radiate})$$

$$Q = \oint_{cyl} \vec{D}_S \cdot d\vec{S} = D_S \int_{sides} dS + 0\int_{top} dS + 0\int_{bottom} dS$$

$$= D_S \int_{sides} dS = D_S \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho d\phi dz$$

$$= D_S 2\pi\rho L$$



$$D_S = D_\rho = \frac{Q}{2\pi\rho L} \qquad \leftarrow \; Q = \rho_L L$$

$$= \frac{\rho_L}{2\pi\rho} \qquad \rightarrow \qquad \vec{D} = \frac{\rho_L}{2\pi\rho}\vec{a}_\rho$$

$$E_\rho = \frac{\rho_L}{2\pi\varepsilon_0\rho} \qquad \rightarrow \qquad \vec{E} = \frac{\rho_L}{2\pi\varepsilon_0\rho}\vec{a}_\rho$$

3.3.3 Coaxial Transmission Line

- Surface charge distribution at outer surface ($\rho = a$) of inner conductor: ρ_S [C/m^2]
- Total electric flux by coaxial cylindrical conductor which is of length L and radius ρ, where $a < \rho < b$:

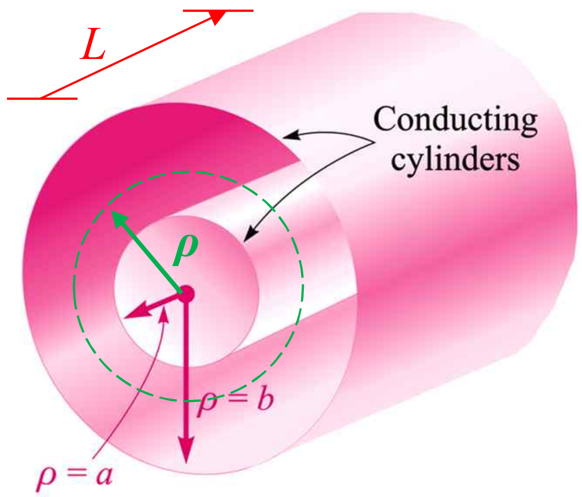
$$\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_0^L \int_0^{2\pi} D_S\, \mathbf{a}_\rho \cdot \underbrace{\mathbf{a}_\rho\, \rho\, d\phi\, dz}_{d\mathbf{S}} = 2\pi\rho D_S L = Q$$

- Total charge on a length L of the inner conductor:

$$Q = \int_{z=0}^{L} \int_{\phi=0}^{2\pi} \rho_S a\, d\phi dz = 2\pi a L \rho_S = D_S 2\pi\rho L$$

$$\therefore D_S = \frac{a\rho_S}{\rho}$$

$$\therefore \vec{D} = \frac{a\rho_S}{\rho} \vec{a}_\rho \quad (a < \rho < b)$$



Coaxial Transmission Line (continued)

- The previous result might be expressed in terms of line charge per unit length.

$$\rho_L = 2\pi a L \rho_S = 2\pi a \rho_S \quad \Leftarrow \quad L = 1 \text{ [m]}$$

$$\rho_S = \frac{\rho_L}{2\pi a}$$

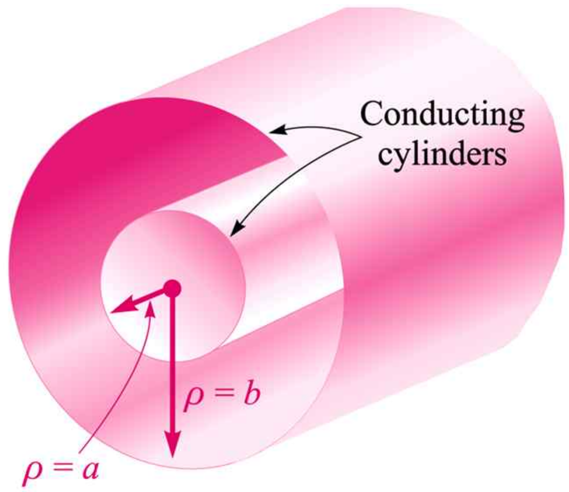
$$\vec{D} = \frac{a\rho_S}{\rho}\vec{a}_\rho = \frac{a\dfrac{\rho_L}{2\pi a}}{\rho}\vec{a}_\rho = \frac{\rho_L}{2\pi\rho}\vec{a}_\rho$$

Coaxial Transmission Line: Exterior Field

- Because every line of electric flux starting from the charge on the inner cylinder must terminate on a negative charge on the inner surface of the outer cylinder

$$Q_{outer\ cyl} = -2\pi aL\rho_{S.inner\ cyl}$$
$$= 2\pi bL\rho_{S.outer\ cyl}$$

$$\rho_{S.outer\ cyl} = -\frac{a}{b}\rho_{S.inner\ cyl}$$



- At $\rho > b$, $0 = D_S 2\pi\rho L \qquad (\rho > b)$

$$D_S = 0 \qquad (\rho > b)$$ (∵ Total enclosed charge would be *zero*.)

