Engineering Electromagnetics

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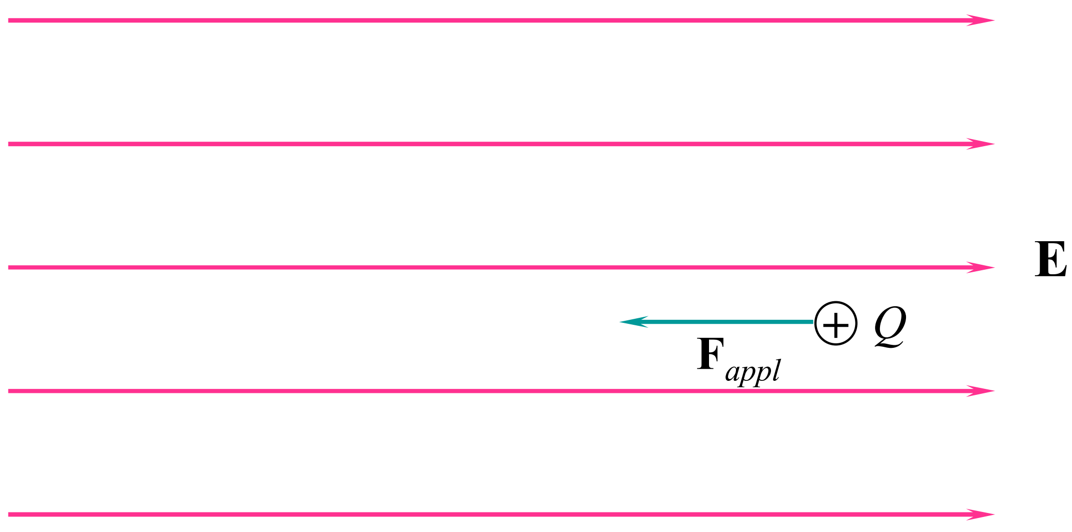
Chapter 4:

Energy and Potential

4. 1 Energy Expended in Moving a Point Charge in an Electric Field

To move charge Q against the electric field, a force must be applied that *counteracts* the force on Q that arises from the electric field.



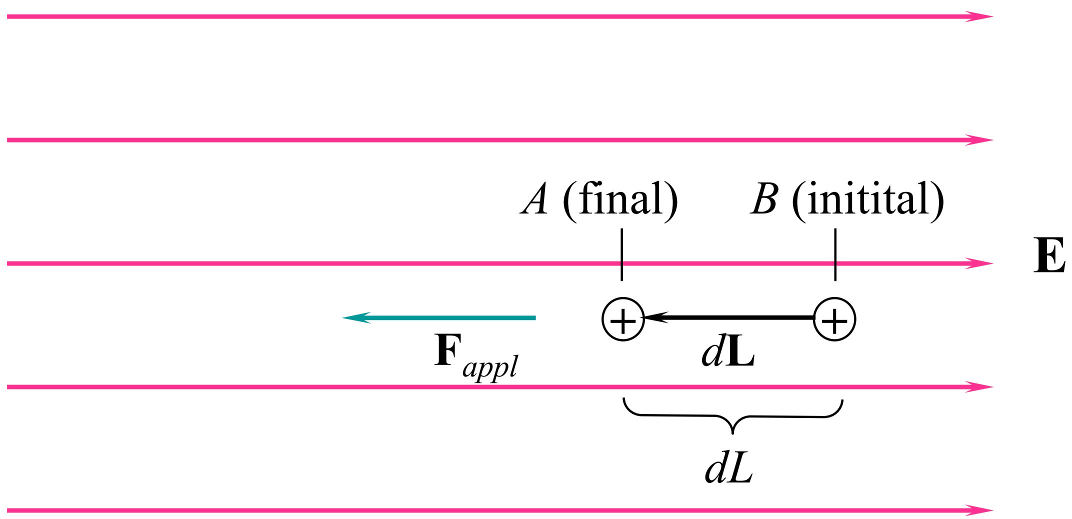
$$\mathbf{F}_{appl} = -Q\,\mathbf{E}$$

Differential Work Done on Moving a Point Charge Against an External Field

In moving point charge Q from initial position B over a differential distance dL (to final position A), the work expended is:

$$dW = F_{appl}\, dL = -QE\, dL = -Q\mathbf{E} \cdot d\mathbf{L} \text{ [J]}$$

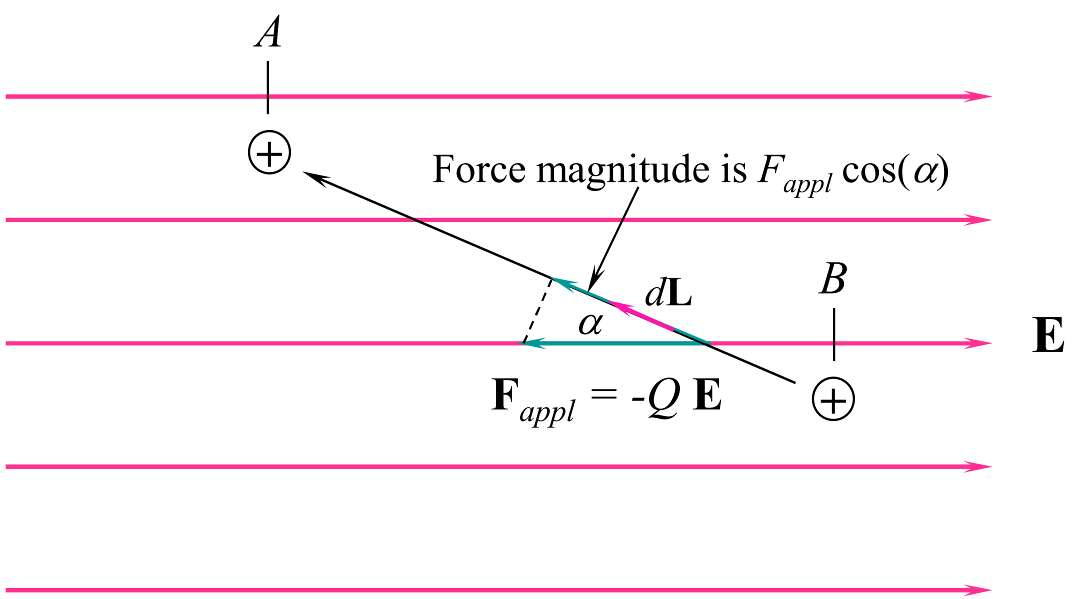
gives positive result if charge is forced *against* the electric field



The path is along an electric field line (in the opposite direction), and over the differential path length, the field can be assumed constant.

Forcing a Charge Against the Field in an Arbitrary Direction

What matters now is the component of force in the direction of motion.



Differential work in moving charge Q through distance dL will be:

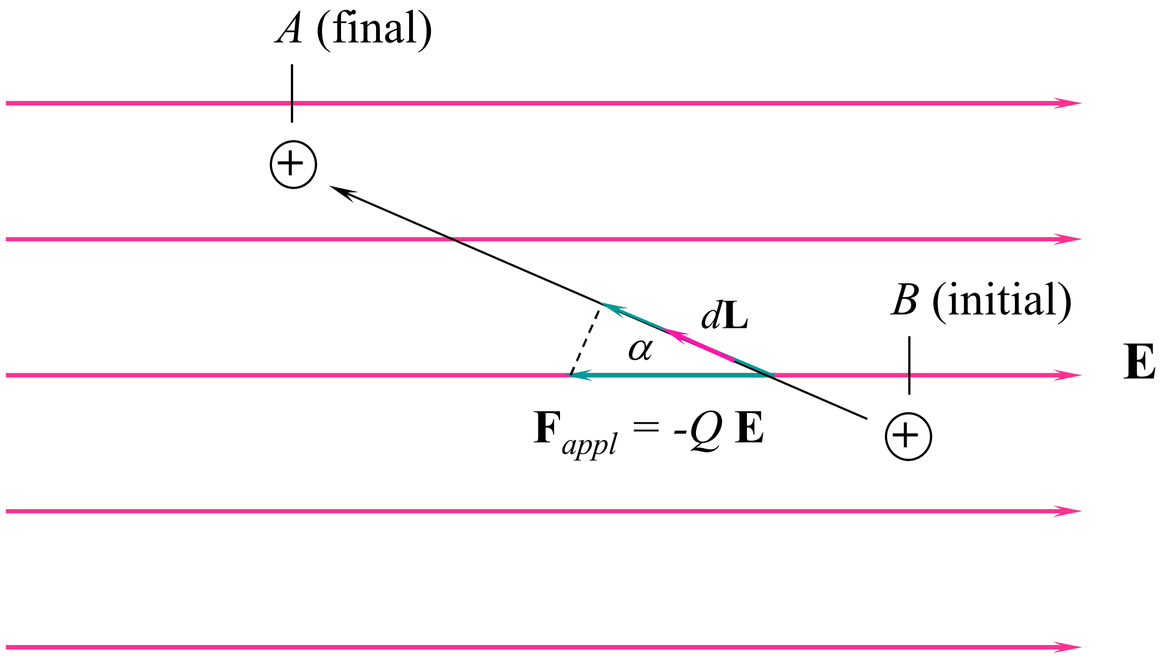
$$dW = F_{appl}\cos(\alpha)\,dL = -Q\mathbf{E}\bullet d\mathbf{L}$$

$$\vec{a}_L\,dL = d\vec{L}$$

Total Work Done

All differential work contributions along the path are summed to give:

