Engineering Electromagnetics

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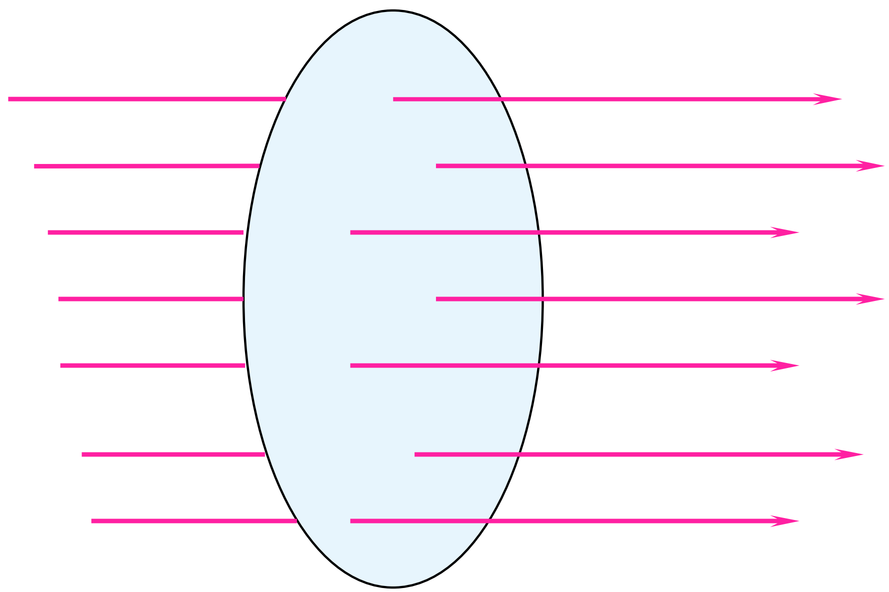
Chapter 5:

Conductors and Dielectrics

5.1 Current and Current Density

- Current I [A] $$I = \frac{dQ}{dt}$$: Movement of positive (hole) and/or negative (electron) charges
  ➔ In textbook, explained in positive charge

- Current density $\vec{J}$: current flowing unit area [A/m^2]
- In case $\vec{J}$ is normal to the surface, $\Delta I = J_N \Delta S$



ΔS

Current Density as a Vector Field

- In reality, the direction of current flow may not be normal to the artificial surface.

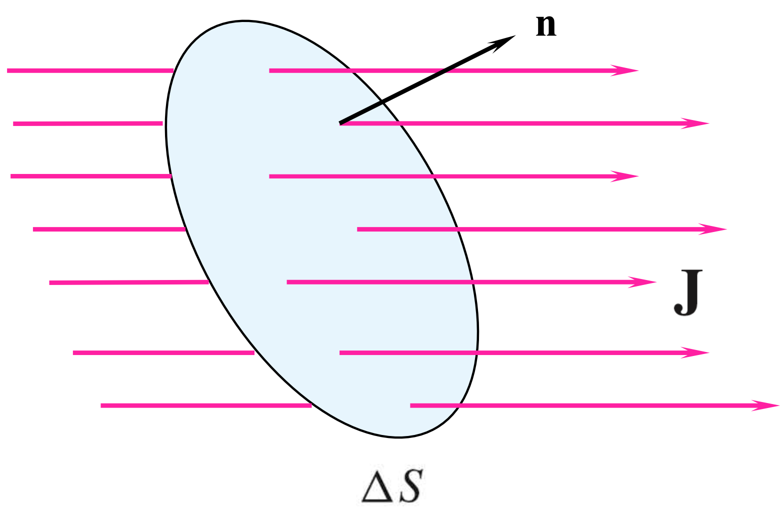
- Increment of current (ΔI) crossing an incremental surface (ΔS) normal to the current density :

$$\Delta I = J_N \Delta S = \vec{J} \cdot \Delta\vec{S} \qquad \text{where } \Delta\vec{S} = \Delta S\overrightarrow{a_N}$$

- Total current

$$\boxed{I = \int_S \mathbf{J} \cdot d\mathbf{S}}$$

: general description even when the current density is not perpendicular to the surface.



Relation of Current to Charge Velocity

- Consider a charge ΔQ, occupying volume Δv, moving in the positive x direction at velocity $\overrightarrow{v_x}$.

$$\Delta Q = \rho_v \Delta v = \rho_v \, \Delta S \, \Delta L$$ in Fig. (a) : 전하량

- For time interval Δt, the element of charge has moved a distance Δx,

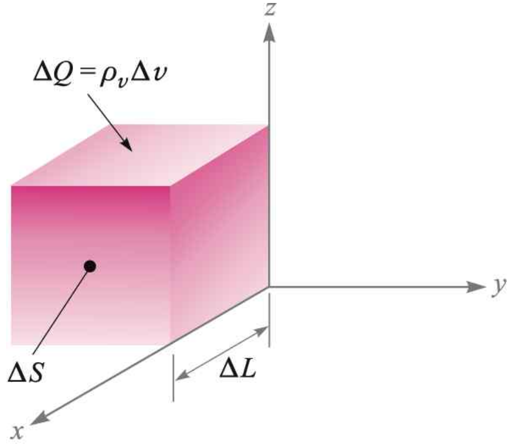
$$\Delta Q = \rho_v \, \Delta S \, \Delta x$$: 전하 증감량 (ΔS 면적이 Δx 만큼 움직여서 체적 Δv가 생겼다고 가정)

- Motion of charge: current

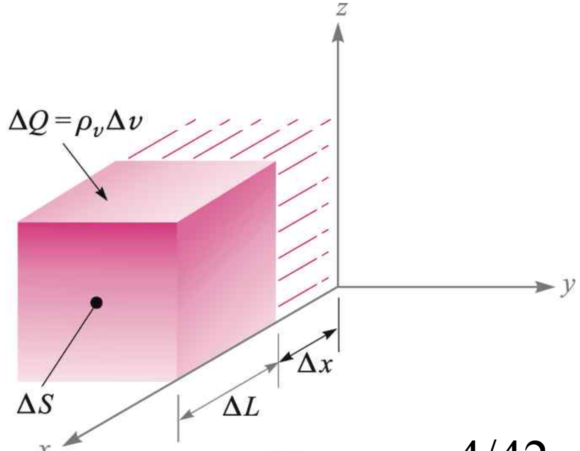
$$\Delta I = \frac{\Delta Q}{\Delta t} = \rho_v \, \Delta S \frac{\Delta x}{\Delta t}$$

$$= \rho_v \Delta S v_x$$

where v_x: x-component of velocity



(a)

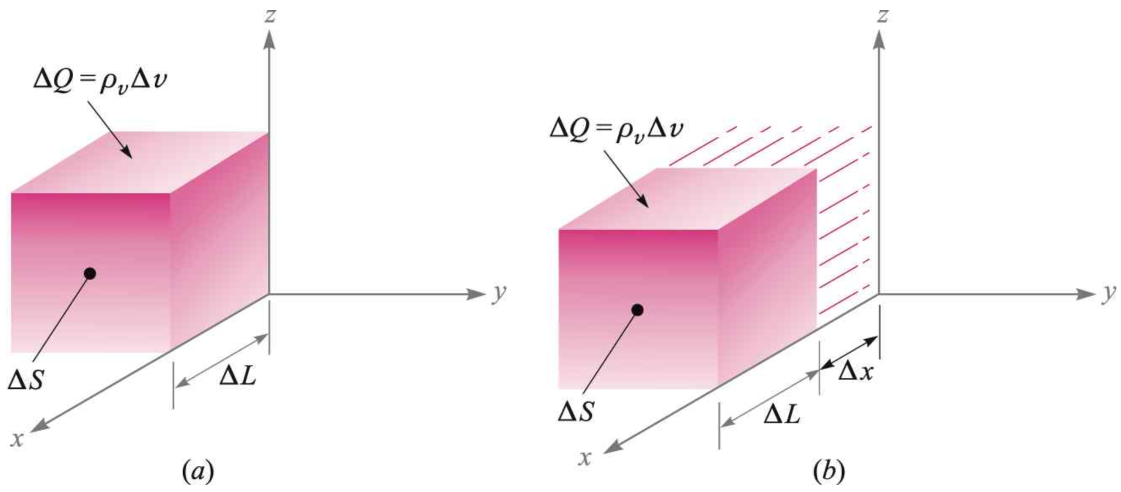


(b)

Relation of Current Density to Charge Velocity

$$\vec{J} = \frac{\vec{I}}{\vec{S}} = \rho_v \vec{v}$$: convection current density

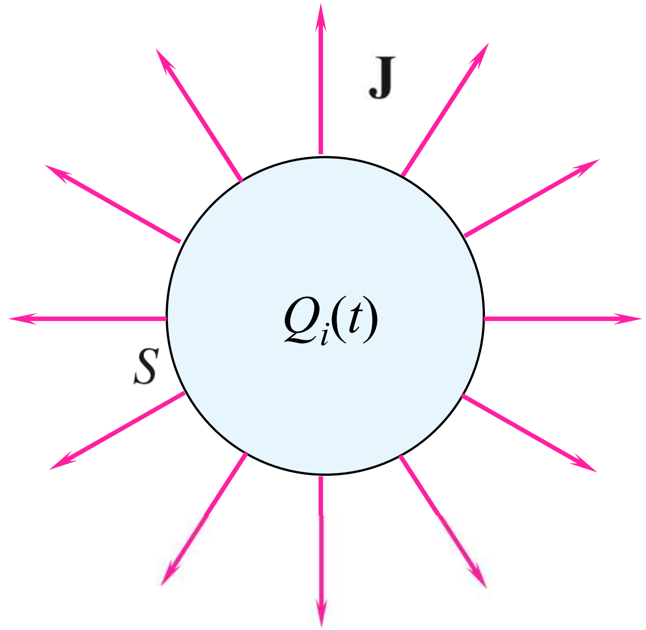
- 전하의 유동이 전류에 미치는 영향: If $\vec{v}\uparrow$, then $\vec{J}\uparrow$
- 터널에서 자동차의 통과속도가 빠르면, 터널 통과 자동차 수도 증가
- 물리전자: $\overrightarrow{J_{p,drift}} = qp\overrightarrow{v_d} = \rho_v\overrightarrow{v_d}$



5.2 Continuity of Current

- Suppose that charge Q_i is escaping from a volume through closed surface S. Total current is:

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} = -\frac{dQ_i}{dt} = -\frac{d}{dt}\int_{\text{vol}} \rho_v \, dv$$



- Outward flow of positive charge
- 임의의 closed surface 밖으로 향하는 전하의 합
- Closed surface 내부 양전하의 감소 (또는 음전하의 증가)

- By the divergence theorem,

$$\left(\leftarrow \int_v \rho_v dv = \int \left(\nabla \cdot \vec{D}\right) dv = \int_S \vec{D} \cdot d\vec{S}\right)$$

$$\oint_S \mathbf{J} \cdot d\mathbf{S} = \int_{\text{vol}} (\nabla \cdot \mathbf{J}) dv = -\frac{dQ_i}{dt} = -\frac{d}{dt}\int_{\text{vol}} \rho_v \, dv$$

$$= \int_{vol} \left(-\frac{d\rho_v}{dt}\right) dv \quad \Rightarrow \quad \boxed{(\nabla \cdot \mathbf{J}) = -\frac{\partial \rho_v}{\partial t}}$$

: Continuity Equation

[Ex.] $\vec{J} = \frac{1}{r} e^{-t} \vec{a}_r$ [A/m^2]