- At $\rho < a$, $0 = D_s 2\pi\rho L$

$$D_s = 0$$ (∵ Total enclosed charge would be *zero*.)

Ex.] $L = 50$ cm, $\rho_{inner} = 1$ mm $(= a)$, $\rho_{outer} = 4$ mm $(= b)$, ε_0 (in intermediate space).

- Total charge on the inner conductor: 30 nC

$$\rho_{S.inner} = \frac{Q_{inner.\,cyl}}{2\pi a L} = \frac{30\times 10^{-9}}{2\pi \times 10^{-3} \times 0.5} = 9.55\ [\mu\text{C/m}^2]$$

- Internal fields:

$$D_\rho = \frac{a\rho_s}{\rho} = \frac{10^{-3}\times 9.55\times 10^{-6}}{\rho} = \frac{9.55}{\rho}\ [\text{nC/m}^2]$$

$$E_\rho = \frac{D_\rho}{\varepsilon_0} = \frac{9.55\times 10^{-9}/\rho}{8.854\times 10^{-12}} = \frac{1079}{\rho}\ [\text{V/m}] \qquad (1 < \rho < 4\ \text{mm})$$

$$E_\rho = D_\rho = 0 \qquad (\rho < 1,\ \rho > 4\ \text{mm})$$

3.4 Gauss's Law in Differential Form: Divergence

- $\vec{D}$ at point P

$$\vec{D}_0 = D_{x0}\vec{a}_x + D_{y0}\vec{a}_y + D_{z0}\vec{a}_z$$

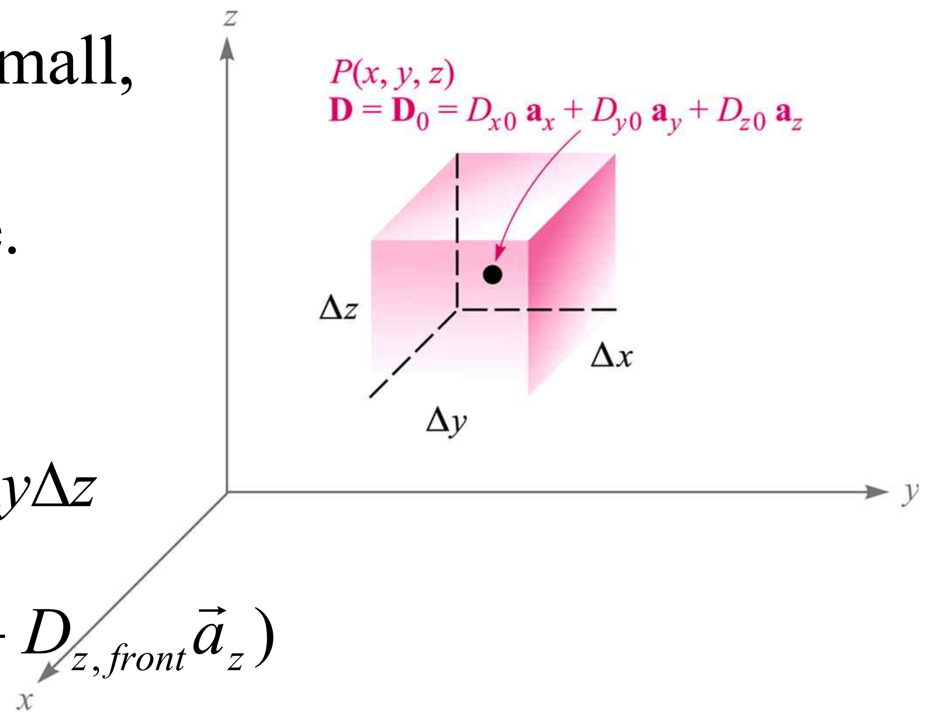
$$Q = \oint_S \vec{D}_S \cdot d\vec{S}$$

$$= \int_{front} \vec{D}_S \cdot d\vec{S} + \int_{back} + \int_{left} + \int_{right} + \int_{top} + \int_{bottom}$$