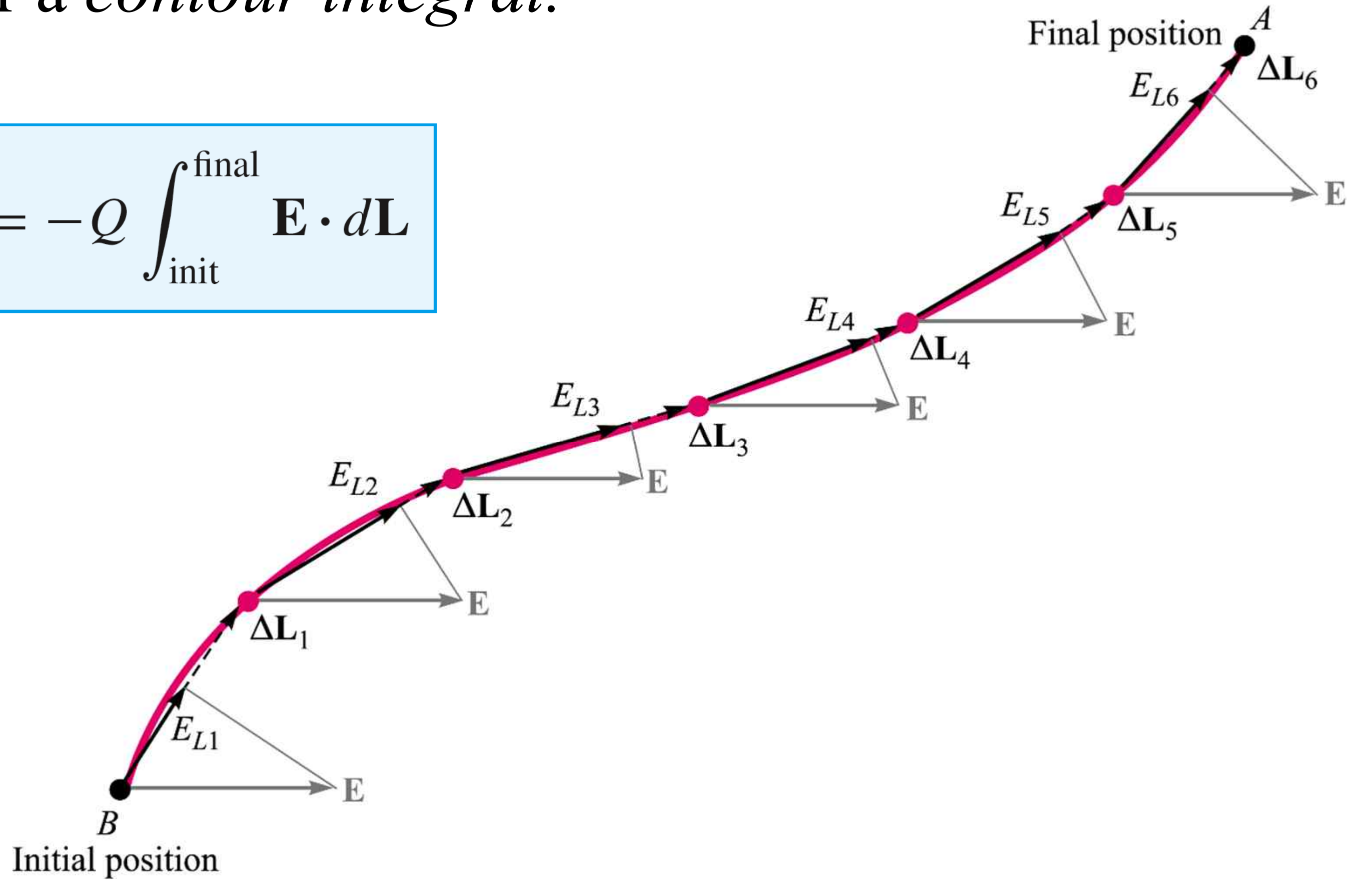
$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$



4.2 The Line Integral: Total Work Done over an Arbitrary Path

- The integral expression involving the scalar product of the (electric) field with a differential path vector is called a *line integral* or a *contour integral*.

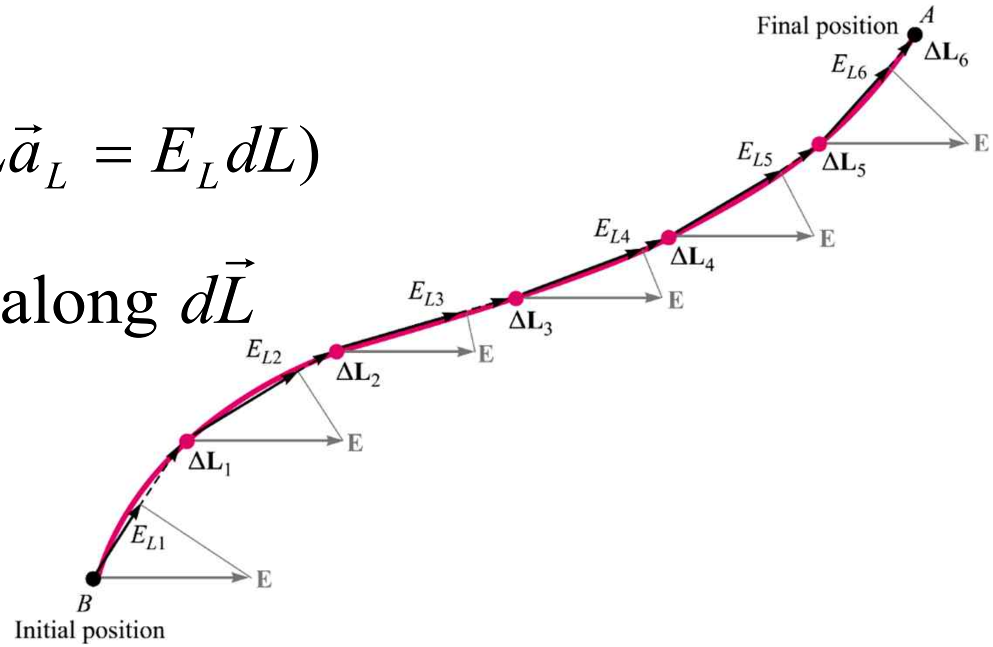
$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$



$$W = -Q\int_{\text{init}}^{\text{final}} \mathbf{E}\cdot d\mathbf{L}$$

$$(\vec{E}\cdot d\vec{L} = (E_L\vec{a}_L + E_{not\ L}\vec{a}_{not\ L})\cdot dL\vec{a}_L = E_L dL)$$

where E_L : component of $\vec{E}$ along $d\vec{L}$



$$W = -Q\int_{init}^{final} E_L dL$$

$$W = -Q(E_{L1}\Delta L_1 + E_{L2}\Delta L_2 + \cdots + E_{L6}\Delta L_6)$$

$$= -Q(\vec{E}_1\cdot\Delta\vec{L}_1 + \vec{E}_2\cdot\Delta\vec{L}_2 + \cdots + \vec{E}_6\cdot\Delta\vec{L}_6)$$

$$= -Q\,\vec{E}\cdot(\Delta\vec{L}_1 + \Delta\vec{L}_2 + \cdots + \Delta\vec{L}_6) \leftarrow \because \vec{E} = \vec{E}_1 = \vec{E}_2 = \cdots = \vec{E}_6$$

$$= -Q\vec{E}\cdot\vec{L}_{BA}$$ (by using vector summation)

* 중간지점의 Point 위치가 중요한 것이 아니라 처음과 끝점의 위치가 중요

$$\therefore \quad W = -Q\int_B^A \vec{E} \cdot d\vec{L}$$

(∵ 단위 line segment를 무한대로 나누어 summation 한다면 line 적분과 동일)

- For an uniform electric field ($\vec{E}$),

$$W = -Q\,\vec{E} \cdot \int_B^A d\vec{L}$$

$$= -Q\,\vec{E} \cdot \vec{L}_{BA} \quad \text{(변수: } Q,\ \vec{E}\ ,\ \vec{L}_{BA}\text{)}$$

→ * 시점과 종점이 중요

- Line integal evaluation example

$$\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

where $\mathbf{E} = E_x \mathbf{a}_x + E_y \mathbf{a}_y + E_z \mathbf{a}_z$

and $d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z$

For $(x_B, y_B, z_B) \rightarrow (x_A, y_A, z_A)$,

$$\int_B^A \mathbf{E} \cdot d\mathbf{L} = \int_B^A (E_x \mathbf{a}_x + E_y \mathbf{a}_y + E_z \mathbf{a}_z) \cdot (dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z)$$

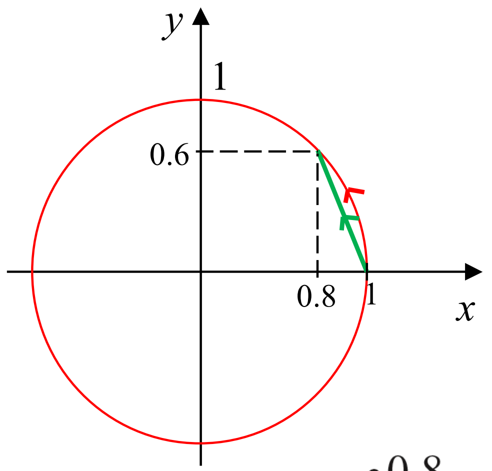
$$= \int_{x_B}^{x_A} E_x\,dx + \int_{y_B}^{y_A} E_y\,dy + \int_{z_B}^{z_A} E_z\,dz$$

Ex. 4.1] $W = ?$ For $\mathbf{E} = y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z$ and $Q = 2$

Contour 1: Shorter arc of the circle given by $x^2 + y^2 = 1 \quad z = 1$

Contour 2: $B\ (1, 0, 1) \rightarrow A\ (0.8, 0.6, 1)$

➔ Method 1) $W = -Q\int_B^A \mathbf{E}\cdot d\mathbf{L}$



$$= -2\int_B^A (y\mathbf{a}_x + x\mathbf{a}_y + 2\mathbf{a}_z)\cdot(dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z)$$

$$= -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$

$$W = -2\int_1^{0.8}\sqrt{1-x^2}\,dx - 2\int_0^{0.6}\sqrt{1-y^2}\,dy - 0 \leftarrow \int\sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$$

$$= -\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_1^{0.8} - \left[y\sqrt{1-y^2} + \sin^{-1}y\right]_0^{0.6}$$

$$= -(0.48 + 0.927 - 0 - 1.571) - (0.48 + 0.644 - 0 - 0)$$

$$= -0.96\,\mathrm{J}$$

➔ Method 2) $y - y_B = \dfrac{y_A - y_B}{x_A - x_B}(x - x_B)$

$$y - 0 = \frac{0.6 - 0}{0.8 - 1}(x - 1) \;\;\Rightarrow\;\; y = -3(x - 1)$$

$$\therefore\; W = -2\int \vec{E}\cdot d\vec{L} = -2\int_B^A (y\vec{a}_x + x\vec{a}_y + 2\vec{a}_z)\cdot(dx\vec{a}_x + dy\vec{a}_y + dz\vec{a}_z)$$

$$= -2\int_1^{0.8} y\,dx - 2\int_0^{0.6} x\,dy - 4\int_1^1 dz$$

$$= 6\int_1^{0.8}(x - 1)\,dx - 2\int_0^{0.6}\left(1 - \frac{y}{3}\right)dy$$

$$= 6\left[\frac{1}{2}x^2 - x\right]_1^{0.8} - 2\left[y - \frac{y^2}{6}\right]_0^{0.6}$$

$$= 6\left[\frac{0.64}{2} - 0.8 - \frac{1}{2} + 1\right] - 2\left(0.6 - \frac{0.36}{6}\right) = -0.96 \quad [\mathrm{J}]$$