- Total outward current @ $t = 1$ sec and $r = 5$ m:

$$I_5 = J_r S = (\frac{1}{5} e^{-1})(4\pi 5^2) = 23.1 \text{ [A]}$$

- Total outward current @ $t = 1$ sec and $r = 6$ m:

$$I_6 = J_r S = (\frac{1}{6} e^{-1})(4\pi 6^2) = 27.7 \text{ [A]} \quad \Rightarrow I_5 < I_6 \qquad \text{Why???}$$

- Because

$$-\frac{\partial \rho_v}{\partial t} = \nabla \cdot \vec{J} = \nabla \cdot (\frac{1}{r} e^{-t} \vec{a}_r) = \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 \frac{1}{r} e^{-t})$$

$$= \frac{1}{r^2} e^{-t} \,, \qquad \leftarrow \text{div } \vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 D_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta D_\theta) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

$$\rho_v = -\int \frac{1}{r^2} e^{-t} dt + K(r) = \frac{1}{r^2} e^{-t} + K(r)$$

- Assumption: $\rho_v \to 0$ as $t \to \infty$

($\vec{J} = \frac{1}{r} e^{-t} \vec{a}_r$ 에서 $t \to \infty$ 이면 $\vec{J} = 0$ $\leftrightarrow$ $\rho = 0$)

$$\therefore \rho_v \Big|_{t=\infty} = K(r) = 0$$

$$\therefore \rho_v = \frac{1}{r^2} e^{-t}$$

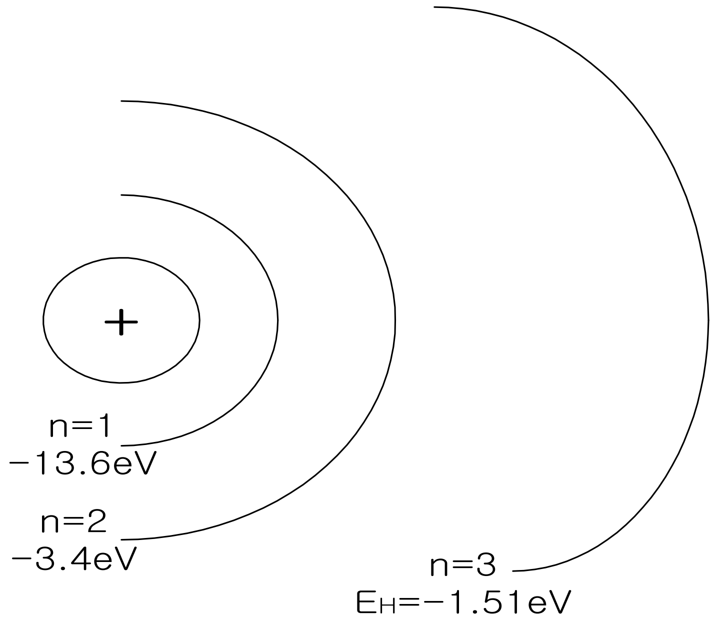
- $\vec{J} = \rho_v \vec{v}$

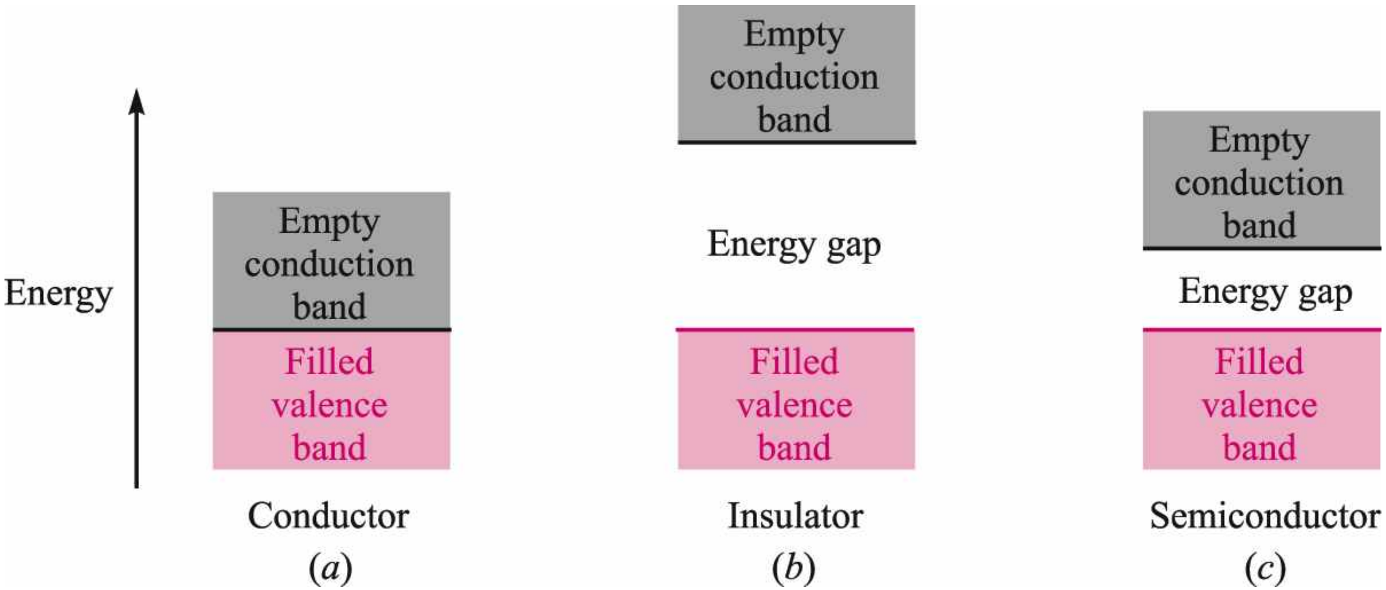
$$v_r = \frac{J_r}{\rho_v} = \frac{\frac{1}{r} e^{-t}}{\frac{1}{r^2} e^{-t}} = r \qquad \Rightarrow (v_r)_{r=5} < (v_r)_{r=6}$$

➔ $I_5 < I_6$ 인 이유는 어떤 보이지 않는 힘에 의해 바깥 방향으로 진행하는 전하의 속도가 가속되기 때문이다.
r = 5 일 때 보다 r = 6 일 때의 v_r이 크므로 r = 6 일 때 convection 전류밀도가 더 큼.

5.3 Metallic conductors

- Quantization energy level





Electron Flow in Conductors

- Applied force on an electron of charge Q = -e : $\mathbf{F} = -e\mathbf{E}$

- $\overrightarrow{v_d}$:drift velocity

(전자에 $\vec{E}$ -field가 주어지면 속도가 생기는데, material 내부 격자와 충돌에 의해 평균 구동속도를 고려해야 함.)

$$\mathbf{v}_d = -\mu_e \mathbf{E},$$

where μ_e [mu] : *mobility* of electron

(Conductivity = Charge density × Electron (or Hole) mobility)

- (Drift) Current density

$$\mathbf{J} = \rho_v \mathbf{v}_d = -\rho_e \mu_e \mathbf{E} = \sigma \mathbf{E},$$

$$\boxed{\mathbf{J} = \sigma \mathbf{E}}$$

where ρ_e: free-electron charge density (negative value)

σ [sigma] : conductivity ([mho/m] or [S/m])

$\boxed{\sigma = -\rho_e \mu_e + \rho_h \mu_h}$ in semiconductor

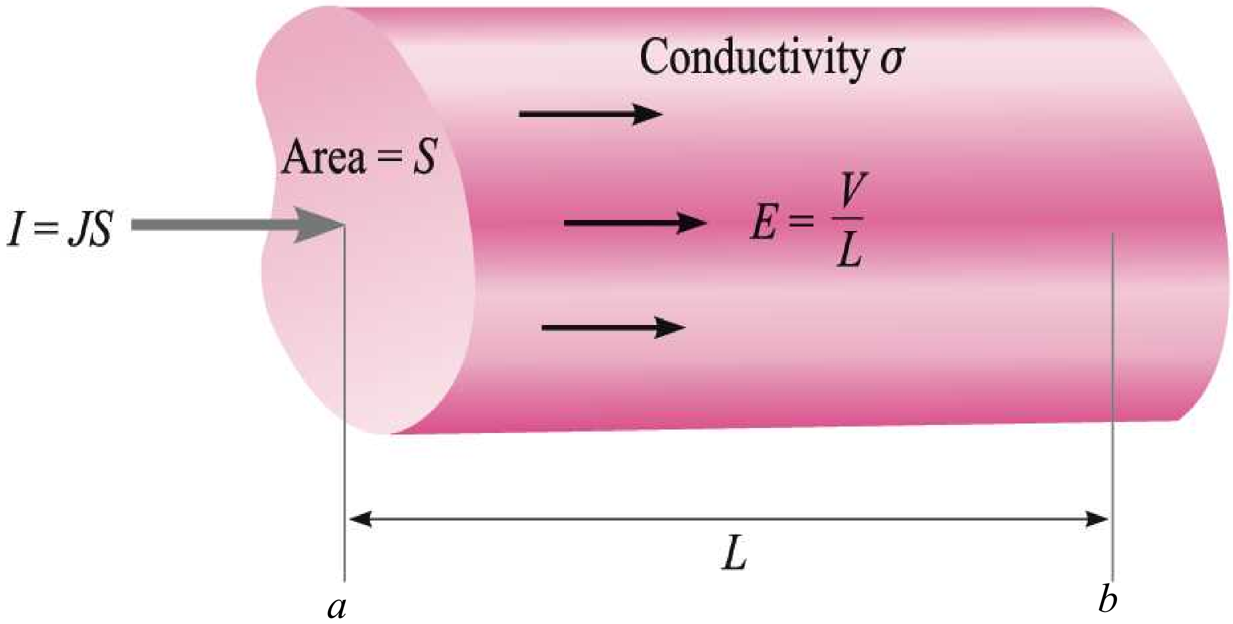
$\sigma =$ 3.8×10^7 for aluminum,

5.8×10^7 for copper, 6.17×10^7 for silver

Resistance

- Consider cylindrical conductor with voltage V applied across ends.
- $\vec{J}$ and $\vec{E}$: uniform

$$I = \int_S \mathbf{J} \cdot d\mathbf{S} = JS$$



$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{L}$$

$$= -\mathbf{E} \cdot \int_b^a d\mathbf{L}$$

$$= -\vec{E} \cdot \vec{L}_{ba} = \vec{E} \cdot \vec{L}_{ab} \quad (\because \vec{E} \text{ : uniform})$$

or $V = EL \;\; (\because \vec{E} \; // \; \vec{L}\,)$

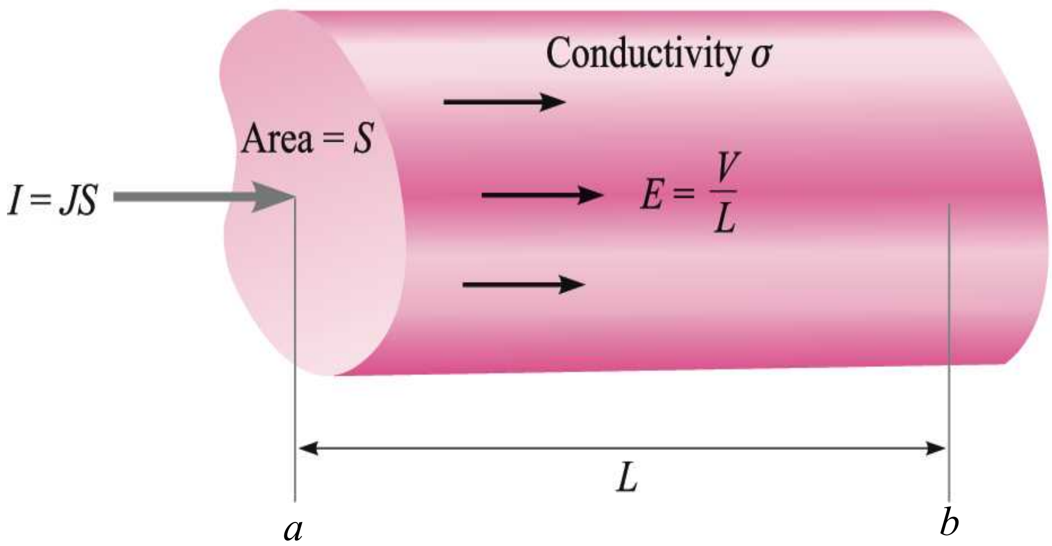
$$\boxed{R = \frac{L}{\sigma S}}$$

$$J = \frac{I}{S} = \sigma E = \sigma \frac{V}{L} \Longrightarrow \underline{V = \frac{L}{\sigma S} I = IR} \text{ : Ohm's law}$$

General Expression for Resistance

$$R = \frac{V_{ab}}{I} = \frac{-\int_b^a \mathbf{E} \cdot d\mathbf{L}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{S}}$$

- Resistance : **Ratio of** potential difference between two ends of cylinder **to** current entering more positive potential end.
(원통 양 단면 사이의 전위차와 높은 전위를 갖는 면으로 입력되는 전류의 비)



[Ex.]