- Since the surface element is very small, $\vec{D}$ is essentially constant over this portion of the entire closed surface.

$$\int_{front} \approx \vec{D}_{front} \cdot \Delta\vec{S}_{front}$$

$$= \vec{D}_{front} \cdot \Delta y \Delta z \vec{a}_x = D_{x,front} \Delta y \Delta z$$

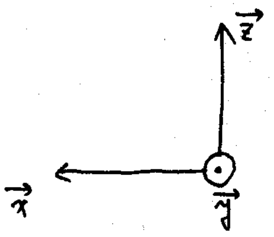


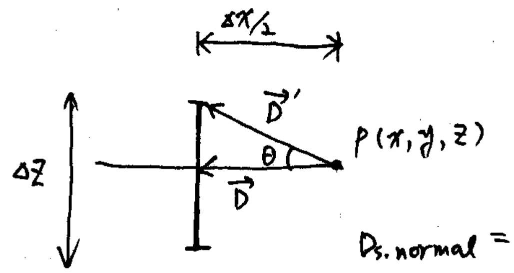
$$(\because \quad \vec{D}_{front} = D_{x,front}\vec{a}_x + D_{y,front}\vec{a}_y + D_{z,front}\vec{a}_z)$$

$$D_{x,\,front} \cong D_{xo} + \frac{\Delta x}{2} \times \text{rate of change } D_x \text{ with } x$$

$$= \; D_{xo} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

$$\therefore \int_{front} \cong (D_{xo} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x})\Delta y \Delta z$$





$D_{s,normal} = D'\cos\theta$

$\Delta z \to 0 \Rightarrow \theta = 0°$

$D_{s,normal} = D'$

- Consider the integral over the back surface,

$$\int_{back} = \vec{D}_{back} \cdot \Delta\vec{S}_{back} = \vec{D}_{back} \cdot (-\Delta y \Delta z \vec{a}_x) \quad \Leftarrow \quad D_{x.back} \cong D_{xo} - \frac{\Delta x}{2}\frac{\partial D_x}{\partial x}$$

$$= -D_{x,back}\Delta y \Delta z$$

$$\therefore \int_{back} \cong (-D_{xo} + \frac{\Delta x}{2}\frac{\partial D_x}{\partial x})\Delta y \Delta z$$

- Therefore: $\displaystyle \int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x}\Delta x \, \Delta y \, \Delta z$

- By exactly the same process,

$$\int_{right} + \int_{left} \cong \frac{\partial D_y}{\partial y} \Delta x \Delta y \Delta z$$

$$\int_{top} + \int_{bottom} \cong \frac{\partial D_z}{\partial z} \Delta x \Delta y \Delta z$$

- All assembled results :

$$\oint_S \vec{D} \cdot d\vec{S} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \Delta y \Delta z$$

$$= \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta v$$

$$= Q$$

- Charge enclosed within volume Δv

$$\cong \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \times \text{ volume } \Delta v$$

Ex. 3.3]

$$\vec{D} = e^{-x} \sin y \vec{a}_x - e^{-x} \cos y \vec{a}_y + 2z\vec{a}_z \ [\mathrm{C/m^2}] \ (= D_x\vec{a}_x + D_y\vec{a}_y + D_z\vec{a}_z)$$

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

$\therefore$ Charge enclosed within volume $\Delta v = 2\Delta v$

If $\Delta v = 10^{-9}$ m^3, then volume charge is 2 nC.

3.4.2 Divergence and Maxwell's First Equation

The divergence of the vector flux density **A** *is the outflow of flux from a small closed surface per unit volume as the volume shrinks to zero.*

$$\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\cong\frac{\oint_s \vec{D}\cdot d\vec{S}}{\Delta v}=\frac{Q}{\Delta v}$$

- As a limit, $\displaystyle \frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}=\lim_{\Delta v\to 0}\frac{\oint_S \vec{D}\cdot d\vec{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{\oint_S Q}{\Delta v}=\rho_v$

$$\text{Divergence of } \mathbf{A} = \text{div } \mathbf{A} = \lim_{\Delta v\to 0}\frac{\oint_S \mathbf{A}\cdot d\mathbf{S}}{\Delta v}$$

$$\left(\frac{\partial D_x}{\partial x}+\frac{\partial D_y}{\partial y}+\frac{\partial D_z}{\partial z}\right)=\lim_{\Delta v\to 0}\frac{\oint_S \mathbf{D}\cdot d\mathbf{S}}{\Delta v}=\lim_{\Delta v\to 0}\frac{Q}{\Delta v}=\underline{\rho_v = \text{div } \mathbf{D}}$$