: Same result

Differential Path Lengths in the Three Coordinate Systems

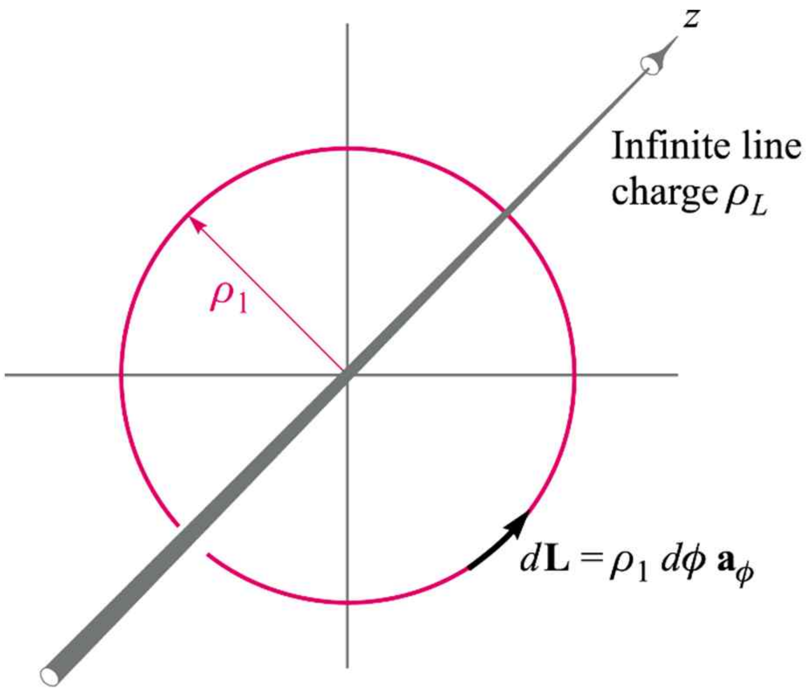
$$d\mathbf{L} = dx\,\mathbf{a}_x + dy\,\mathbf{a}_y + dz\,\mathbf{a}_z \qquad \text{(rectangular)}$$

$$d\mathbf{L} = d\rho\,\mathbf{a}_\rho + \rho\,d\phi\,\mathbf{a}_\phi + dz\,\mathbf{a}_z \qquad \text{(cylindrical)}$$

$$d\mathbf{L} = dr\,\mathbf{a}_r + r\,d\theta\,\mathbf{a}_\theta + r\sin\theta\,d\phi\,\mathbf{a}_\phi \qquad \text{(spherical)}$$

Final illustration] Evaluating work within a line charge field

1) Work for moving a charge along the circular path



(a)

$$\mathbf{E} = E_\rho \mathbf{a}_\rho = \frac{\rho_L}{2\pi\epsilon_0 \rho}\mathbf{a}_\rho$$

$$W = -Q\int_{\text{init}}^{\text{final}} \frac{\rho_L}{2\pi\epsilon_0\rho_1}\mathbf{a}_\rho \cdot \rho_1\, d\phi\, \mathbf{a}_\phi$$

$$= -Q\int_0^{2\pi} \frac{\rho_L}{2\pi\epsilon_0} d\phi\, \mathbf{a}_\rho \cdot \mathbf{a}_\phi = 0$$

➔ No work !!!

2) Work for moving a charge along a radial path ($a \rightarrow b$)

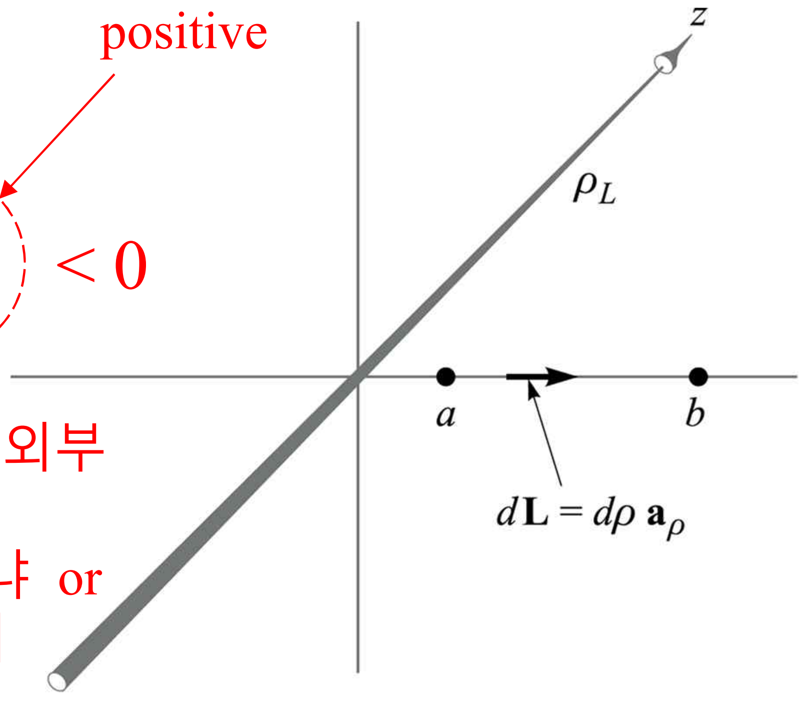
$$W = -Q \int_{init}^{final} \frac{\rho_L}{2\pi\varepsilon_o \rho} \vec{a}_\rho \cdot d\rho \vec{a}_\rho$$

$$= -Q \int_a^b \frac{\rho_L}{2\pi\varepsilon_o \rho} d\rho = -\frac{Q\rho_L}{2\pi\varepsilon_o} \ln\frac{b}{a} < 0$$

positive

(Line change ρ_L가 $+Q$를 이동시키는 것이므로 외부 source로부터 energy 를 받는 형태)

Ex.] 태풍이 불 때 바람이 부는 방향으로 가느냐 or 바람에 거슬리는 방향으로 가느냐의 문제



(b)

3) Work for moving a charge along a radial path ($b \rightarrow a$)

$$W = -Q\int_b^a \frac{\rho_L}{2\pi\varepsilon_0 \rho} \vec{a}_\rho \cdot d\rho \vec{a}_\rho = \frac{Q\rho_L}{2\pi\varepsilon_0} \ln\frac{b}{a} > 0$$

(Line change ρ_L이 있는 상태에서 $+Q$를 이동시키는 것이므로 일이 필요. 외부 source가 일을 하는 형태)

4.3 Definition of Potential Difference and Potential

- Work done by an external source in moving a charge Q from initial to final positions in an electric field :

$$W = -Q \int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L}$$

← $\vec{F} \leftrightarrow \vec{E}$ and $W \leftrightarrow V$

- *Potential difference* ([V] or [J/C]) is a work done (by an external source) in moving a *unit* positive charge from one point to another in an electric field.

$$\text{Potential Difference} = \frac{W}{Q} = -\int_{\text{init}}^{\text{final}} \mathbf{E} \cdot d\mathbf{L} \quad \text{Volts}$$

- Potential difference in moving a unit positive charge from B to A:

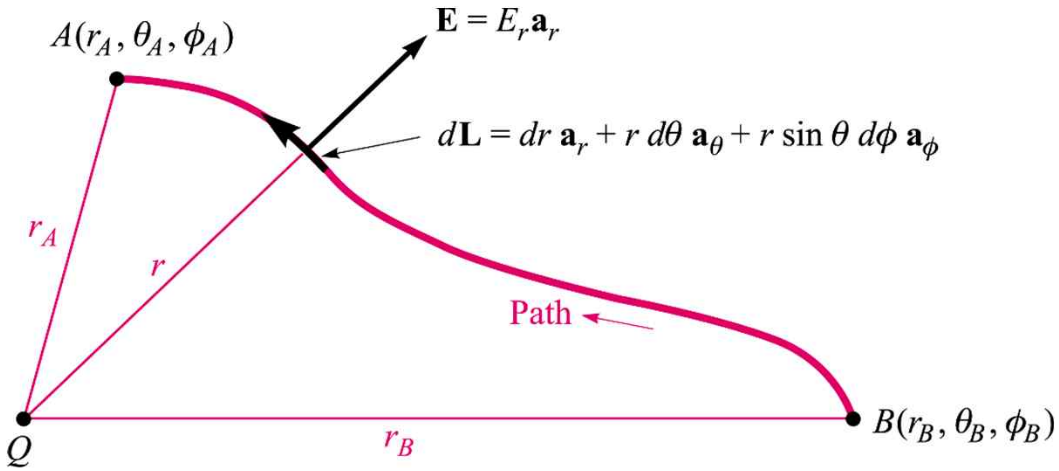
$$V_{AB} = -\int_{B}^{A} \vec{E} \cdot d\vec{L} \qquad \text{[J/C] or [Volt]} \quad \leftrightarrow \quad \vec{E} = \frac{\vec{F}}{Q_t}$$

A → target, B → source

- If $V_{AB} > 0$, work is done in carrying the positive charge from B to A. If $V_{AB} < 0$, the external source moving the charge receive energy.

- In case of $W = \dfrac{Q\rho_L}{2\pi\varepsilon_0} \ln\dfrac{b}{a}$, $\quad V_{ab} = \dfrac{W}{Q} = \dfrac{\rho_L}{2\pi\varepsilon_0} \ln\dfrac{b}{a}$

- A, B: radial distance r_A and r_B from a point charge Q
  Q is at an origin.



$$\vec{E} = E_r \overrightarrow{a_r} = \frac{Q}{4\pi\varepsilon_0 r^2} \overrightarrow{a_r}$$

$$d\vec{L} = dr\overrightarrow{a_r} + rd\theta\overrightarrow{a_\theta} + r\sin\theta\, d\phi\overrightarrow{a_\phi}$$

$$\therefore\ V_{AB} = -\int_B^A \vec{E}\cdot d\vec{L} = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

➔ If $r_B > r_A$, $V_{AB} > 0$: Energy is expended to bring a charge on $\vec{E}$.

- *Potential* or *absolute potential* is a potential difference with respect to a specified reference point that we consider to have zero potential.