- AWG (American Wire Gauge) #16: $d = 0.0508' = 1.291 \times 10^{-3}$ [m]
  (: **d**iameter)

$$S = \pi r^2 = \pi \times (1.291 \times 10^{-3} / 2)^2 = 1.308 \times 10^{-6} \ [\mathrm{m}^2]$$

- Resistance of a wire 1 mile (1609m):

$$R = \frac{1609}{(5.8 \times 10^{7}) \times (1.308 \times 10^{-6})} = 21.2 \ [\Omega]$$

- If I = 10 [A],

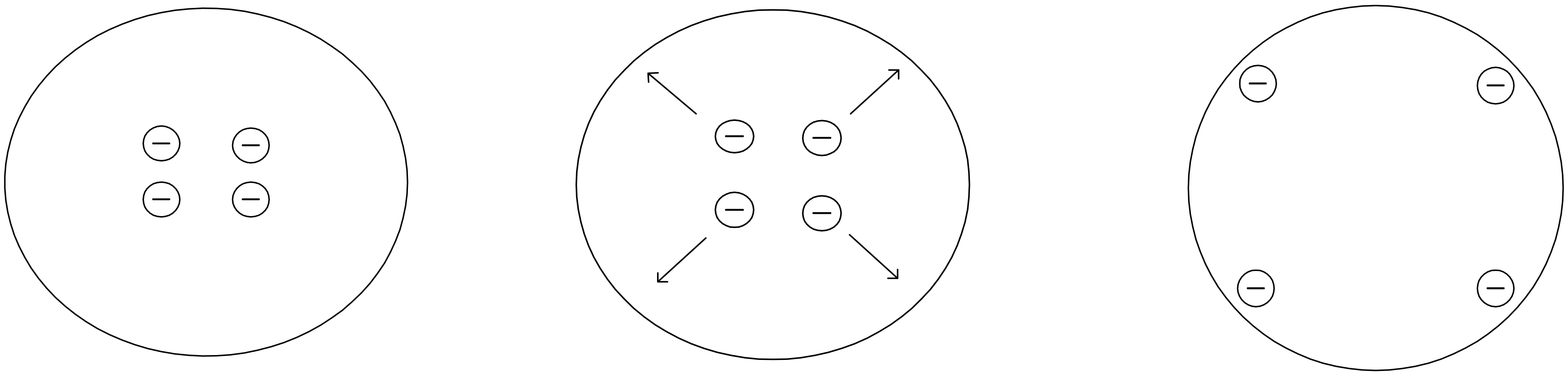
$$J = \frac{I}{S} = \frac{10}{1.308 \times 10^{-6}} = 7.65 \times 10^{6} [\mathrm{A/m}^2] = 7.65 [\mathrm{A/mm}^2]$$

$$V = IR = 10 \times 21.2 = 212 \ [\mathrm{V}], \ E = V / L = 0.132 \ [\mathrm{V/m}]$$

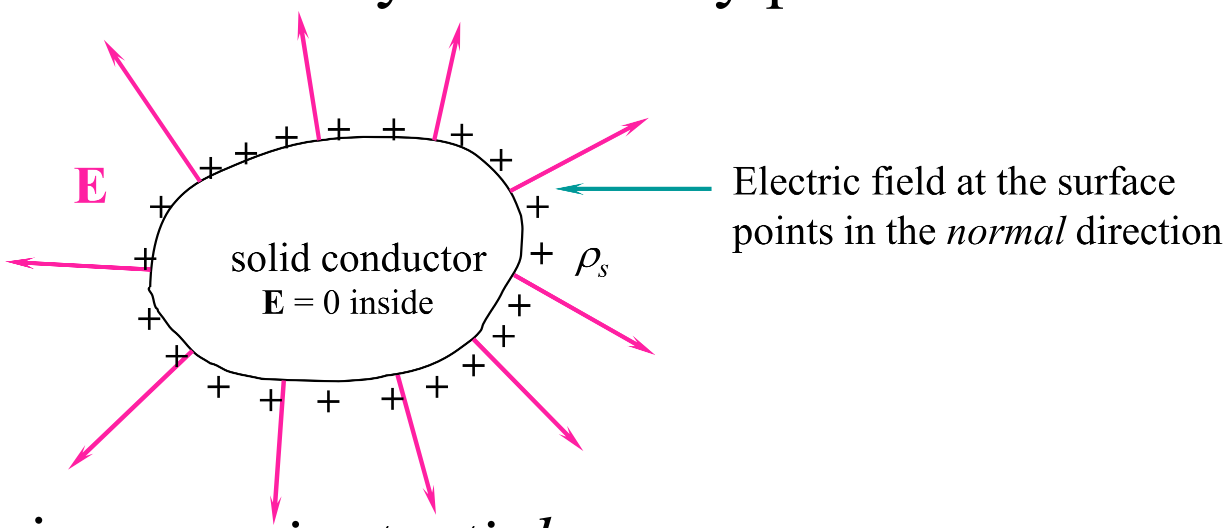
5.4 Conductor properties and boundary conditions

▪ Characteristics of a good conductor

1) Charge can exist only on the surface as a surface charge density, ρ_s.



2) In static condition, no electric field may exist at any point within a conducting material.



3) The surface of a conductor is an *equipotential*.

- Relationship between external fields and the charge on the surface of the conductor

- External electric field intensity

  1. Tangential component to the conductor surface

  $$E_t \vec{a}_t = 0$$

  (만약 '0'이 아니라면 $\vec{E}$- field의 접선성분이 표면전하에 영향을 미쳐 움직임을 일으키고 non-static이 됨)

  2. Normal component: $D_n = \rho_S$ by Gauss's law

  (표면의 미소면적을 통과하는 $\vec{D}$ 는 미소표면에 존재하는 전하량과 동일)

Proof 1] Tangential Electric Field Boundary Condition

- $\vec{D} = 0 = \vec{E}$ at inside of conductor

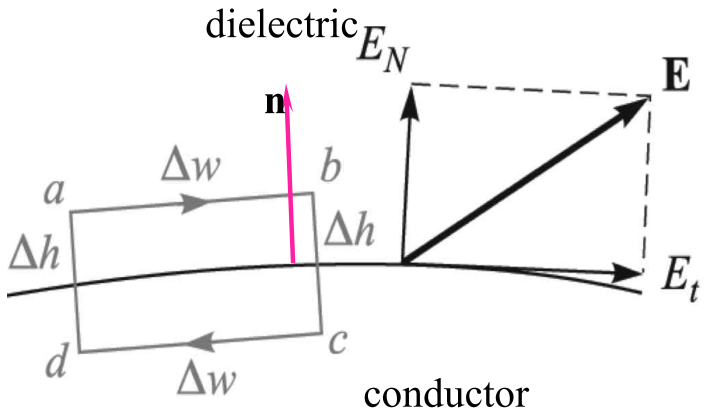
$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

(close path 주변으로 전하를 한 바퀴 이동시키는데 일이 필요 없음)

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0 \leftarrow \Delta w = \overline{ab} = \overline{cd}, \quad \Delta h = \overline{bc} = \overline{ad}$$

$$E_t \Delta w \underbrace{- E_{N,\text{at}\, b} \frac{1}{2} \Delta h + E_{N,\text{at}\, a} \frac{1}{2} \Delta h}_{} = 0$$

These become negligible as Δh approaches zero.



- As $\Delta h \rightarrow 0$, $E_t \Delta w = 0$ ➔ $\boxed{E_t = 0}$ in free space

- More formally: $\boxed{\mathbf{E} \times \mathbf{n}\big|_s = 0}$

Proof 2] Boundary Condition for the Normal Component of **D**

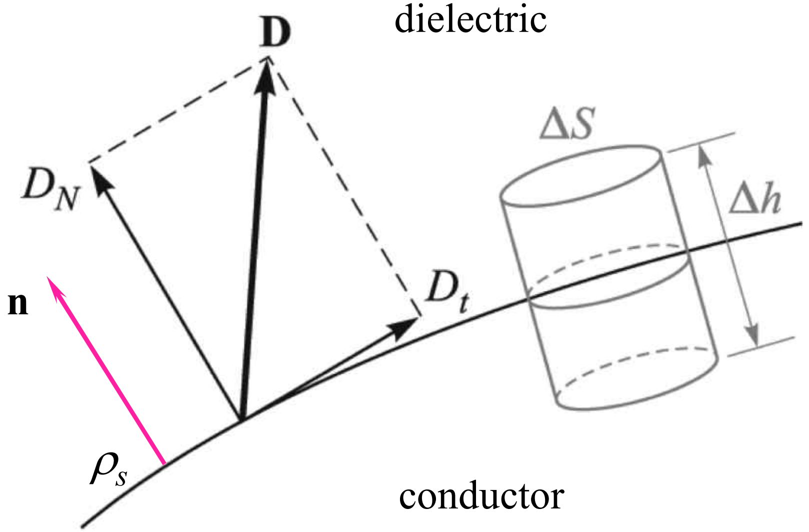
- Gauss' Law is applied to the cylindrical surface shown below:

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_{\text{top}} + \cancel{\int_{\text{bottom}}} + \cancel{\int_{\text{sides}}} = Q$$

$$\rightarrow \int_{bottom} = 0 \;\left(\because\; \vec{D}_{inside} = 0\right),\;\; \int_{side} = 0 \;\left(\because\; \vec{D}_t = 0 \;\&\Delta h \rightarrow 0\right)$$

$$\therefore\; D_N \Delta S = Q = \rho_S \Delta S$$

$$\therefore\; \boxed{D_N = \rho_S}$$



- More formally:

$$\boxed{\mathbf{D}\cdot\mathbf{n}\Big|_S = \rho_s}$$

Summary

- Desired boundary conditions for the conductor–free space boundary in electrostatics

1. The static electric field intensity inside a conductor is zero. $\rightarrow \boldsymbol{D_t = E_t = 0}$
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface. $\rightarrow \boldsymbol{D_N = \varepsilon_0 E_N = \rho_S}$
3. The conductor surface is an equipotential surface. $\rightarrow \boldsymbol{E_t = 0 = -\nabla V}$

- At the surface:

$$\mathbf{E} \times \mathbf{n}\big|_s = 0$$: Tangential E is zero

$$\mathbf{D} \cdot \mathbf{n}\big|_s = \rho_s$$: Normal D is equal to the surface charge density

[Ex.] ▪ $V = 100(x^2 - y^2)$

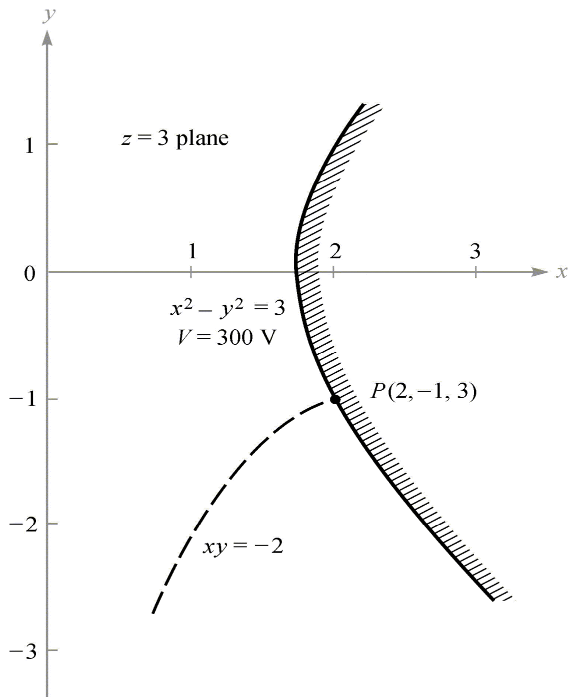
▪ P (2, -1, 3) : in boundary between conductor and free space

➔ $V_P = 100 \times (2^2 - 1^2) = 300$ [V]

▪ Conductor surface is an equipotential plane of $V_P = 300$ [V].