Point form of Gauss's law:
Maxwell's first equation

Divergence Expressions in the Three Coordinate Systems

$$\text{div}\,\mathbf{D} = \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \qquad \text{(rectangular)}$$

$$\text{div}\,\mathbf{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical)}$$

$$\text{div}\,\mathbf{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi} \qquad \text{(spherical)}$$

Ex. 3.4] $\vec{D} = e^{-x}\sin y\vec{a}_x - e^{-x}\cos y\vec{a}_y + 2z\vec{a}_x$

$$\text{div}\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = -e^{-x}\sin y + e^{-x}\sin y + 2 = 2$$

$$\operatorname{div}\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\frac{\partial D_x \vec{a_x} \cdot \partial y \partial z \vec{a_x}}{\Delta v} = \frac{\partial D_x \cancel{\partial y} \cancel{\partial z}}{\partial x \cancel{\partial y} \cancel{\partial z}} = \frac{\partial D_x}{\partial x}$$

$$\operatorname{div}\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial\phi}$$

$$\frac{\partial D_r \vec{a_r} \cdot r^2 \sin\theta\, \partial\theta\, \partial\phi\, \vec{a_r}}{\Delta v} = \frac{\sin\theta\, \partial\theta\, \partial\phi\, \{\partial(D_r \cdot r^2)\}}{r^2 \sin\theta\, \partial r\, \partial\theta\, \partial\phi}$$

$$= \frac{\partial(r^2 D_r)}{r^2\, \partial r}$$

3.4.3 Maxwell's First Equation: Gauss's Law in Point Form

$$\mathrm{div}\vec{D} = \rho_v$$

[Ex.] $\vec{D} = \dfrac{Q}{4\pi r^2}\vec{a}_r$

Since $\mathrm{div}\ \vec{D} = \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(r^2 D_r) + \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial\theta}(\sin\theta D_\theta) + \dfrac{1}{r\sin\theta}\dfrac{\partial D_\phi}{\partial\phi}$,

$$\mathrm{div}\ D = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Q}{4\pi}\right) = 0$$

$\therefore \rho_v = 0$ @ $r \neq 0$ (everywhere except at the origin, where it is infinite)

3.5 Divergence Theorem
3.5.1 The Del Operator

- The del operator (∇) is a *vector differential operator*, and defined as,

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z}$$

Note that:

$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \text{div}\,\mathbf{D}$$

- In other coordinate systems,

$$\nabla \cdot \vec{D} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho D_\rho) + \frac{1}{\rho}\frac{\partial D_\phi}{\partial \phi} + \frac{\partial D_z}{\partial z} \qquad \text{(cylindrical coordinate)}$$

$$\nabla \cdot \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\, D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

(spherical coordinate)

3.5.3 Divergence Theorem

- Maxwell's first equation (or the point form of Gauss' Law) :

$$\mathrm{div}\,\mathbf{D} = \nabla \cdot \mathbf{D} = \rho_v$$

- Gauss's Law in large-scale (or integral) form

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\mathrm{vol}} \rho_v dv = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv$$