- Universal zero reference points

  ① Ground (물리적으로는 확실한 zero potential임. 지표면. 그러나 거리적인 측면에서 보면 깊은 바다의 해수면에 있을 때는 부적절)

  ② Infinity point ($r = \infty$)

  ③ Cylindrical surface (side-wall of cylinder in $\rho = \infty$)

  ④ Outer conductor of coaxial cable

- If the potential at point A is V_A and that at B is V_B,

$$V_{AB} = V_A - V_B = (V_A - 0) - (V_B - 0) = \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_A} - \frac{1}{\infty}\right] - \frac{Q}{4\pi\varepsilon_o}\left[\frac{1}{r_B} - \frac{1}{\infty}\right]$$

where V_A and V_B have the same zero reference point

4.4 Potential Difference in a Point Charge Field

- A general path between general points B and A in the field of a point charge Q at the origin

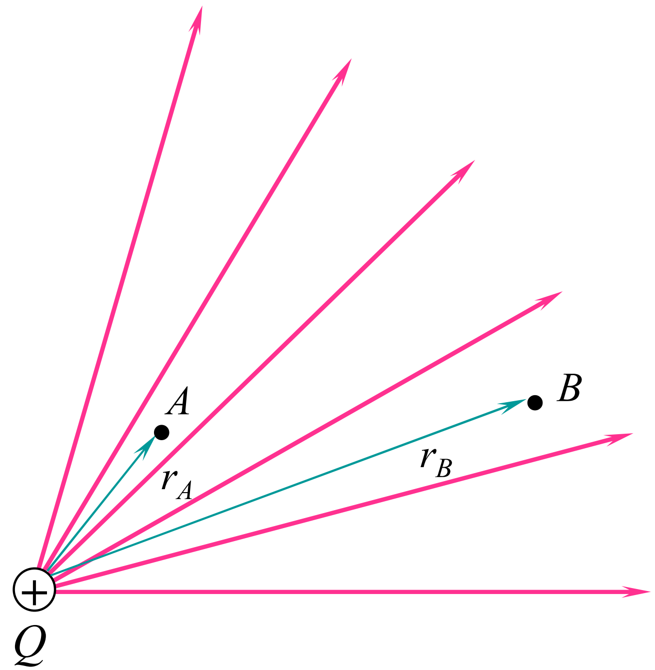
$$V_{AB} = -\int_{r_B}^{r_A} \vec{E} \cdot d r \vec{a}_r = -\int_{r_B}^{r_A} E_r dr = -\int_{r_B}^{r_A} \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right)$$

point charge가 원점에 있으므로 E-field가 $\vec{a_r}$ 방향으로만 존재

- Let $V = 0$ at infinity and $r = r_B = \infty$.

$$V_A = \frac{Q}{4\pi\varepsilon_0 r_A}$$

$$\leftarrow \left(V_B = \frac{Q}{4\pi\varepsilon_O r_B} = 0 \quad @ r_B = \infty\right)$$



$$V = \frac{Q}{4\pi\varepsilon_0 r}$$

(전하 Q 가 원점에 있는 상태에서 1[C]의 전하량을 무한대의 위치에서 원점으로부터 r meter 거리에 있는 점으로 옮겨놓는데 드는 일)

- A conventional method to express the potential without selecting a specific *zero* reference

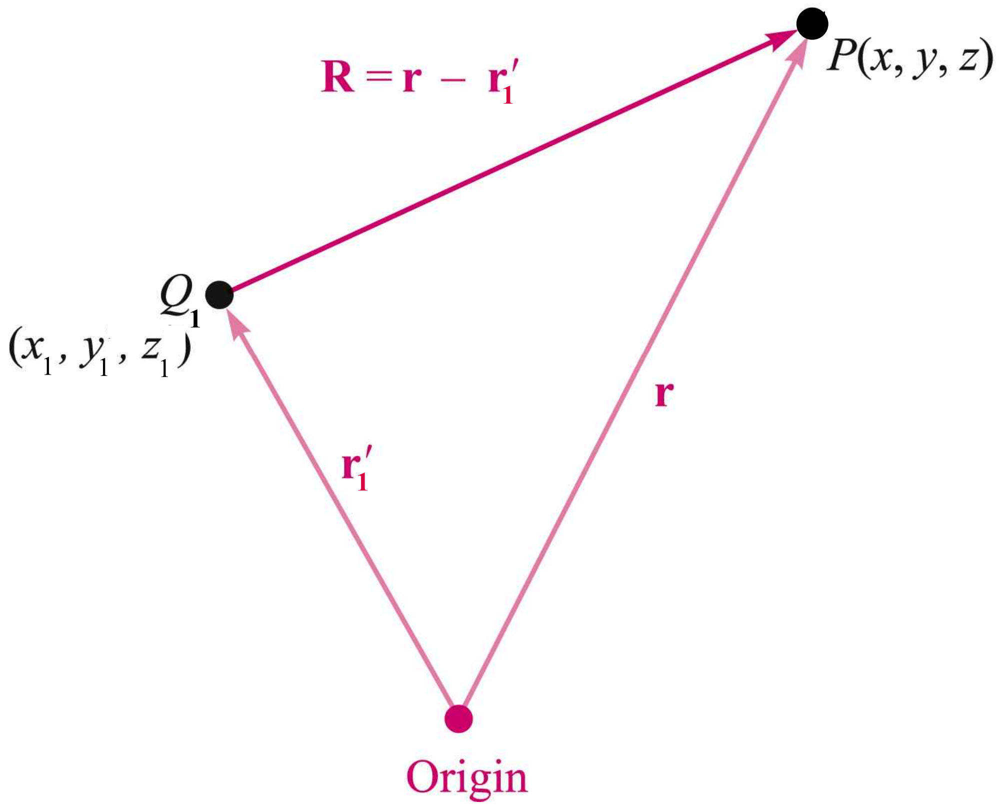
$$V = \frac{Q}{4\pi\varepsilon_0 r} + C_1$$

Boundary conditions: $\begin{cases} V = V_0 & \text{at } r = r_0 \\ V = 0 & \text{at } r : \text{any desired value} \end{cases}$

- Equipotential surface: A surface composed of all those points having the same potential value.

Ex.] Sphere centered at point charge

4.5 The Potential Field of a System of Charges: Conservative Property
4.5.1 Potential Field of an Ensemble of Point Charge Off-Origin



$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r} - \vec{r}_1|}$$

- Zero reference at infinity

- $\vec{r_1}$ 인 지점에 Q_1이 있어서 $\vec{E}$ - field를 발생시킬 때, test 전하 1 C를 무한대(infinity)에서 $\vec{r}$ 로 옮기는데 드는 일)

Potential Field Arising From Two or More Point Charges

- Potential due to two charges in condition of Q_1 at $\vec{r}_1$ and Q_2 at $\vec{r}_2$

$$V(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_1|} + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}_2|}$$

- Potential due to *n* point charges

$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0|\vec{r} - \vec{r}_1|} + \frac{Q_2}{4\pi\varepsilon_0|\vec{r} - \vec{r}_2|} + \cdots + \frac{Q_n}{4\pi\varepsilon_0|\vec{r} - \vec{r}_n|} = \sum_{m=1}^{n} \frac{Q_m}{4\pi\varepsilon_0|\vec{r} - \vec{r}_m|}$$

4.5.2 Potential Field of a Continuous Charge Distributions

- If each point charge is a continuous volume charge distribution $\rho_v \Delta v$,

$$V(\mathbf{r}) = \frac{\rho_v(\mathbf{r}_1)\Delta v_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|} + \frac{\rho_v(\mathbf{r}_2)\Delta v_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|} + \cdots + \frac{\rho_v(\mathbf{r}_n)\Delta v_n}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_n|}$$

$$V(\vec{r}) = \sum_{m=1}^{n} \frac{\rho_v(r_m)\Delta v_m}{4\pi\varepsilon_0\left|\vec{r}-\vec{r}_m\right|}$$

- $n \longrightarrow \infty$

$$V(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}$$

Potential Functions Associated with Line, Surface, and Volume Charge Distributions

Line Charge: $$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\, dL'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

Surface Charge: $$V(\mathbf{r}) = \int_S \frac{\rho_S(\mathbf{r}')\, dS'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

Volume Charge: $$V(\mathbf{r}) = \int_{\mathrm{vol}} \frac{\rho_v(\mathbf{r}')\, dv'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|}$$

cf.) $$\mathbf{E}(\mathbf{r}) = \int_{\mathrm{vol}} \frac{\rho_v(\mathbf{r}')\, dv'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

Ex. 4.3] V =? on the z axis

- Uniform line charge ρ_L in the form of a ring $\rho = a$ in the x-y plane

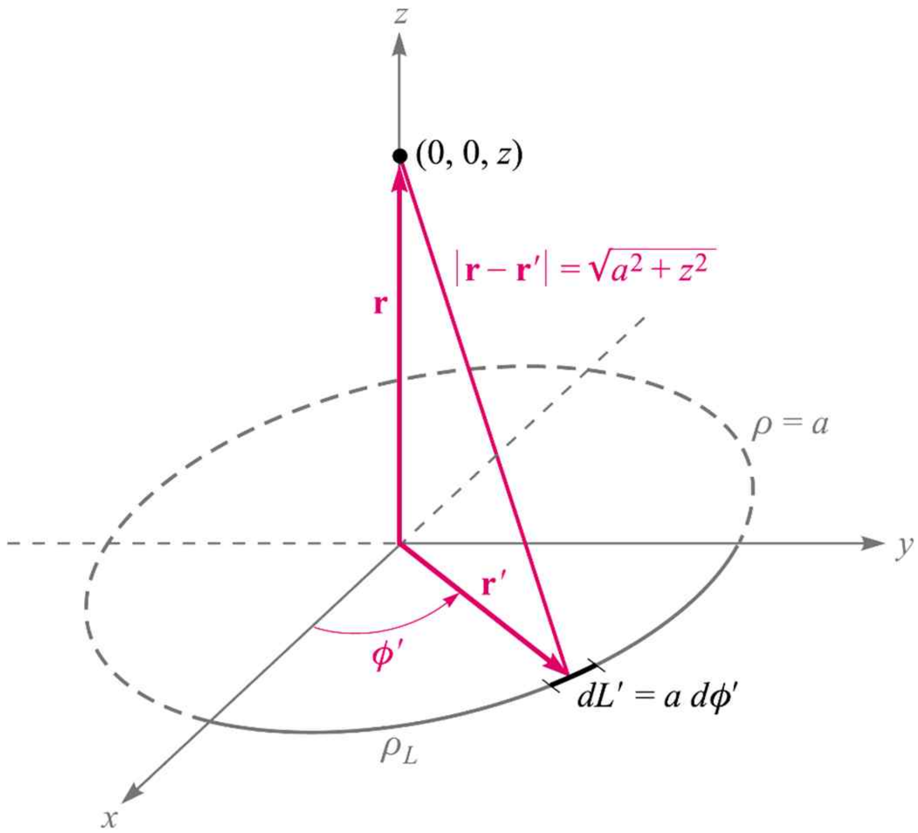
$$V(\mathbf{r}) = \int \frac{\rho_L(\mathbf{r}')\, dL'}{4\pi\epsilon_0|\mathbf{r} - \mathbf{r}'|}$$

with $\quad dL' = a d\phi'$

$$\mathbf{r} = z\mathbf{a}_z$$

$$\mathbf{r}' = a\mathbf{a}_\rho$$

$$|\mathbf{r}-\mathbf{r}'| = \sqrt{a^2 + z^2}$$



$$\therefore\ V = \int_0^{2\pi} \frac{\rho_L a d\phi'}{4\pi\varepsilon_0\sqrt{a^2 + z^2}} = \frac{\rho_L a}{2\varepsilon_0\sqrt{a^2 + z^2}} \cdot \frac{[\phi']_0^{2\pi}}{2\pi} = \frac{\rho_L a}{2\varepsilon_0\sqrt{a^2 + z^2}}$$