➔ $300 = 100(x^2 - y^2)$, $x^2 - y^2 = 3$: equipotential equation

$$\vec{E} = -\nabla V = -100\nabla(x^2 - y^2) = -200x\vec{a}_x + 200y\vec{a}_y$$

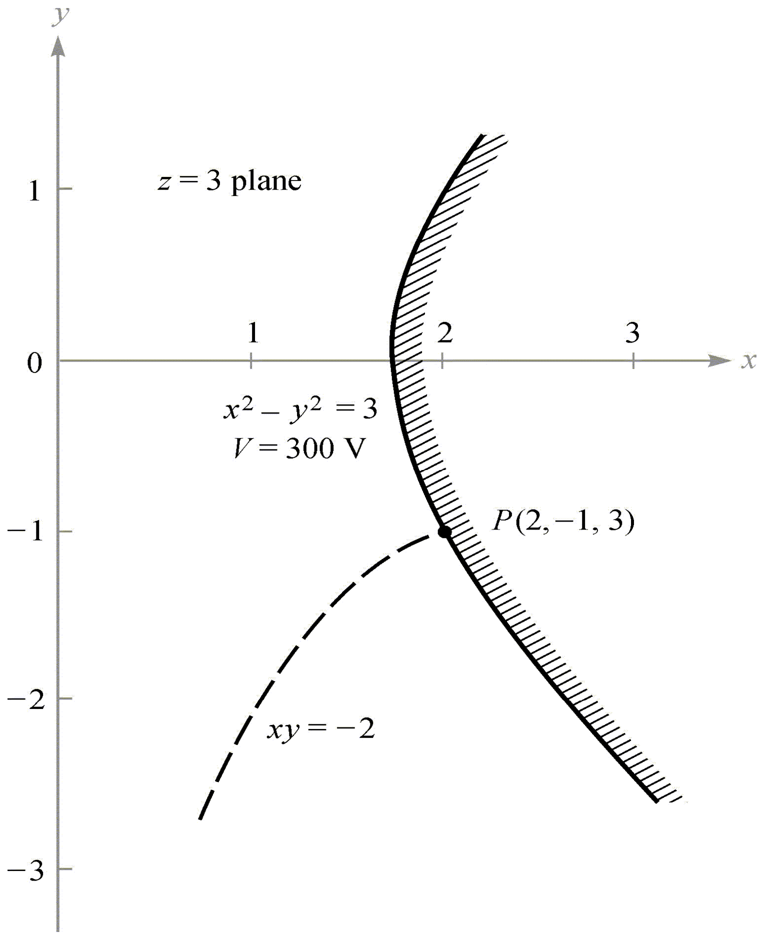


$$\vec{E}_p = -400\vec{a}_x - 200\vec{a}_y$$

$$\vec{D}_P = \varepsilon_0 \vec{E}_P = 8.854 \times 10^{-12}\,\vec{E}_P$$

$$= -3.54\vec{a}_x - 1.771\vec{a}_y \,[\mathrm{nC/m^2}]$$

$$\rho_{s.p} = D_N = 3.96 \ \text{[nC/m}^2\text{]} \ : \text{surface charge density}$$



$$\frac{E_y}{E_x} = \frac{200y}{-200x} = -\frac{y}{x} = \frac{dy}{dx}$$

$$\frac{dy}{y} + \frac{dx}{x} = 0$$

$$\ln y + \ln x = C_1$$

$$\therefore xy = C_2 \qquad \leftarrow P\,(2, \text{-}1, 3)$$

$$C_2 = 2 \times (-1) = -2$$

$\therefore \quad xy = \text{-}2$ $\quad$ ($\vec{E}$ -field의 방향)

5.5 Method of Images

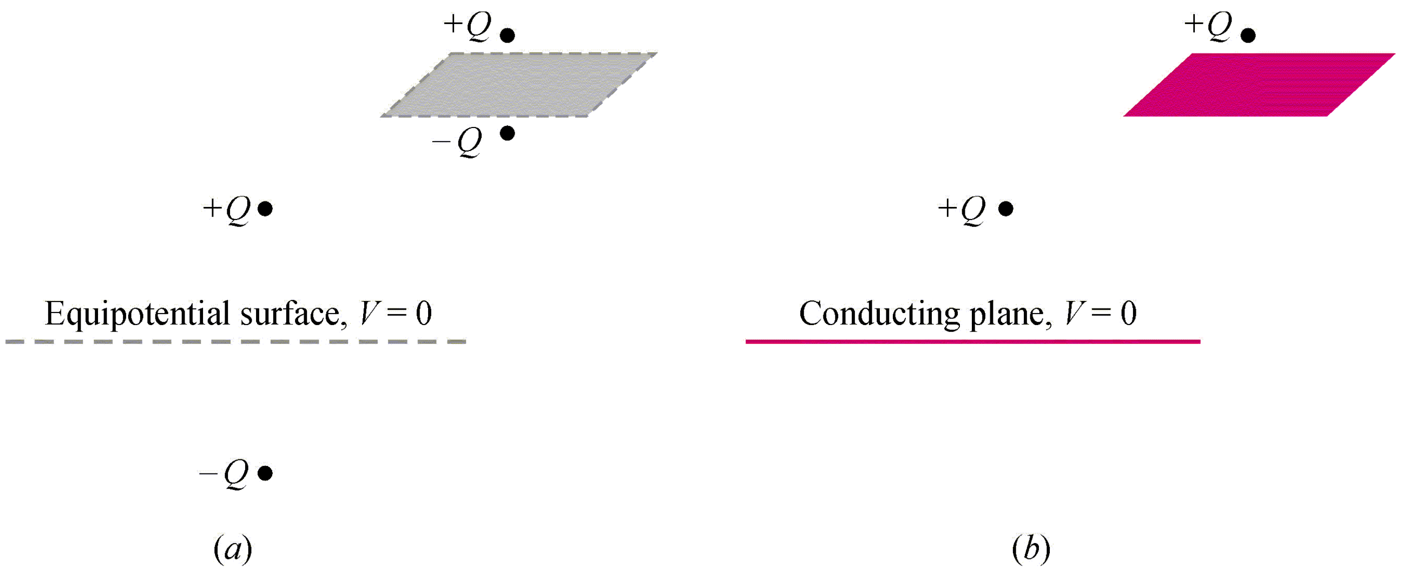


Figure 5.6 (a) Two equal but opposite charges may be replaced by (b) a single charge and a conducting plane without affecting the fields above the $V = 0$ surface.

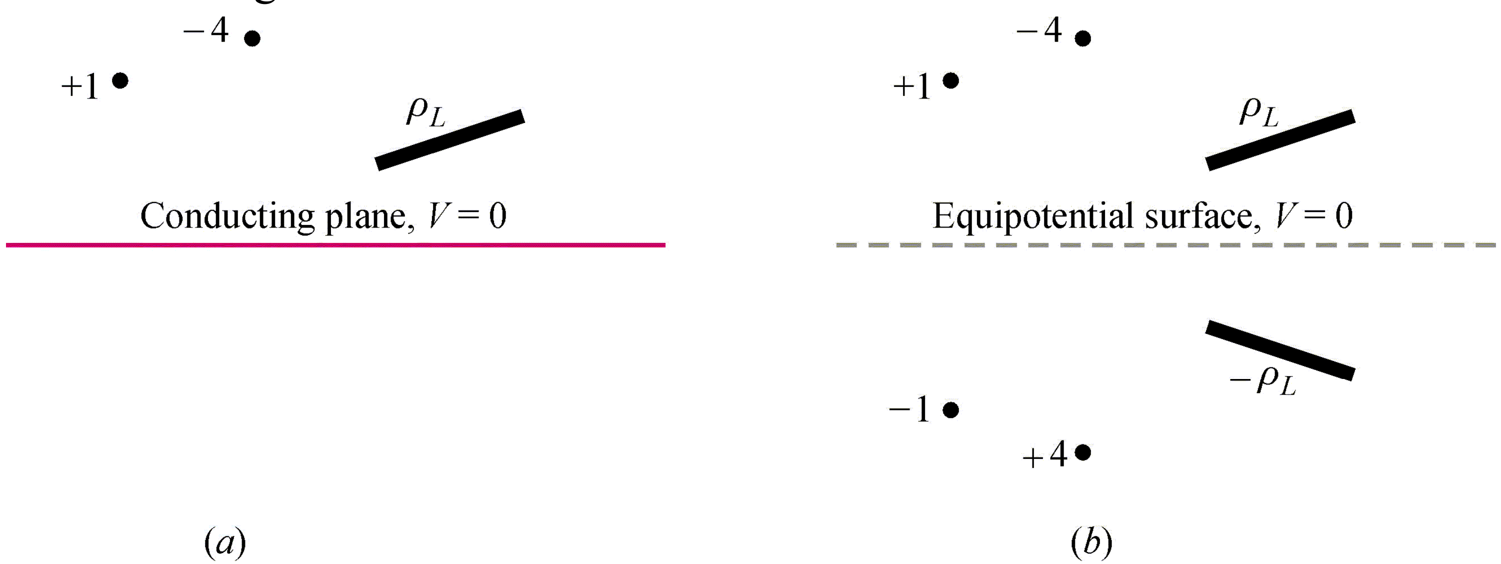
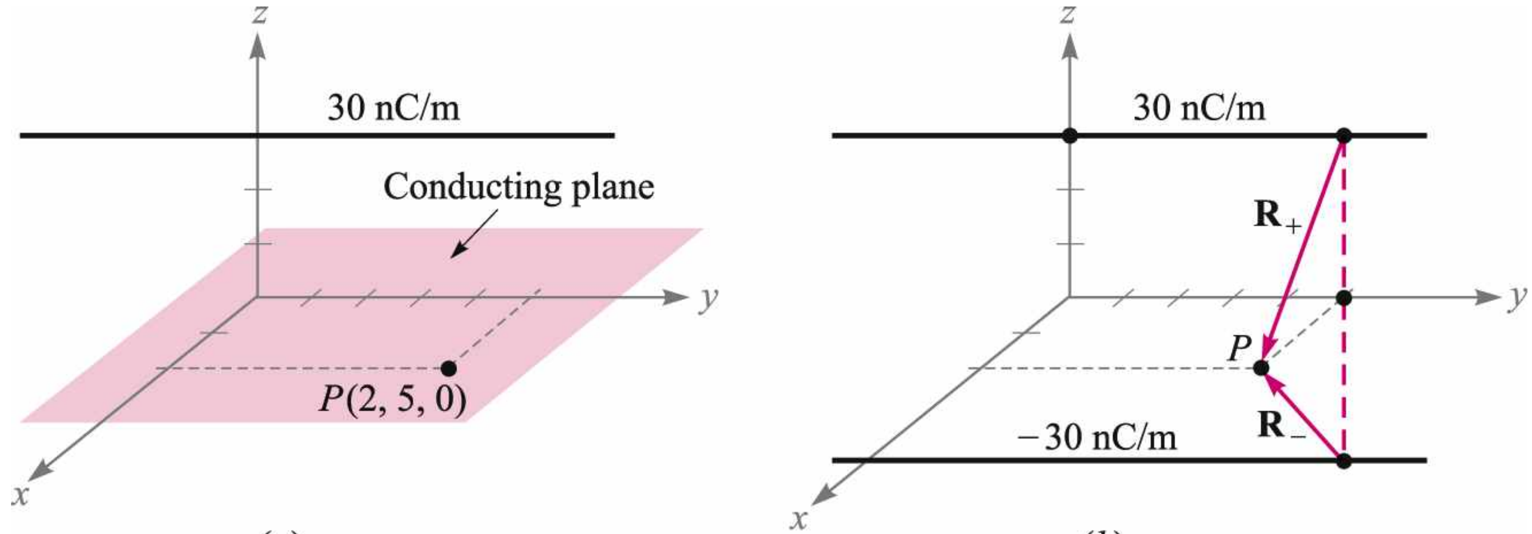