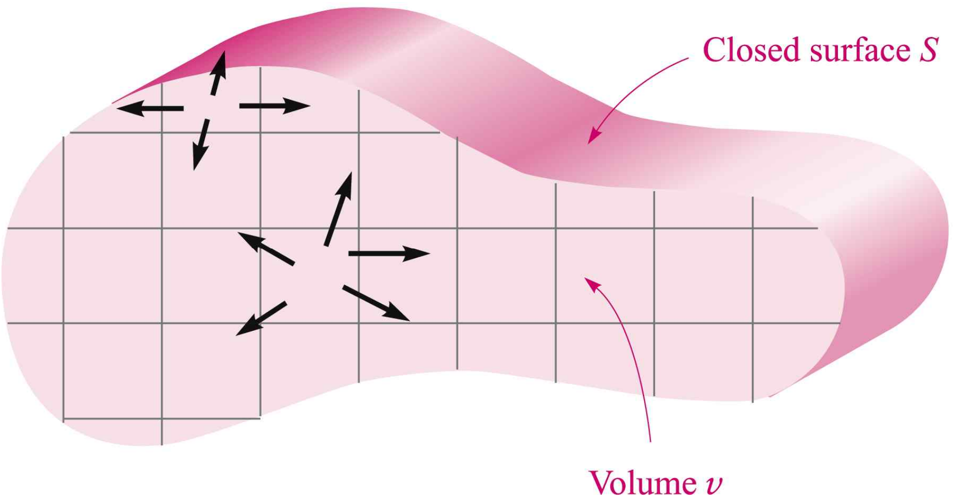
- **Divergence theorem**

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv}$$

Statement of the Divergence Theorem

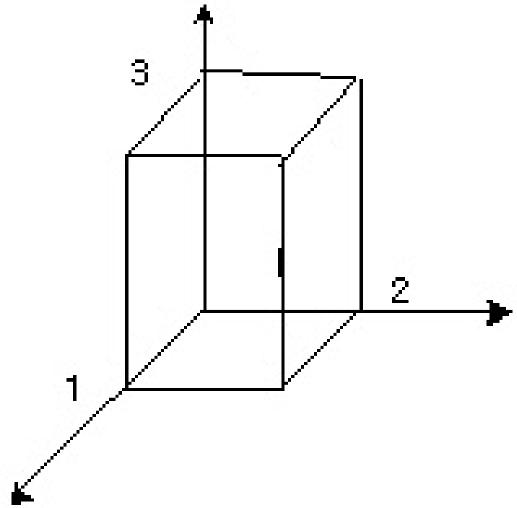
The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv$$



(면적분 ⇔ 체적적분, 이중적분 ⇔ 삼중적분)

Ex. 3.5] $\vec{D} = 2xy\vec{a}_x + x^2\vec{a}_y \quad [\mathrm{C/m^2}]$



Solution I)

$$\vec{D}\cdot d\vec{S}\Big|_{\substack{z=0\\z=3}} = 0\vec{a}_z \cdot (\pm dxdy\vec{a}_z) = 0$$

$$\oint_s \vec{D}\cdot d\vec{S} = \int_{back} + \int_{front} + \int_{left} + \int_{right}$$

$$= \int_0^3\int_0^2 \vec{D}\Big|_{x=0}\cdot(-dydz\vec{a}_x) + \int_0^3\int_0^2 \vec{D}\Big|_{x=1}\cdot(dydz\vec{a}_x)$$

$$+ \int_0^3\int_0^1 \vec{D}\Big|_{y=0}\cdot(-dxdz\vec{a}_y) + \int_0^3\int_0^1 \vec{D}\Big|_{y=2}\cdot(dxdz\vec{a}_y)$$

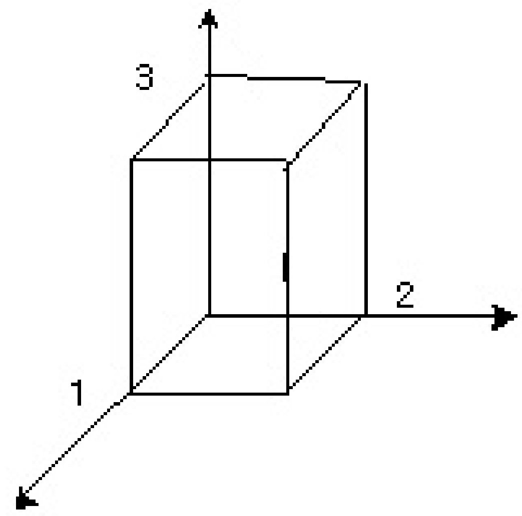
$$= -\int_0^3\int_0^2 (D_x)_{x=0}\,dydz + \int_0^3\int_0^2 (D_x)_{x=1}\,dydz$$

$$-\int_0^3\int_0^1 (D_y)_{y=0}\,dxdz + \int_0^3\int_0^1 (D_y)_{y=2}\,dxdz$$

Since $(D_x)_{x=0} = 0$ and $(D_y)_{y=0} = (D_y)_{y=2}$,

$$\oint_S \vec{D} \cdot d\vec{S} = \int_0^3 \int_0^2 (2xy)_{x=1} dy\, dz$$

$$= \int_0^3 [y^2]_0^2 dz = [4z]_0^3 = 12$$



Solution II)

$$\nabla \cdot \vec{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

$$\oint_{vol} \nabla \cdot \vec{D} dv = \int_0^3 \int_0^2 \int_0^1 2y dx dy dz$$

$$= \int_0^3 \int_0^2 2y dy dz = \int_0^3 [y^2]_0^2 dz = [4z]_0^3 = 12$$

(∴ 직육면체 안에 12 [C]의 전하가 존재)