4.5.3 The Conservative Nature of the Static Electric Field

- Potential with zero reference at infinity

$$V_A = -\int_{\infty}^{A} \vec{E} \cdot d\vec{L}$$

- Potential difference : *independent on path chosen* for line integral

$$V_{AB} = V_A - V_B = -\int_{B}^{A} \vec{E} \cdot d\vec{L}$$

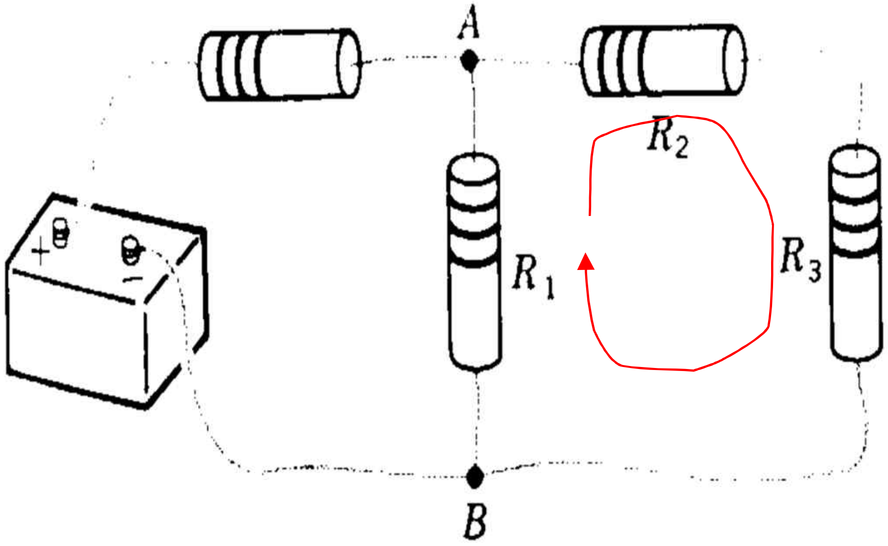
- No work is done in carrying the unit charge around any closed path.

$\oint \vec{E} \cdot d\vec{L} = 0$: conservative field

(∵ Work is not done because the destination point is same with the starting point.)

➔ Only static field (For time-varying field, it is not sufficient (necessary).)

Ex.]



$$V_{R2} + V_{R3} + V_{R1} = 0$$

➔ Kirchhoff's voltage law (KVL)

Ex.] $\vec{F} = \sin \pi \, \rho \, \vec{a}_\phi$ around a circular path of an radius $\rho = \rho_1$

$$d\vec{L} = \rho_1 \, d\phi \, \vec{a}_\phi$$

$$\therefore \oint \vec{F} \cdot d\vec{L} = \int_0^{2\pi} \sin(\pi\rho_1) \vec{a}_\phi \cdot \rho_1 d\phi \, \vec{a}_\phi = \int_0^{2\pi} \rho_1 \sin(\pi\rho_1) d\phi = 2\pi\rho_1 \sin \pi\rho_1$$

$$\neq 0 \quad (@ \, \rho_1 : \text{non-integer})$$

($\because$ $\sin\omega t$ 처럼 $\sin(\pi\rho)$는 $\rho = n$ (n : 정수)일 때만 $\sin(\pi\rho) = 0$)

➔ Non-conservative field will be concerned at Ch. 10.

4.6 Potential Gradient

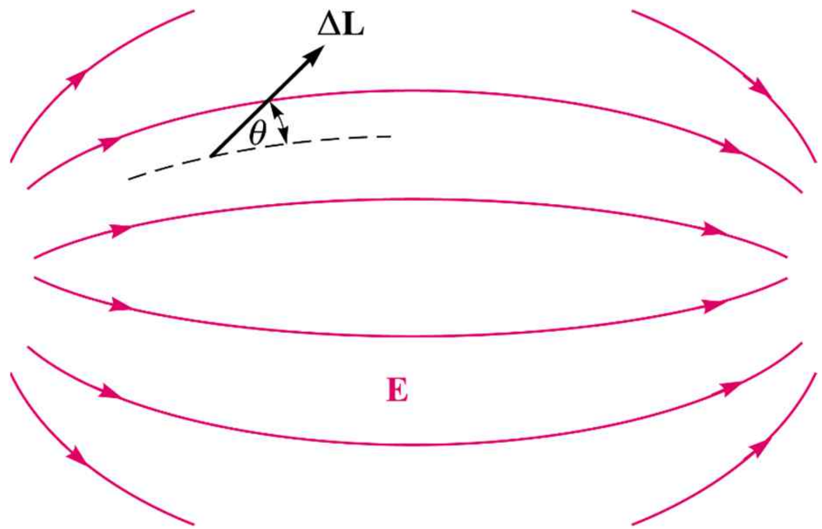
4.6.1 General Relation between the Electric and Potential Field

- Potential : $V = -\int \vec{E} \cdot d\vec{L} \quad \leftarrow \quad W = -Q\int \vec{E} \cdot d\vec{L}$

$$= \int_{vol} \frac{\rho_v(\vec{r}\,')\,dv}{4\pi\varepsilon_0 \left| \vec{r} - \vec{r}\,' \right|}$$

$\Rightarrow V = f(\vec{E} \text{ or } \rho_v)$

- Apply to a very short element of length $\Delta\vec{L}$ along constant $\Delta\vec{E}$



$$\Delta V = -\vec{E} \cdot \Delta\vec{L}$$

$$\Delta\vec{L} = \Delta L\vec{a}_L \qquad \Delta V \approx -(\vec{E}) \cdot (\Delta L\vec{a}_L) = -E\Delta L\cos\theta$$

- As $\Delta L \to 0$, $\left.\frac{\Delta V}{\Delta L}\right|_{\Delta L \to 0} = \frac{dV}{dL} = -E\cos\theta$

  $\therefore \left.\frac{dV}{dL}\right|_{max} = -E\cos\theta \,\Big|_{\theta=\pi} = E$

  (cos θ = -1 일 때, 또는 $\vec{E} \sim \vec{L}$ 사이의 방향이 180^{o}일 때, 거리에 따른 potential 변화가 최대)

- Relation between $\vec{E}$ and V at any point ← $\vec{E} = g(V)$

  1) The magnitude of $\vec{E}$ is given by the maximum value of the rate of change of potential with distance.

  (Ex.] 역풍 바람에 따른 yacht의 진행방향)

  2) The direction of $\vec{E}$ is opposite to the direction in which the potential is increasing the most rapidly.

  $$\leftarrow \quad \frac{dV}{dL} = -E\cos\theta$$

4.6.2 Static Electric Field as the Negative Gradient of Potential

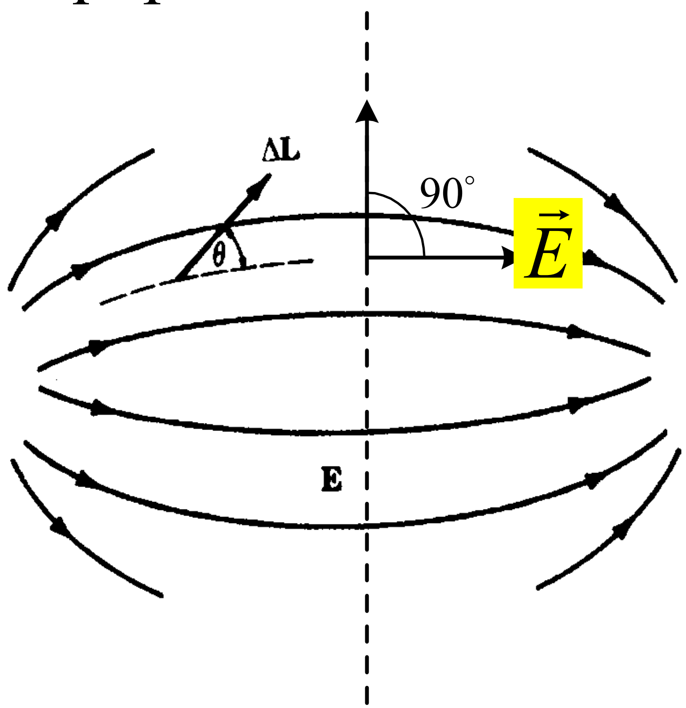
- If ΔL is directed along an equipotential,

  then $\Delta V = 0$, $\quad \Delta V = -\vec{E} \cdot \Delta \vec{L} = 0$

  ➔ Since $\left|\vec{E}\right| \neq 0$, $\Delta L \neq 0$,

  $\vec{E}$ must be perpendicular to this $\vec{L}$ or equipotentials.

  $\Rightarrow \cos\theta = 0, \quad \theta = 90^{\circ} \qquad \Rightarrow \vec{E} \perp \Delta \vec{L}$



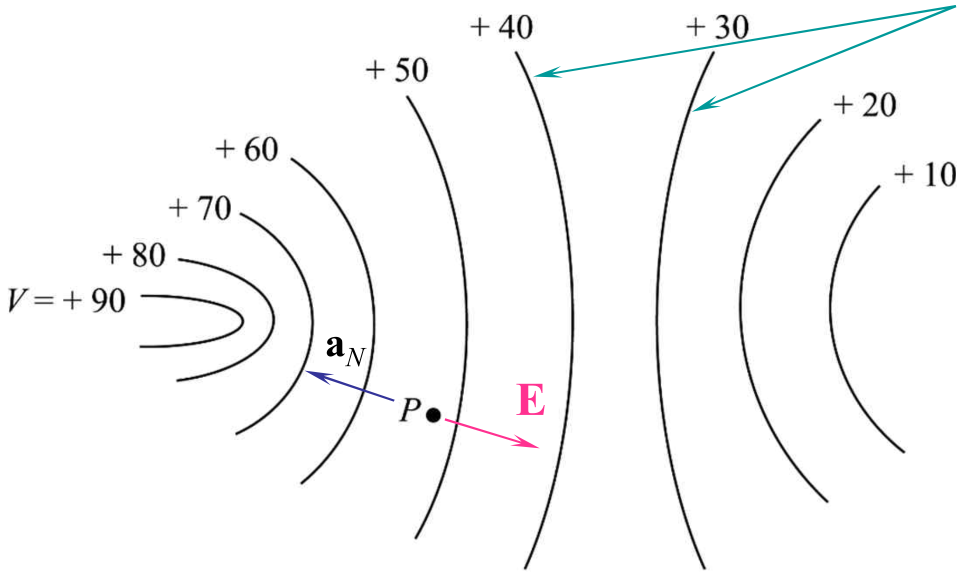
- Since $\left.\dfrac{dV}{dL}\right|_{max}$ occurs when ΔL is in the direction of $\overrightarrow{a_N}$,

$$\left.\frac{dV}{dL}\right|_{\max} = \frac{dV}{dN} \qquad \rightarrow \qquad \vec{E} = -\frac{dV}{dN}\,\vec{a}_N$$

- The maximum rate of *increase* in potential should occur in a direction *exactly* *opposite* the electric field:

$$\mathbf{E} = -\left.\frac{dV}{dL}\right|_{\max} \mathbf{a}_N$$

$\mathbf{a}_N$: unit vector normal to an equipotential surface and in the direction of *increasing* potential