Figure 5.7 (a) A given configuration above an infinite conducting plane may be replaced by (b) the given charge configuration plus the image configuration, without the conducting plane.

[Ex.]



(a) (b)

$$\mathbf{R}_+ = 2\mathbf{a}_x - 3\mathbf{a}_z \qquad \mathbf{R}_- = 2\mathbf{a}_x + 3\mathbf{a}_z$$

$$\therefore\ \mathbf{E}_+ = \frac{\rho_L}{2\pi\epsilon_0 R_+}\mathbf{a}_{R+} = \frac{30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x - 3\mathbf{a}_z}{\sqrt{13}}$$

$$\mathbf{E}_- = \frac{-30\times 10^{-9}}{2\pi\epsilon_0\sqrt{13}}\,\frac{2\mathbf{a}_x + 3\mathbf{a}_z}{\sqrt{13}}$$

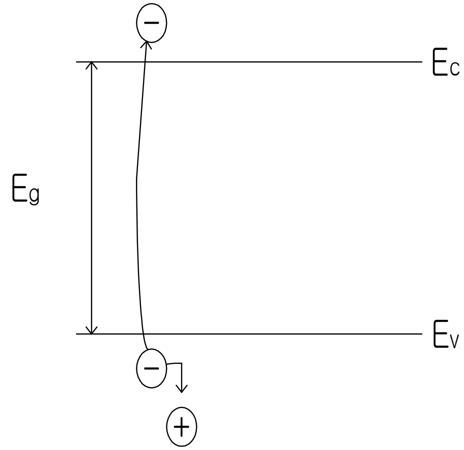
$$\mathbf{E} = \frac{-180\times 10^{-9}\mathbf{a}_z}{2\pi\epsilon_0(13)} = -249\mathbf{a}_z \text{ V/m}$$

$$\vec{D} = \varepsilon_0\vec{E} = -2.2\vec{a}_z \ \ [\text{nC/m}^2]$$

$$\therefore\ \rho_s = -2.2 \ \ [\text{nC/m}^2] \ \ \text{at } P$$

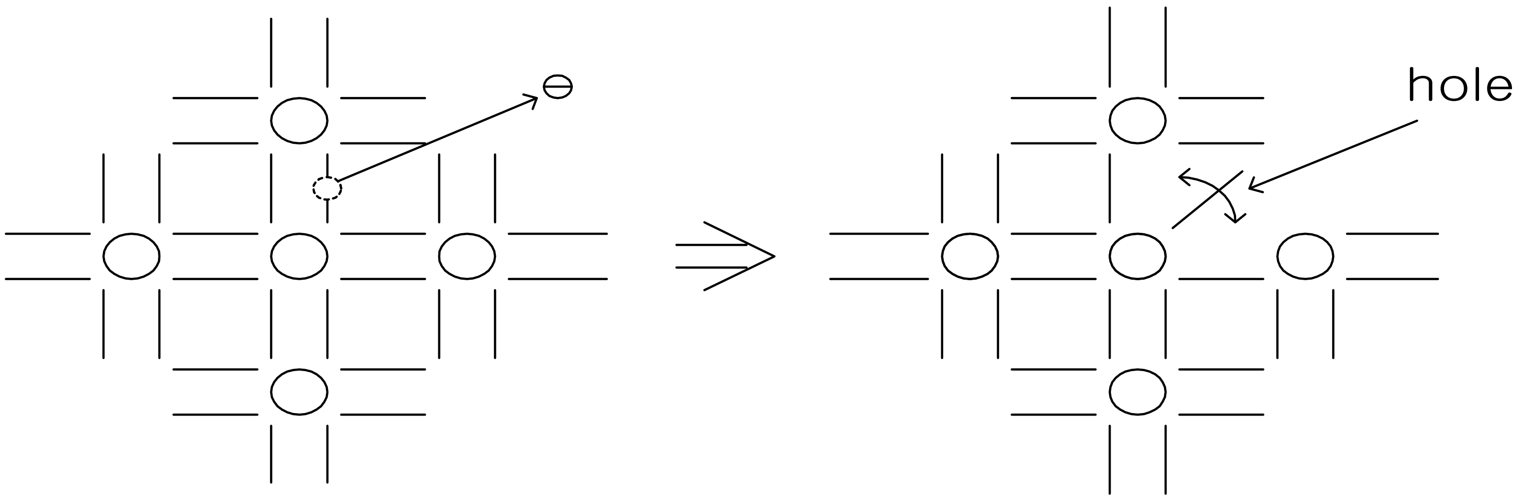
5.6 Semiconductors

- Valanced band에 있는 전자가 외부로부터 충분한 에너지를 받아 Conduction band로 올라가 자유롭게 움직이는 전자(-*e*)가 되고, 자가 천이함으로써 빈자리가 된 것을 hole(+*e*)라고 함.

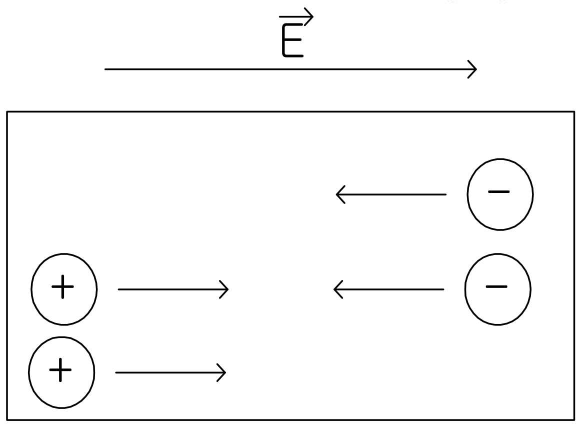


Mobility: μ_h
Mass: 전자와 거의 동일

- Bonding model of intrinsic semiconductor



- Both carriers (holes and electrons) move in an electric field ($\vec{E}$), and they move in opposite directions.



$$I = -\frac{dQ_i}{dt}$$

➔ 전류를 양전하의 감소로 보면, 전류의 방향은 hole의 방향과 동일 (전자의 방향과는 반대, 음전하의 증가이므로)

- When both carriers contribute a component of the total current,

$$\sigma = -\rho_e \mu_e + -\rho_h \mu_h$$

Conductivity | due to electron | due to hole

Ex.] Intrinsic Si : $\mu_e = 0.12$, $\mu_h = 0.025$ [$m^2/V\cdot sec$] ($\rightarrow \mu_e > \mu_h$)

Ge : $\mu_e = 0.36$, $\mu_h = 0.17$ [$m^2/V\cdot sec$]

Si : $-\rho_e = \rho_h = 0.0024$ [C/m^2] and Ge : $-\rho_e = \rho_h = 3.0$ @ 300^oK

$\sigma_{Si} = -(-0.0024) \times 0.12 + 0.024 \times 0.025 = 3.48 \times 10^{-4}$ [mho/m]

$\sigma_{Ge} = 3.0 \times (0.36 + 0.17) = 1.59$ [mho/m]

- Extrinsic semiconductor

Donor (electron increasing)

IV + V = VIII + I

IV + III = VIII - I

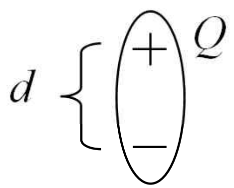
Acceptor (hole increasing)

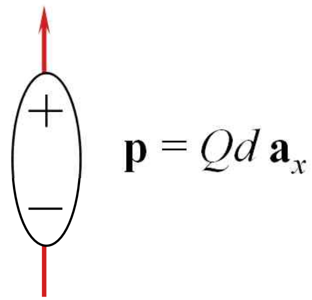
5.7 The Nature of Dielectric Materials
Electric Dipole and Dipole Moment

- In dielectric, charges are held in position (bound charges, and ideally cannot move as not like free charges) and form a current.
- Atoms and molecules may be polar (having separated positive and negative charges), and be polarized by the external electric field.
- From equation (36) in sec. 4.7