Equipotential surfaces

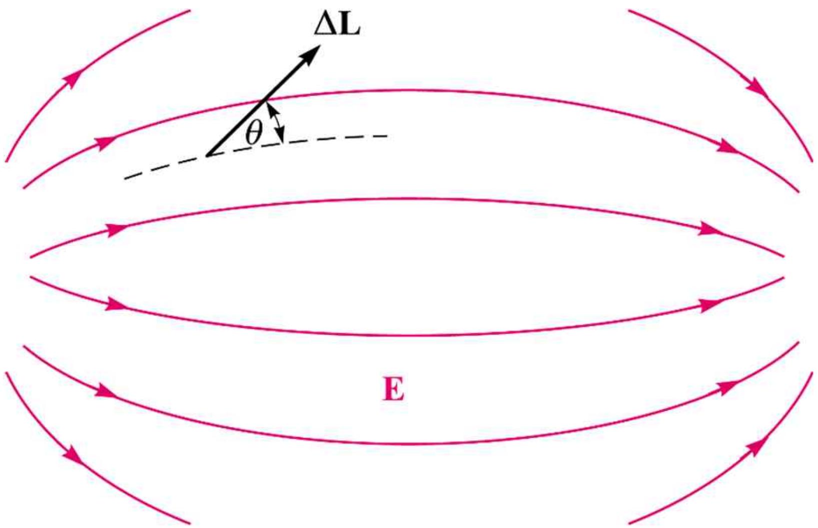
E points in the direction of maximum rate of *decrease* in potential -- in the direction of the *negative gradient* of *V.*

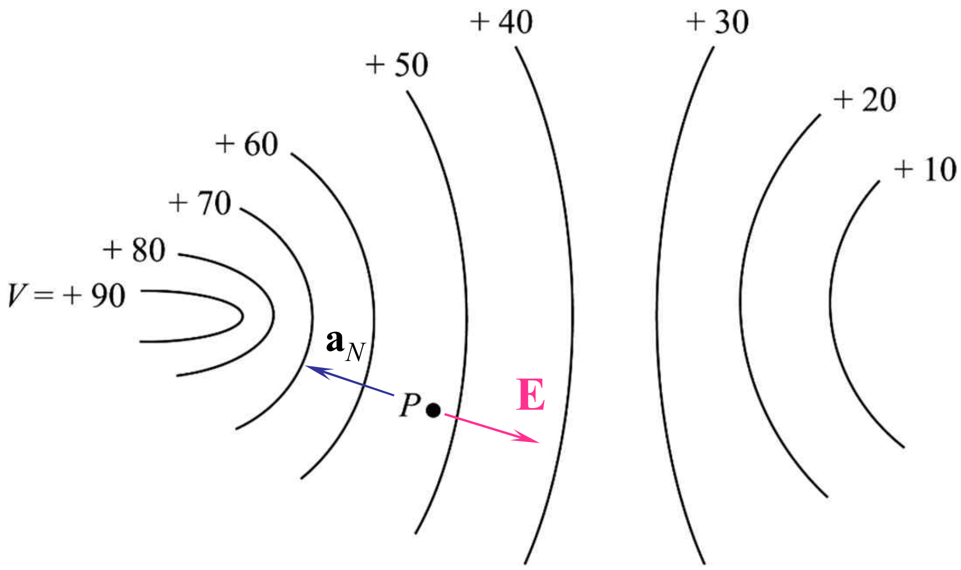
- Gradient: operation on V by which $-\vec{E}$ is obtained

$$\text{Gradient } T = \text{grad } T = \frac{dT}{dN}\vec{a}_N$$

Unit vector normal to the equipotential surfaces and toward higher value

$$\vec{E} = -\text{grad } V$$





4.6.3 Computation of the Gradient

(V 은 무한대에 zero potential 이고 종점(x, y, z)에 의해 그 값이 결정됨 ➔ $V = f(x, y, z)$)

- The differential voltage change can be written as the sum of changes of V in the three coordinate directions.

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

$$dV = -\vec{E}\cdot d\vec{L} = -(E_x\vec{a}_x + E_y\vec{a}_y + E_z\vec{a}_z)\cdot(dx\vec{a}_x + dy\vec{a}_y + dz\vec{a}_z)$$

$$= -E_x dx - E_y dy - E_z dz \quad \leftarrow \quad V = -\int \vec{E}\cdot d\vec{L}$$

$$E_x = -\frac{\partial V}{\partial x} \qquad E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

$$\vec{E} = E_x\vec{a}_x + E_y\vec{a}_y + E_z\vec{a}_z = -\left(\frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z\right) = -\text{grad}\, V$$

$$\text{grad } V = \frac{\partial V}{\partial x}\vec{a}_x + \frac{\partial V}{\partial y}\vec{a}_y + \frac{\partial V}{\partial z}\vec{a}_z$$ ← grad (scalar) = vector

cf.) $\nabla \cdot \vec{D} = \rho_v$ (scalar)

- Gradient: Maximum space rate of change of a scalar quantity and the direction in which this maximum occurs

$$\nabla = \frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z$$

$$\nabla T = \frac{\partial T}{\partial x}\vec{a}_x + \frac{\partial T}{\partial y}\vec{a}_y + \frac{\partial T}{\partial z}\vec{a}_z$$

$$= \text{grad } \ T$$

cf.) $$\nabla \cdot \vec{D} = \left(\frac{\partial}{\partial x}\vec{a}_x + \frac{\partial}{\partial y}\vec{a}_y + \frac{\partial}{\partial z}\vec{a}_z\right) \cdot \left(D_x\vec{a}_x + D_y\vec{a}_y + D_z\vec{a}_z\right)$$

$$= \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

$$\vec{E} = -\nabla V$$ $\leftarrow \vec{E} = f(V)$

$$V = -\int_{init}^{fin} \vec{E} \cdot d\vec{L}$$ $\leftarrow V = g(\vec{E})$

Gradient of V in the Three Coordinate Systems

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(rectangular)}$$

$$\nabla V = \frac{\partial V}{\partial \rho}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi + \frac{\partial V}{\partial z}\mathbf{a}_z \qquad \text{(cylindrical)}$$

Unit vector

$$\nabla V = \frac{\partial V}{\partial r}\mathbf{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\mathbf{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\mathbf{a}_\phi \qquad \text{(spherical)}$$

Spatial variation

[Ex.] $V = 2x^2y - 5z$ @ $P(-4, 3, 6)$

$$V_p = 2(-4)^2(3) - 5(6) = 66 \quad [\mathrm{V}]$$

$$\vec{E} = -\nabla V = -4xy\vec{a}_x - 2x^2\vec{a}_y + 5\vec{a}_z \qquad [\mathrm{V/m}]$$

$$\vec{E}_p = -[4\times(-4)\times 3]\vec{a}_x - [2\times(-4)^2]\vec{a}_y - 5\vec{a}_z = 48\vec{a}_x - 32\vec{a}_y + 5\vec{a}_z$$

$$\left|\vec{E}_p\right| = \sqrt{48^2 + 32^2 + 5^2} = 57.9 \qquad [\mathrm{V/m}]$$

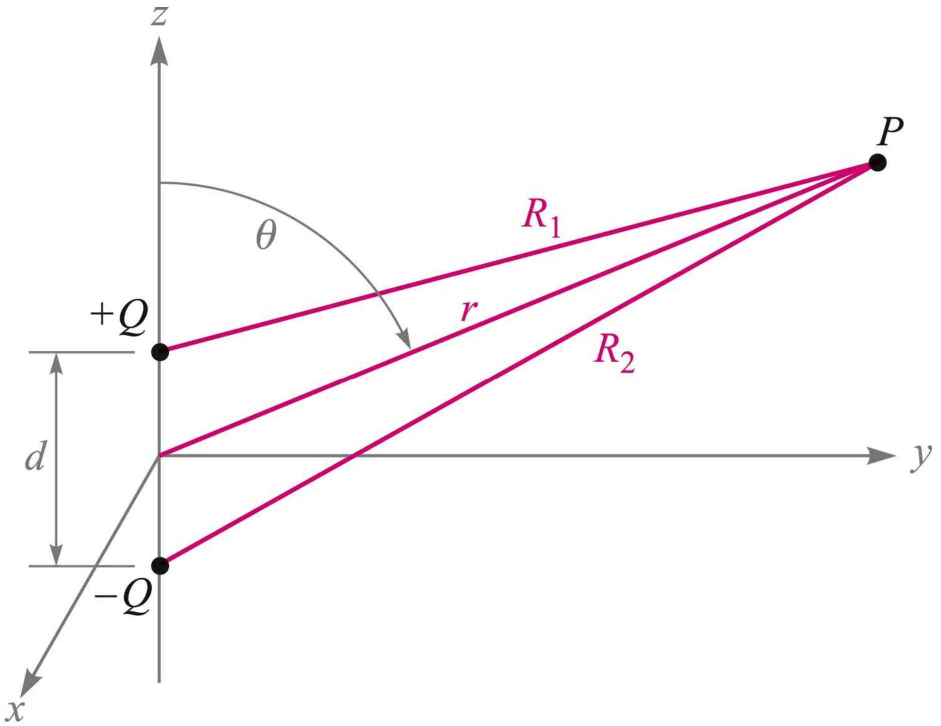
$$\vec{a}_{E.p} = \frac{48\vec{a}_x - 32\vec{a}_y + 5\vec{a}_z}{57.9} = 0.829\vec{a}_x - 0.553\vec{a}_y + 0.086\vec{a}_z$$

$$\vec{D} = \varepsilon_0\vec{E} = -35.4xy\vec{a}_x - 17.7x^2\vec{a}_y + 44.3\vec{a}_z \qquad [\mathrm{pC/m^2}]$$

$$\rho = \nabla\cdot\vec{D} = -35.4y \qquad [\mathrm{pC/m^3}]$$

4.7 The Electric Dipole

- Potential due to both charges at *P* by using superposition theorem



$$V = \frac{Q}{4\pi\varepsilon_0}\frac{1}{R_1} + \frac{(-Q)}{4\pi\varepsilon_0}\frac{1}{R_2}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{Q}{4\pi\varepsilon_0}\frac{R_2 - R_1}{R_1 R_2}$$