$$\vec{p} = Q\vec{d}$$

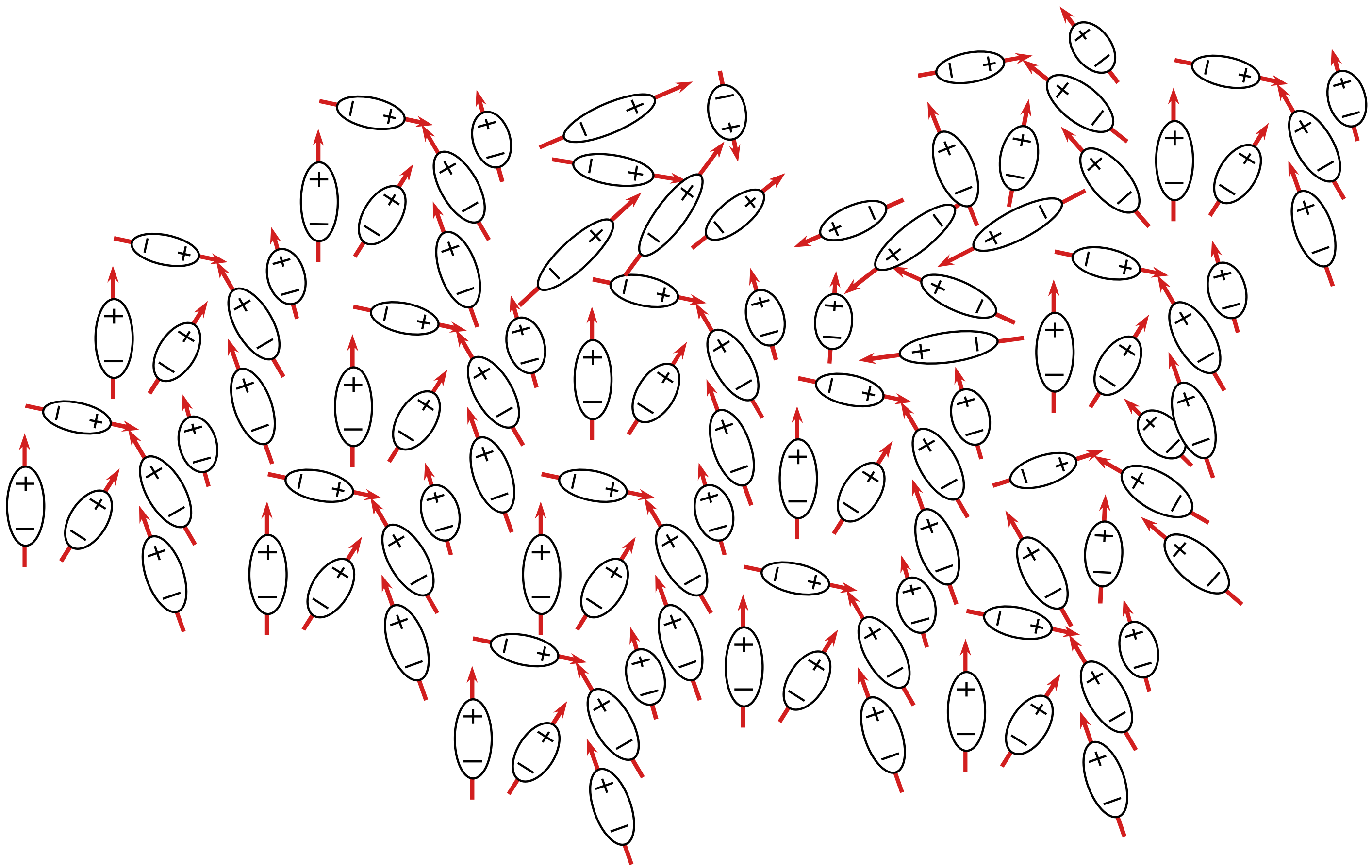
where $\vec{p}$: dipole momentum
Q : positive charge of two bound charges composing dipole
$\vec{d}$: vector from negative to positive charges





Model of a Dielectric

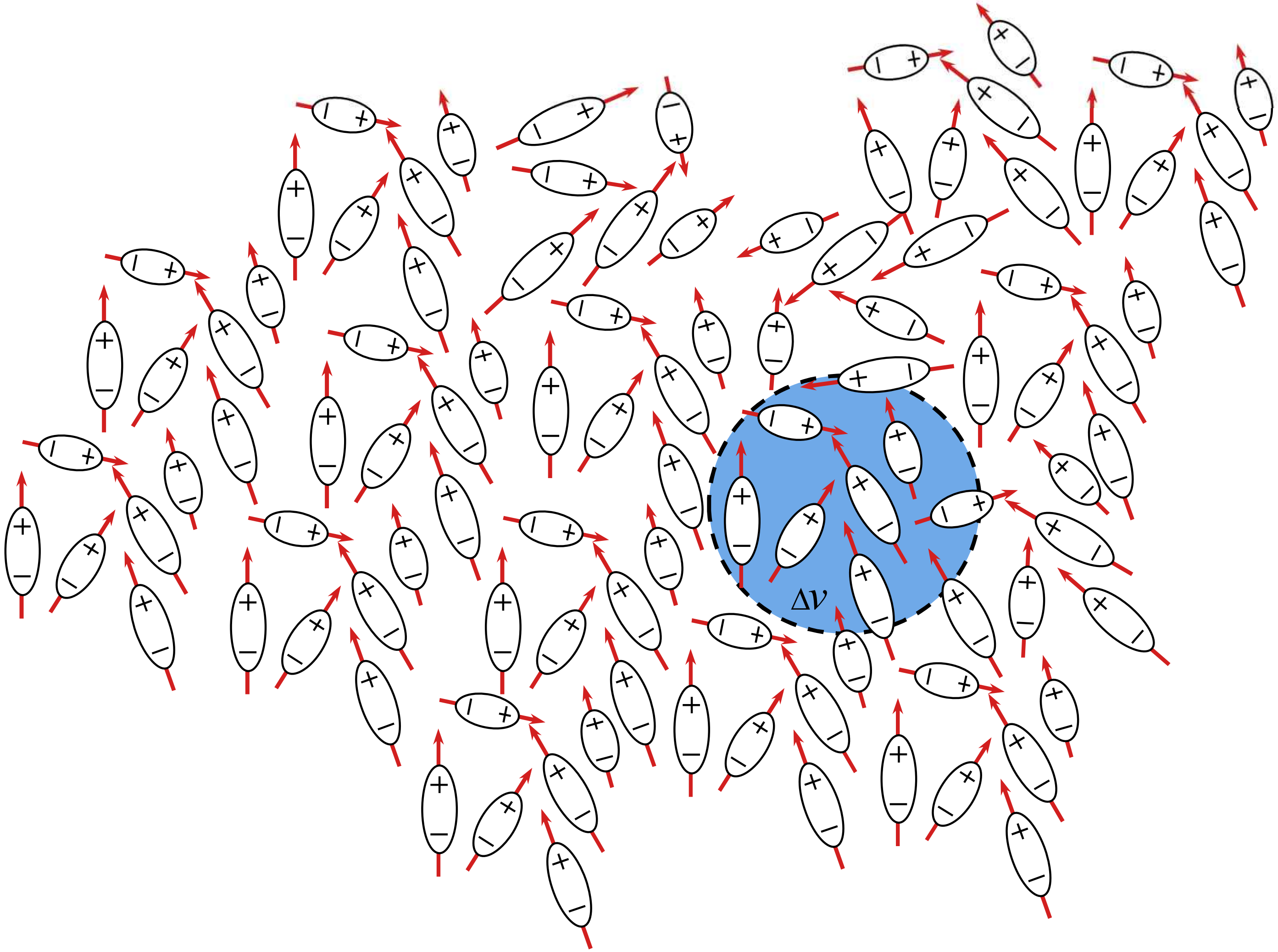
- A dielectric can be modeled as an ensemble of bound charges in free space, associated with the atoms and molecules that make up the material.
- Some of these may have intrinsic dipole moments, others not. In some materials (such as liquids), dipole moments are in **random directions**.



- If there are n dipoles per unit volume and we deal with a volume Δv, then there are $n\Delta v$ dipoles. Total dipole moment is below.

$$\vec{\mathrm{p}}_{\mathrm{total}} = \sum_{i=1}^{n\Delta v} \vec{\mathrm{p}}_i$$ ← 전체 volume을 Δv의 합산으로 가정

$$= 0 \quad @ \text{ without } \vec{E}\text{ - field} \qquad \text{(random align)}$$



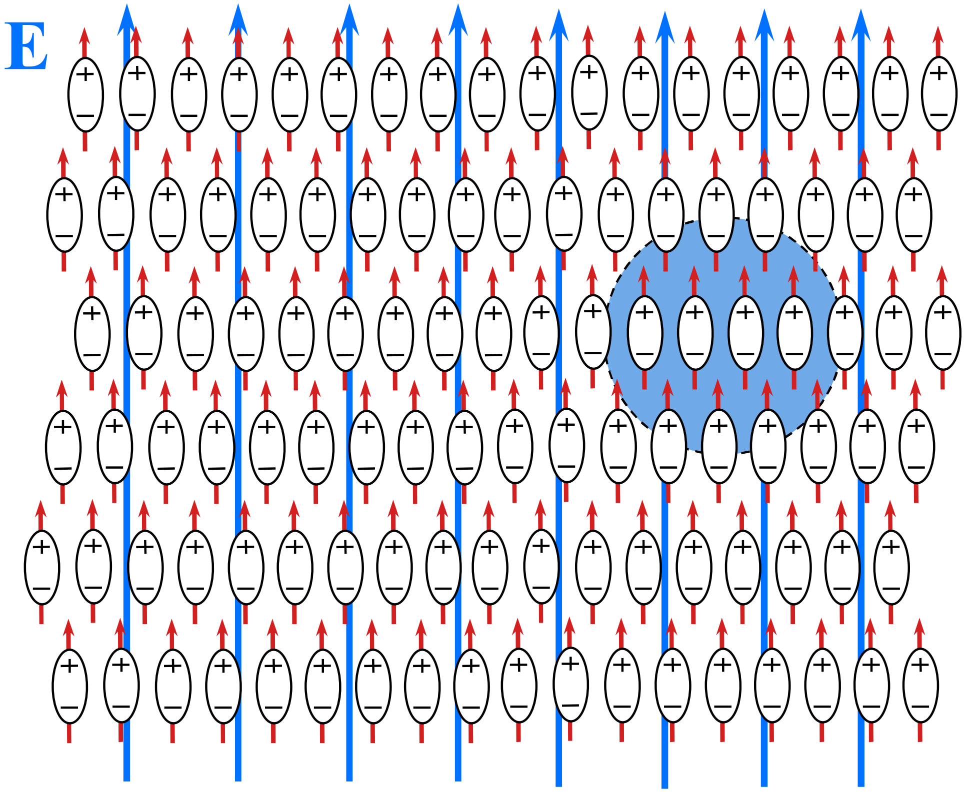
- *Polarization* $\vec{P}$: dipole moment per unit volume

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

$[\mathrm{C/m^3}]$

Polarization Field (with Electric Field Applied)

- By introducing an electric field, the charge separation in each dipole possibly re-orient dipoles so that there is some aggregate alignment, as shown here.



- The effect is to increase **P**.

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\,\Delta v} \mathbf{p}_i$$

$$= n\mathbf{p}$$

← Assumption: all dipoles are identical

- **Our immediate goal is to show that the bound volume charge (density) acts like the free volume charge density in producing an additional external field. ➔ Similar to Gauss's law**

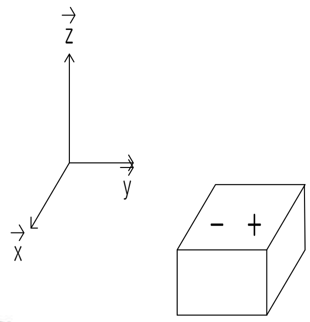
Migration of Bound Charge

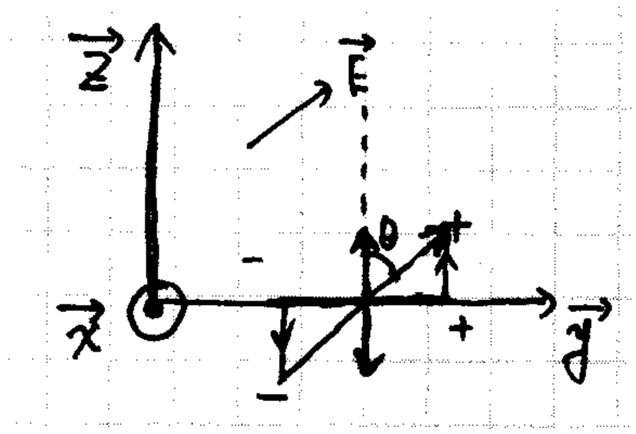
- Assume that a dielectric contains nonpolar molecules before applying $\vec{E}$ - filed.

$$\vec{\mathrm{P}} = 0$$

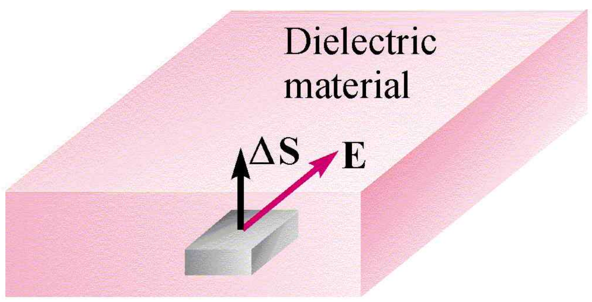
- After applying $\vec{E}$ - filed.

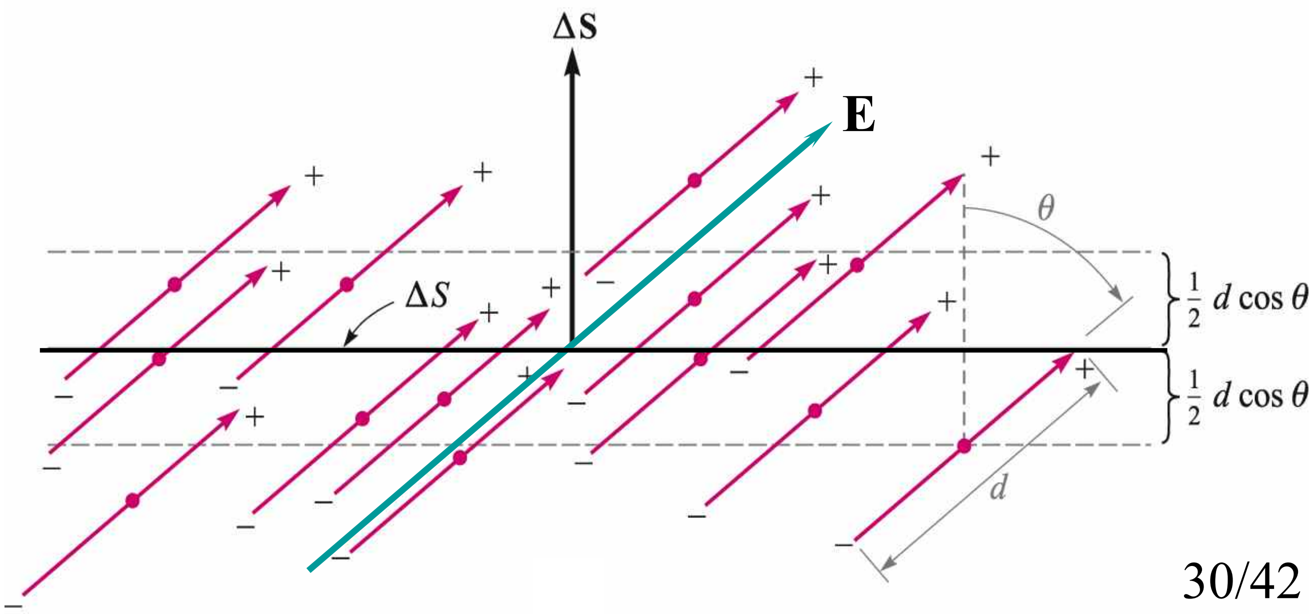
$$\vec{p} = Q\vec{d} \quad : \text{Dipole momentums}$$





Dipole on the surface (red dots) will transfer charge across the surface about (1/2) $d\cos\theta$ above or below.





- Net total bound charge that crosses the elementary surface in an upward direction:

$$\Delta Q_b = nQ\underbrace{d\cos\theta\Delta S}_{\text{volume}} = nQ\mathbf{d}\cdot\Delta\mathbf{S} = \mathbf{P}\cdot\Delta\mathbf{S} \qquad \leftarrow n \text{ [molecules/m}^3\text{]}$$

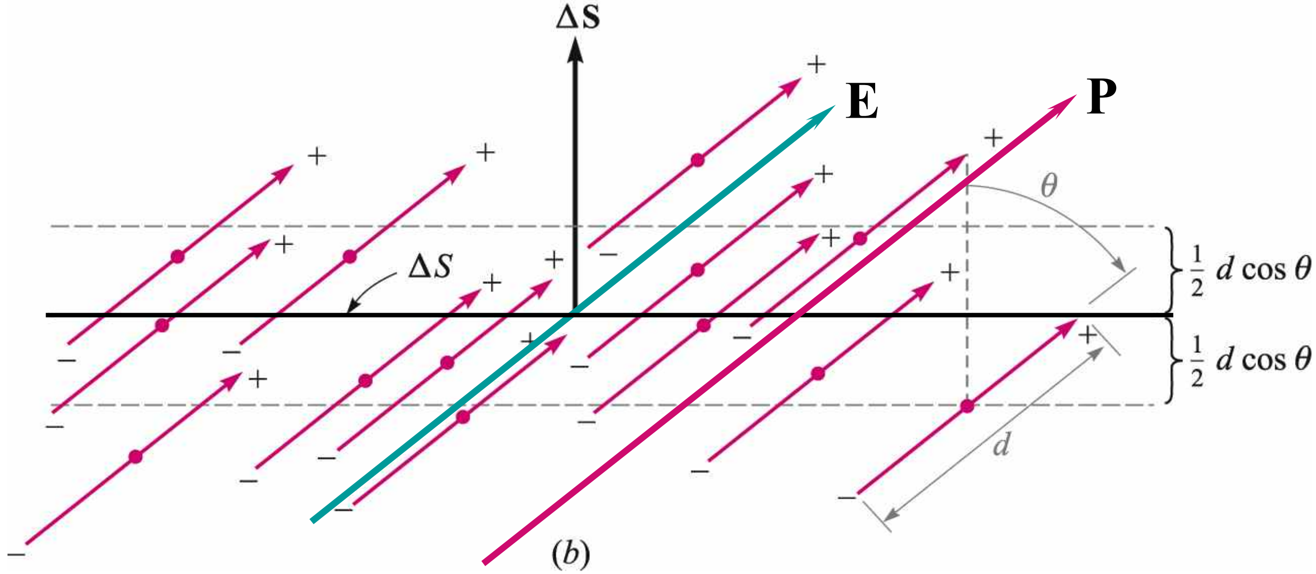
Bound charge
Not free charge

Dipole momentum이
단면적 ΔS의 normal
방향으로 미치는 성분

Polarization (∵ 단위
volume에 n개의 dipole이
있다고 가정하였으므로)

where ΔS: an element of closed surface inside dielectric material

direction of ΔS: outward

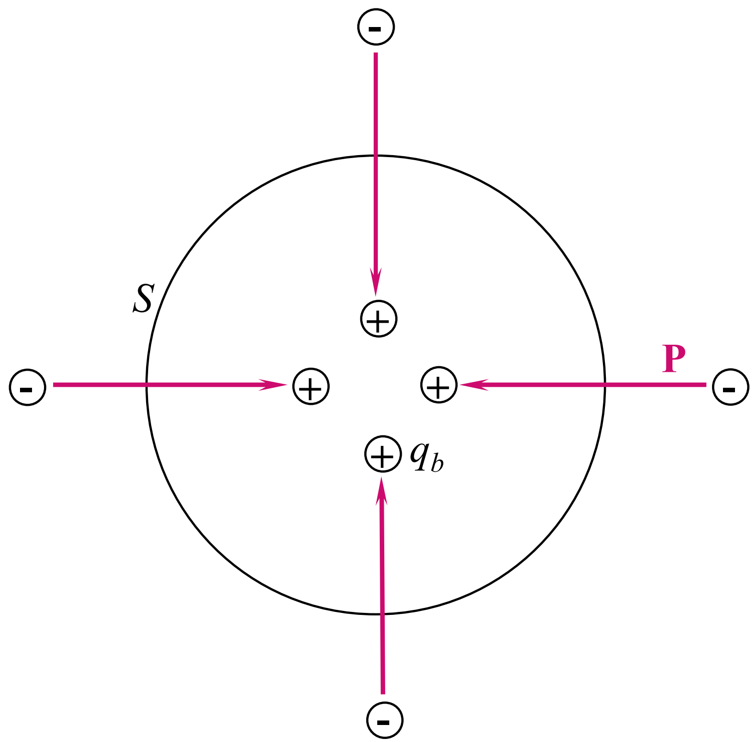


(b)

Polarization Flux Through a Closed Surface

- The accumulation of positive bound charge within a closed surface means that the polarization vector must be pointing *inward*.

$$Q_b = -\oint_S \mathbf{P} \cdot d\mathbf{S}$$



- Total enclosed charge:

$$Q_T = \oint_S \vec{D} \cdot d\vec{S} \quad \leftarrow \text{Gauss's Law}$$

$$= \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S}$$

$$= Q_b + Q$$

bound charge free charge

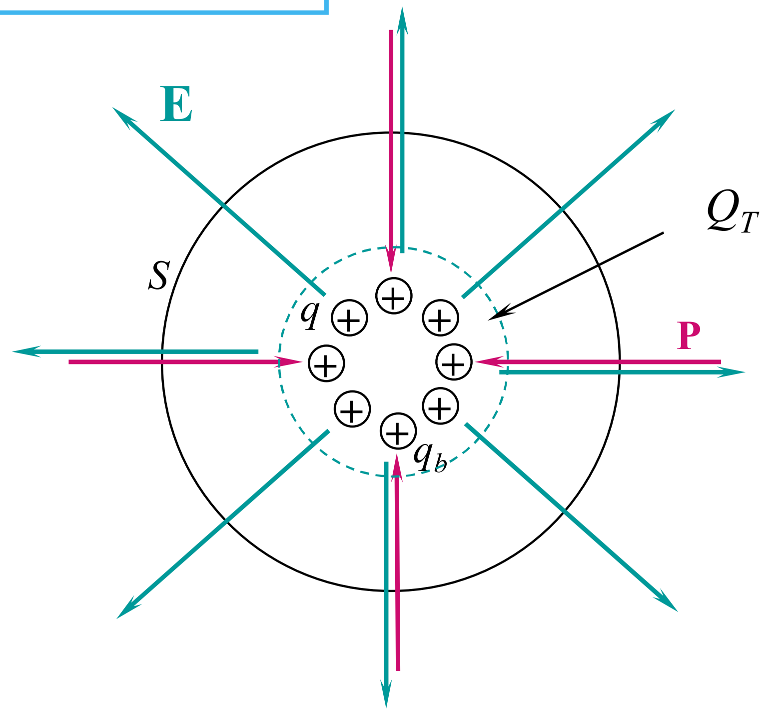
where Q: total free charge enclosed by surface

Bound and Free Charge

- Now consider the charge within the closed surface consisting of bound charges, q_b, *and* free charges, q.
- Free charge: $\boxed{Q = Q_T - Q_b = \oint_S (\epsilon_0 \mathbf{E} + \mathbf{P}) \cdot d\mathbf{S}}$
- Let define $\vec{D}$ in more general form

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}}$$

$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$: free charge enclsed



▪ Several volume charge densities with divergence theorem

Bound Charge: $$Q_b = \int_v \rho_b \, dv = -\oint_S \mathbf{P} \cdot d\mathbf{S} = \int -(\nabla \cdot \vec{P}) dv \longrightarrow \boxed{\nabla \cdot \mathbf{P} = -\rho_b}$$

Total Charge: $$Q_T = \int_v \rho_T \, dv = \oint_S \epsilon_0 \mathbf{E} \cdot d\mathbf{S} = \int (\nabla \cdot \varepsilon_0 \vec{E}) dv \longrightarrow \boxed{\nabla \cdot \epsilon_0 \mathbf{E} = \rho_T}$$

Free Charge: $$Q = \int_v \rho_v \, dv = \oint_S \mathbf{D} \cdot d\mathbf{S} = \int (\nabla \cdot \vec{D}) dv \longrightarrow \boxed{\nabla \cdot \mathbf{D} = \rho_v}$$

Electric Susceptibility and the Dielectric Constant

- A stronger electric field results in a larger polarization in the medium.
- Relation between **P** and **E** in a *linear* medium (isotropic material)

$$\boxed{\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}} \qquad \rightarrow \vec{P} \; // \; \vec{E}$$

where χ_e [chi] : electric *susceptibility* of material

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \chi_e \epsilon_0 \mathbf{E} = (\chi_e + 1)\epsilon_0 \mathbf{E}$$