- At the $z = 0$ plane, midway between two point charges,

$$V = 0 \qquad (\because R_1 = R_2)$$

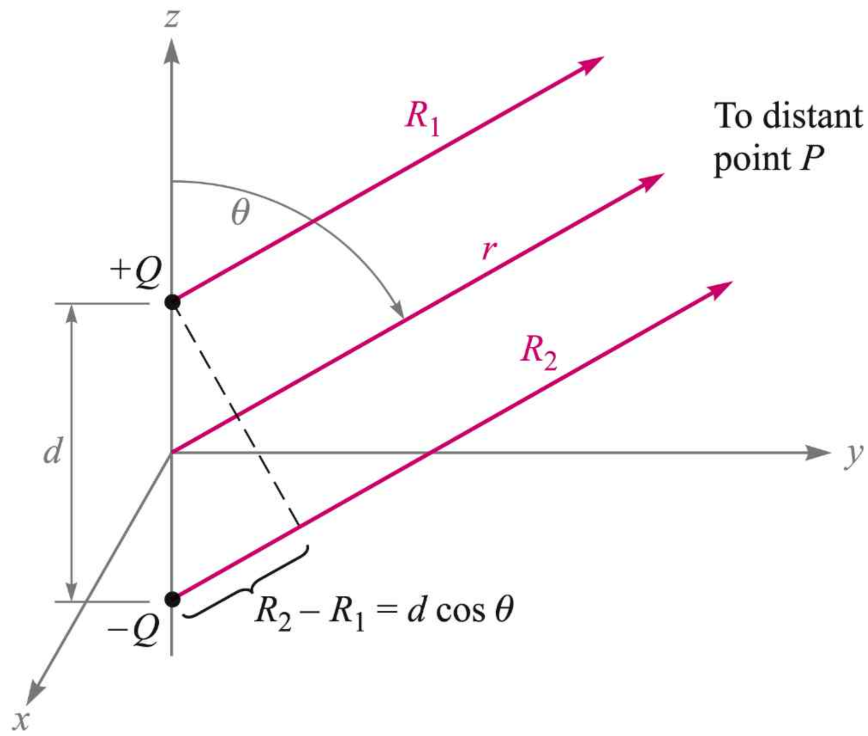
Far-Field Approximation

- If point P is located far away relative to distance between dipoles ($r >> d$),

$$R_2 - R_1 \doteq d\cos\theta$$

$$V = \frac{Q}{4\pi\varepsilon_0}\frac{R_2 - R_1}{R_1 R_2} = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2}$$

$$V\Big|_{z=0\ (\text{or}\ \theta=90^\circ)} = 0$$



Electric Field of the Dipole

$$\vec{E} = -\nabla V = -\left(\frac{\partial V}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\vec{a}_\phi \right)$$

$$= -\left(-\frac{Qd\cos\theta}{2\pi\varepsilon_0 r^3}\vec{a}_r - \frac{Qd\sin\theta}{4\pi\varepsilon_0 r^3}\vec{a}_\theta \right)$$

$$= \frac{Qd}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\vec{a}_r + \sin\theta\,\vec{a}_\theta) = \frac{Qd}{4\pi\varepsilon_0}(\frac{2}{r^3}\cos\theta\,\vec{a}_r + \frac{\sin\theta}{r^3}\vec{a}_\theta)$$

Electric Dipole Field and Equipotentials

[Ex.] $\dfrac{Qd}{4\pi\varepsilon_0} = 1 \quad \Rightarrow \quad V = \dfrac{\cos\theta}{r^2} \qquad \cos\theta = Vr^2$

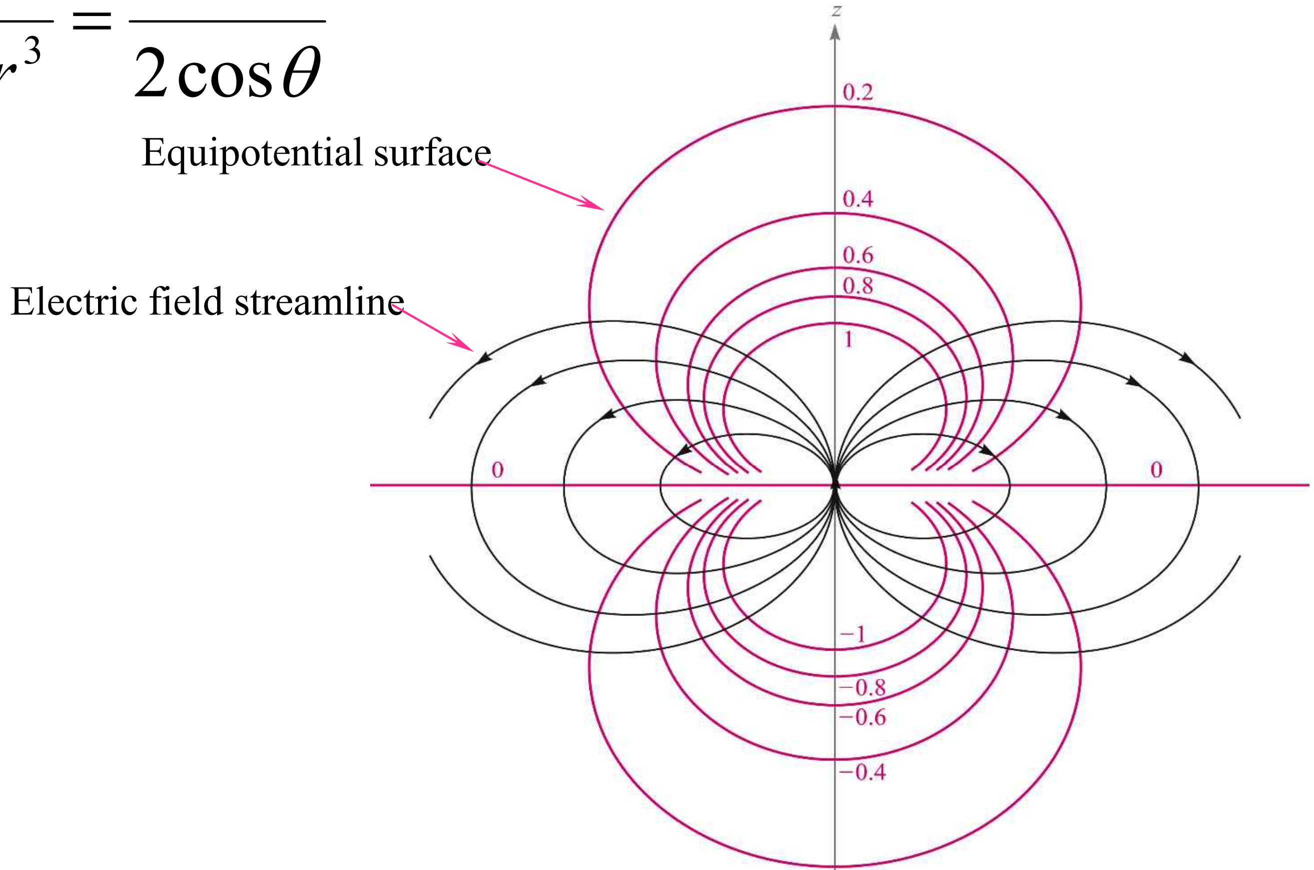
$$\frac{E_\theta}{E_r} = \frac{r\,d\theta}{dr} = \frac{\sin\theta / r^3}{2\cos\theta / r^3} = \frac{\sin\theta}{2\cos\theta}$$

$$\frac{dr}{r} = 2\cot\theta\, d\theta$$

$$\ln r = 2\ln\sin\theta + \ln C_1$$

$$= \ln C_1 \sin^2\theta$$

$$\therefore\ r = C_1 \sin^2\theta$$



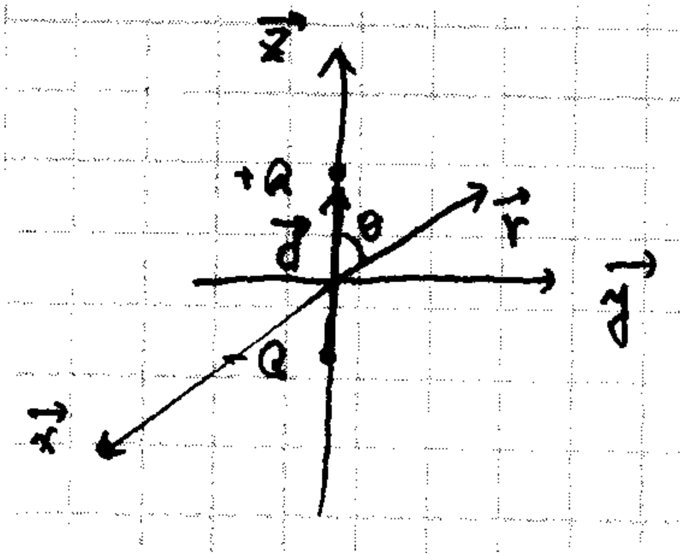
➔ Black stream lines are for C_1 = 1, 1.5, 2, and 2.5

4.7.3 Rewriting the Potential Field due to Electric Dipole Moment

- Dipole moment

$$\mathbf{p} = Q\mathbf{d}$$

where $\vec{d}$: vector length -Q to +Q



$$\mathbf{d} \cdot \mathbf{a}_r = d \cos\theta$$

$$V = \frac{Qd\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{P}\cdot\vec{a}_r}{4\pi\varepsilon_0 r^2}$$

- Generalized form (in case center of dipole is moved to $\vec{r}'$):

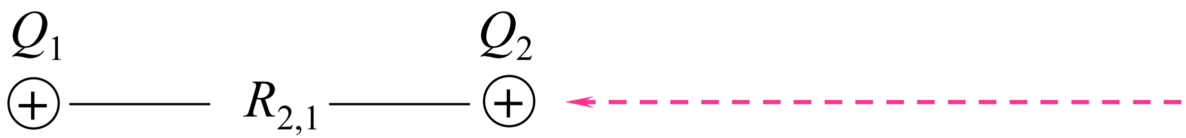
$$V = \frac{1}{4\pi\varepsilon_0 \left|\vec{r}-\vec{r}'\right|^2}\vec{P}\cdot\frac{(\vec{r}-\vec{r}')}{\left|\vec{r}-\vec{r}'\right|}$$

where $\vec{r}$: field measurement point
$\vec{r}'$: dipole center

4.8 Electrostatic Energy
4.8.1 Stored Energy in a Distribution of Charge

- Bringing a charge Q_1 from infinity to any position region requires no work. (In empty (or no charge) space, $W = 0$. $\because \vec{E} = 0$)

- Work required for positioning of Q_2 at a point 2 in the field of Q_1



Charge Q_2 is brought into position from infinity.

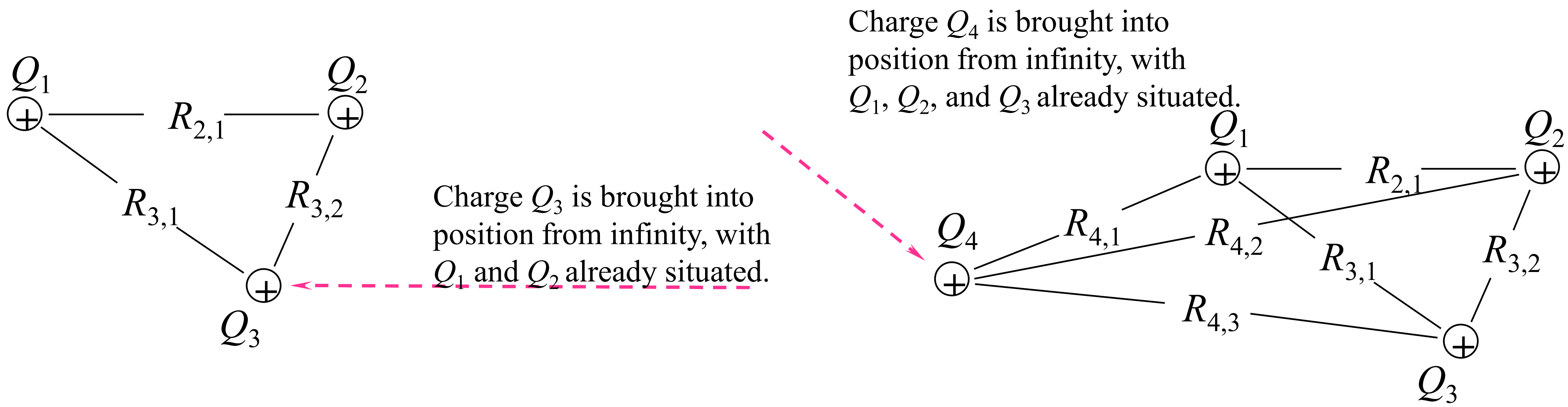
$$W_E(2 \text{ charges}) = Q_2 V_{2,1} = \frac{Q_2 Q_1}{4\pi\epsilon_0 R_{2,1}}$$

where $V_{2,1}$: 특정 지점(1) Q_1 전하에 의한 $\vec{E}$ 가 있는 상태에서 1 C 의 전하를 무한대 지점에서 특정 지점(2)으로 옮기는데 드는 일

- Work required to locate each additional charge in the field of all those already present;

Work for position $Q_3 = Q_3\, V_{3,1} + Q_3\, V_{3,2} +$

“ $Q_4 = Q_4\, V_{4,1} + Q_4\, V_{4,2} + Q_4\, V_{4,3}$



where $V_{3,1} = \dfrac{Q_1}{4\pi\epsilon_0 R_{3,1}}$, $V_{3,2} = \dfrac{Q_2}{4\pi\epsilon_0 R_{3,2}}$,

$V_{4,1} = \dfrac{Q_1}{4\pi\epsilon_0 R_{4,1}}$, $V_{4,2} = \dfrac{Q_2}{4\pi\epsilon_0 R_{4,2}}$, and $V_{4,3} = \dfrac{Q_3}{4\pi\epsilon_0 R_{4,3}}$

- The total work (or energy) is obtained by adding each contribution.

Total positioning work = potential energy of field

$$= W_E = Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \cdots\cdots \quad \text{- ①}$$

Since $Q_3 V_{3,1} = Q_3 \dfrac{Q_1}{4\pi\varepsilon_0 R_{13}} = Q_1 \dfrac{Q_3}{4\pi\varepsilon_0 R_{31}} = Q_1 V_{1,3}$ and

$R_{13} = R_{31}$ (: scalar distance),

$$W_E = Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \cdots\cdots \quad \text{- ②}$$

Eq. ① + Eq. ②:

$$2W_E = \left(Q_2 V_{2,1} + Q_3 V_{3,1} + Q_3 V_{3,2} + Q_4 V_{4,1} + Q_4 V_{4,2} + Q_4 V_{4,3} + \cdots\right)$$
$$+ \left(Q_1 V_{1,2} + Q_1 V_{1,3} + Q_2 V_{2,3} + Q_1 V_{1,4} + Q_2 V_{2,4} + Q_3 V_{3,4} + \cdots\cdots\right)$$

$$
\begin{aligned}
2W_E &= Q_1\left(V_{1,2} + V_{1,3} + V_{1,4} + \cdots \right) \\
&+ Q_2\left(V_{2,1} + V_{2,3} + V_{2,4} + \cdots \right) \\
&+ Q_3\left(V_{3,1} + V_{3,2} + V_{3,4} + \cdots \right) + \cdots\cdots
\end{aligned}
$$

where ( ): combined potential due to all the charge except for the charge at the point where this combined potential is being found. (결합 전위가 발견될 위치의 전하를 제외한 나머지 정전하에 의한 결합 전위)

- Define the "local" potentials:

$$
\begin{cases}
V_1 = V_{1,2} + V_{1,3} + V_{1,4} + \cdots \\
V_2 = V_{2,1} + V_{2,3} + V_{2,4} + \cdots \\
V_3 = V_{3,1} + V_{3,2} + V_{3,4} + \cdots \\
V_4 = V_{4,1} + V_{4,2} + V_{4,3} + \cdots
\end{cases}
$$

$$
W_E = \frac{1}{2}\left(\underline{Q_1 V_1} + Q_2 V_2 + Q_3 V_3 + \cdots \right) = \frac{1}{2}\sum_{m=1}^{m=N} Q_m V_m
$$

Q_1 이라는 전하를 점1에 위치시키기 위해 나머지 전하들에 의해 드는 일(또는 에너지)의 합

- Energy stored in a region of continuous charge distribution:

$$W_E = \frac{1}{2}\int_{vol} \rho_v V dv$$

By using vector identity $\nabla\cdot(V\vec{D}) \equiv V(\nabla\cdot\vec{D}) + \vec{D}\cdot(\nabla V)$,

$$W_E = \frac{1}{2}\int_{vol} \rho_v V\, dv = \frac{1}{2}\int_{vol} (\nabla\cdot\vec{D}) V\, dv = \frac{1}{2}\int_{vol} V(\nabla\cdot\vec{D})\, dv$$

$$= \frac{1}{2}\int_{vol}\left[\nabla\cdot(V\vec{D}) - \vec{D}\cdot(\nabla V)\right] dv$$

$$= \frac{1}{2}\oint_s (V\vec{D})\cdot d\vec{S}$$

$$-\frac{1}{2}\int_{vol} \vec{D}\cdot(\nabla V) dv$$

By divergence theorem

$$\oint_{vol} \nabla\cdot\vec{D}\, dv = \oint_s \vec{D}\cdot d\vec{S} ,$$

여기서 S와 v는 임의 체적안에 있는 모든 전하를 포함할 수 있는 폐곡면(closed surface) 과 $r = \infty$ ($\leftrightarrow$ $V = 0$)인 체적을 고려

$$\nabla V = -\vec{E}$$

Stored Energy in the Electric Field

$$\therefore \; W_E = \frac{1}{2}\int_{vol} \vec{D}\cdot\vec{E}\,dv = \frac{1}{2}\int_{vol} \varepsilon_0 E^2\,dv$$

(Since $V = \dfrac{Q}{4\pi\varepsilon_0 r}$, $\vec{D} = \dfrac{Q}{4\pi r^2}\vec{a}_r$, $\vec{S} = 4\pi r^2 \vec{a}_r$,

$$\frac{1}{2}\oint (V\vec{D})\cdot d\vec{S} = \frac{1}{2}\oint \frac{Q}{4\pi\varepsilon_0 r}\frac{Q}{4\pi r^2}\overrightarrow{a_r}\cdot 4\pi r^2\overrightarrow{a_r} = \frac{1}{8\pi\varepsilon_0}\oint \frac{1}{r}\,dr \cong 0$$

As $r \to \infty$, $\dfrac{1}{2}\oint (V\vec{D})\cdot d\vec{S} \approx 0$)

- Differential Basis: $dW_E = \dfrac{1}{2}\vec{D}\cdot\vec{E}\,dv$

$$\frac{dW_E}{dv} = \frac{1}{2}\vec{D}\cdot\vec{E}$$

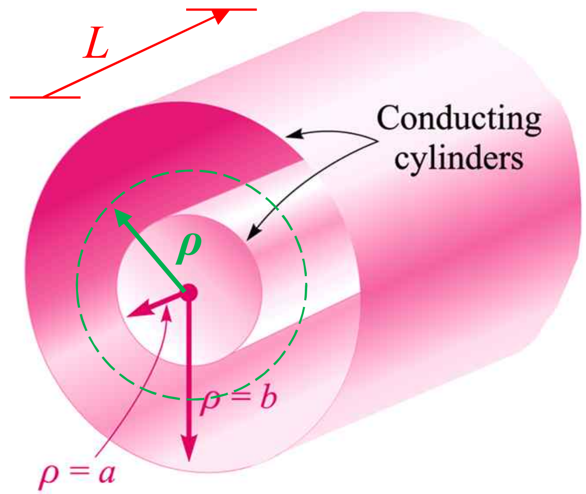
4.8.3 Field Energy in a Coaxial Cable

- Calculating the energy stored in the electrostatic field of a section of a coaxial cable or capacitor of length L

$$\vec{E} = \frac{a\rho_S}{\varepsilon_0 \rho}\overrightarrow{a_\rho} \leftarrow \text{From Ch. 3 page 16} \quad \vec{D} = \frac{a\rho_S}{\rho}\overrightarrow{a_\rho} \quad (a < \rho < b)$$

$$W_E = \frac{1}{2}\int_0^L \int_0^{2\pi} \int_a^b \varepsilon_0 \frac{a^2 \rho_s^2}{\varepsilon_0^2 \rho^2} \rho \, d\rho \, d\phi \, dz$$

$$= \frac{a^2 \rho_s^2}{2\varepsilon_0} [z]_0^L [\phi]_0^{2\pi} [\ln \rho]_a^b = \frac{\pi L a^2 \rho_s^2}{\varepsilon_0} \ln \frac{b}{a}$$



- Another method

$$V_a = -\int_b^a E_\rho d\rho = -\int_b^a \frac{a\rho_s}{\varepsilon_0 \rho} d\rho = \frac{a\rho_s}{\varepsilon_0} \ln\frac{b}{a}$$

← zero-potential reference (outer conductor)

$$W_E = \frac{1}{2}\int_{vol} \rho_v V dv = \frac{1}{2}\int_0^L \int_{a-\frac{t}{2}}^{a+\frac{t}{2}} \int_0^{2\pi} \frac{\rho_s}{t} \frac{a\rho_s}{\varepsilon_0} (\ln\frac{b}{a}) \rho \, d\rho \, d\phi \, dz$$

← $\rho = a$ 에서의 체적전하밀도는 ρ_S/t 로 가정 가능.

$a - (t/2) \le \rho \le a + (t/2), \; t << a$

$$= \frac{1}{2}\frac{a\rho_s^2}{\varepsilon_0 t} \ln\frac{b}{a} [z]_0^L \, [\phi]_0^{2\pi} \, [\frac{1}{2}\rho^2]_{a-\frac{t}{2}}^{a+\frac{t}{2}} = \frac{a\rho_s^2}{\varepsilon_0 t} \ln\frac{b}{a} L\pi \frac{1}{2}\left[(a+\frac{t}{2})^2 - (a-\frac{t}{2})^2\right]$$

$$= \frac{a^2 \rho_s^2 \ln\left(\frac{b}{a}\right)}{\varepsilon_0} \pi L$$

(same with the previous page result)