- Let $\boxed{\epsilon_r = \chi_e + 1}$: *relative permittivity* or *dielectric constant*

$$\vec{D} = \varepsilon_0 \varepsilon_r \vec{E} = \varepsilon \vec{E}$$ where $\varepsilon = \varepsilon_0 \varepsilon_r$: permittivity

Isotropic vs. Anisotropic Materials

- The dielectric constant of *Anisotropic* materials will vary as the electric field is rotated in certain directions.
  ➔ dielectric *tensor*.

$$\vec{D} = \varepsilon\vec{E}$$

$$\begin{bmatrix} D_x \\ D_y \\ D_z \end{bmatrix} = \begin{bmatrix} \varepsilon_{xx} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{yx} & \varepsilon_{yy} & \varepsilon_{yz} \\ \varepsilon_{zx} & \varepsilon_{zy} & \varepsilon_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix},$$

$$D_x = \epsilon_{xx}E_x + \epsilon_{xy}E_y + \epsilon_{xz}E_z$$
$$D_y = \epsilon_{yx}E_x + \epsilon_{yy}E_y + \epsilon_{yz}E_z$$
$$D_z = \epsilon_{zx}E_x + \epsilon_{zy}E_y + \epsilon_{zz}E_z$$

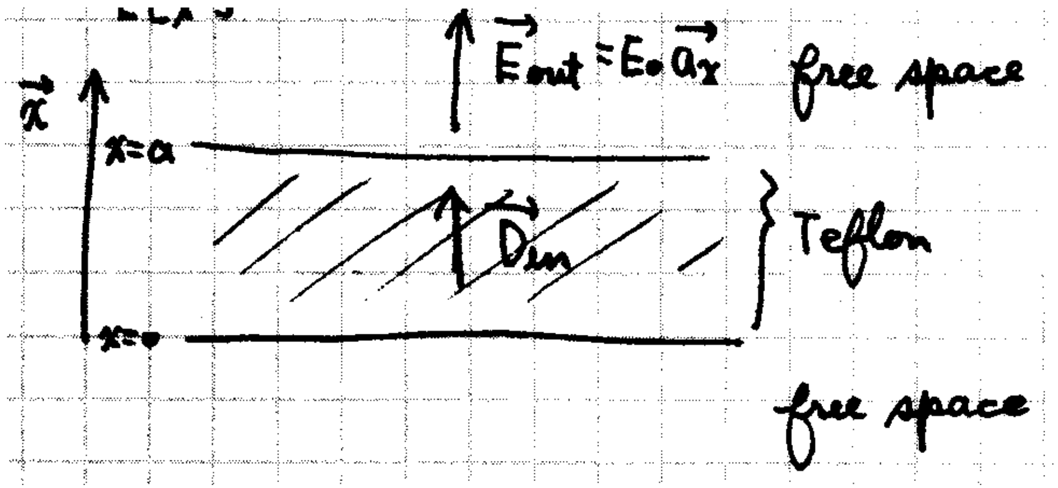
$\vec{D} = \varepsilon_0\vec{E} + \vec{P}$: valid for all isotropic materials

- Gauss's law can be applied to the dieletric materials.

$$\nabla \cdot \vec{D} = \rho_v$$

$$\oint \vec{D} \cdot d\vec{S} = Q$$

[Ex.] Teflon slab



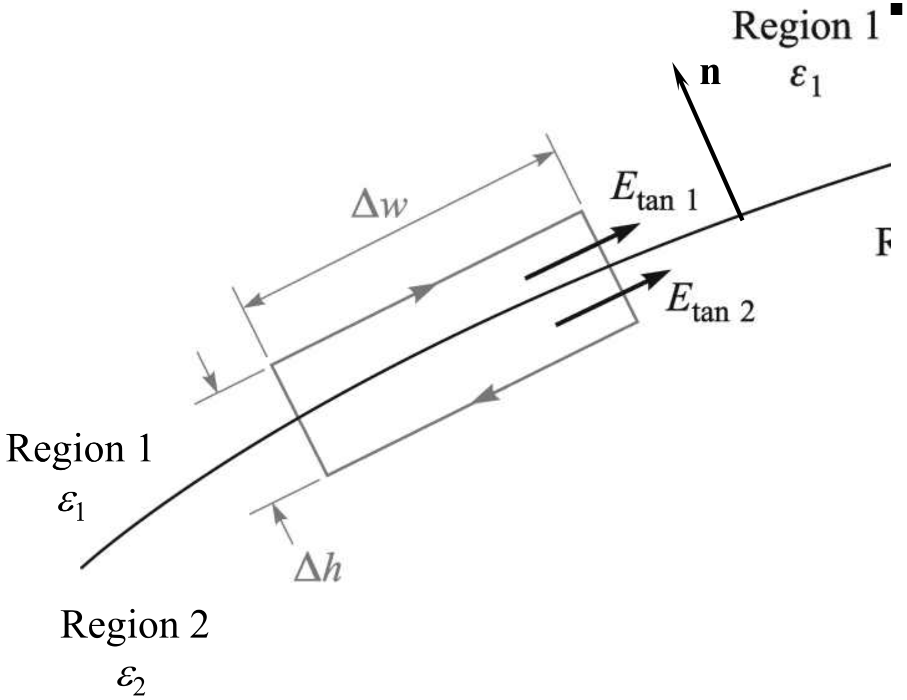
$$\vec{D}_{out} = \varepsilon_0 E_0 \vec{a}_x\ , \quad \vec{P}_{out} = 0 \quad (\because \chi_{\mathrm{out}} = 0)$$

$$\vec{D}_{in} = \varepsilon_r \varepsilon_0 E_0 \vec{a}_x = 2.1\varepsilon_0 E_0 \vec{a}_x = 2.1\varepsilon_0 \vec{E}_{in}$$

$$\vec{P}_{in} = \vec{D}_{in} - \varepsilon_0 \vec{E}_{in} = (2.1-1)\varepsilon_0 \vec{E}_{in} = 1.1\varepsilon_0 \vec{E}_{in}$$

- If we know one of unknown variables ($\vec{E}_{in}$, $\vec{D}_{in}$, $\vec{P}_{in}$), we can know others.

5.8 Boundary Condition for perfect dielectric materials



- Let use the fact that **E** is conservative.

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

$$\int_a^b (\ ) + \int_b^c (\ ) + \int_c^d (\ ) + \int_d^a (\ ) = 0$$

$$E_{\tan 1}\Delta w - E_{nor}\Delta h - E_{\tan 2}\Delta w + E_{nor}\Delta h = 0$$

- As $\Delta h \rightarrow 0$, $E_{\tan 1}\,\Delta w - E_{\tan 2}\,\Delta w = 0$

$$\therefore \quad \boxed{E_{\tan 1} = E_{\tan 2}}$$

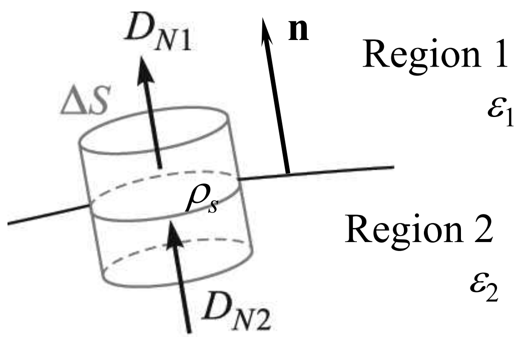
$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

$$\boxed{\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2}}$$

- General vectorial form: $\boxed{(\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{n} = 0}$

▪ Boundary Condition for Normal Electric Flux Density

Gauss' Law to the cylindrical volume which its height approaches zero and charge density on the surface is ρ_s.



$$Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$$

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

$$(\leftarrow (\Delta h \rightarrow 0))$$

$$\boxed{D_{N1} - D_{N2} = \rho_S}$$

General vectorial form: $\boxed{(\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{n} = \rho_s}$

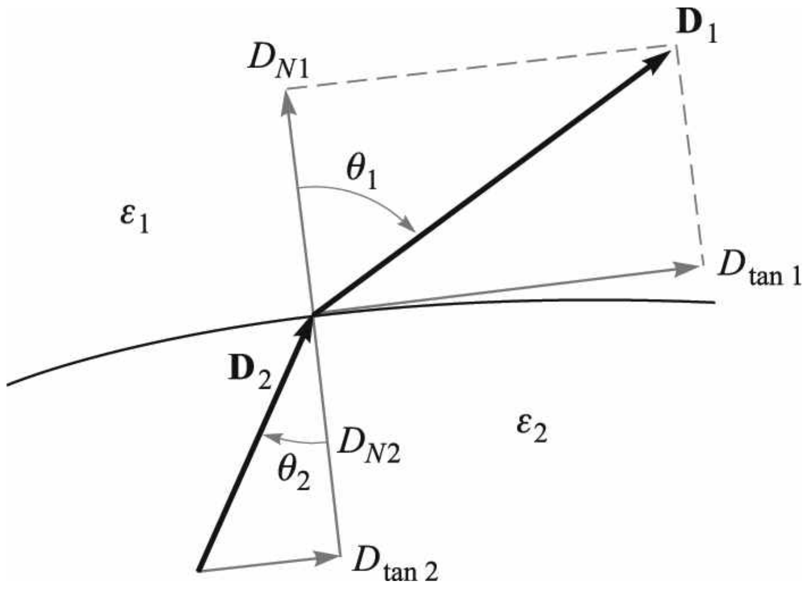
If $\rho_S = 0$, $D_{N1} - D_{N2} = 0$

➔ $\boxed{D_{N1} = D_{N2}}$ and $\varepsilon_1 E_{N1} = \varepsilon_2 E_{N2}$

- Reflection of $\vec{D}$ at a dielectric interface

1) Normal components of **D** will be continuous across the boundary.

$$D_{N1} = D_1 \cos\theta_1 = D_2 \cos\theta_2 = D_{N2}$$



2) Tangential components of **E** will be continuous across the boundary.

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{D_1 \sin\theta_1}{D_2 \sin\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

By (1) $\left(D_1 = D_2 \dfrac{\cos\theta_2}{\cos\theta_1} \right)$,

$$\varepsilon_2 D_1 \sin\theta_1 = \varepsilon_2 D_2 \frac{\cos\theta_2}{\cos\theta_1} \sin\theta_1 = \varepsilon_1 D_2 \sin\theta_2$$

$$\boldsymbol{\frac{\sin\theta_1 / \cos\theta_1}{\sin\theta_2 / \cos\theta_2} = \frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}}$$

→ Reflection and transmission angles are decided by dielectric constants of composing boundary dielectric materials.

- By assumption that $\varepsilon_1 > \varepsilon_2$ and $\theta_1 > \theta_2$,

$$\boldsymbol{D}_2 = \sqrt{(\boldsymbol{D}_{2.\tan})^2 + (\boldsymbol{D}_{2.Nor})^2} = \sqrt{\left(\frac{\boldsymbol{\varepsilon}_2}{\boldsymbol{\varepsilon}_1}\boldsymbol{D}_{1.\tan}\right)^2 + (\boldsymbol{D}_{1.Nor})^2}$$

$$= \sqrt{\left(\frac{\varepsilon_2}{\varepsilon_1} D_1 \sin\theta_1\right)^2 + (D_1 \cos\theta_1)^2} = D_1\sqrt{\cos^2\theta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \sin^2\theta_1} < D_1$$

$$\varepsilon_2 E_2 = \varepsilon_1 E_1 \sqrt{\cos^2\theta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \sin^2\theta_1}$$

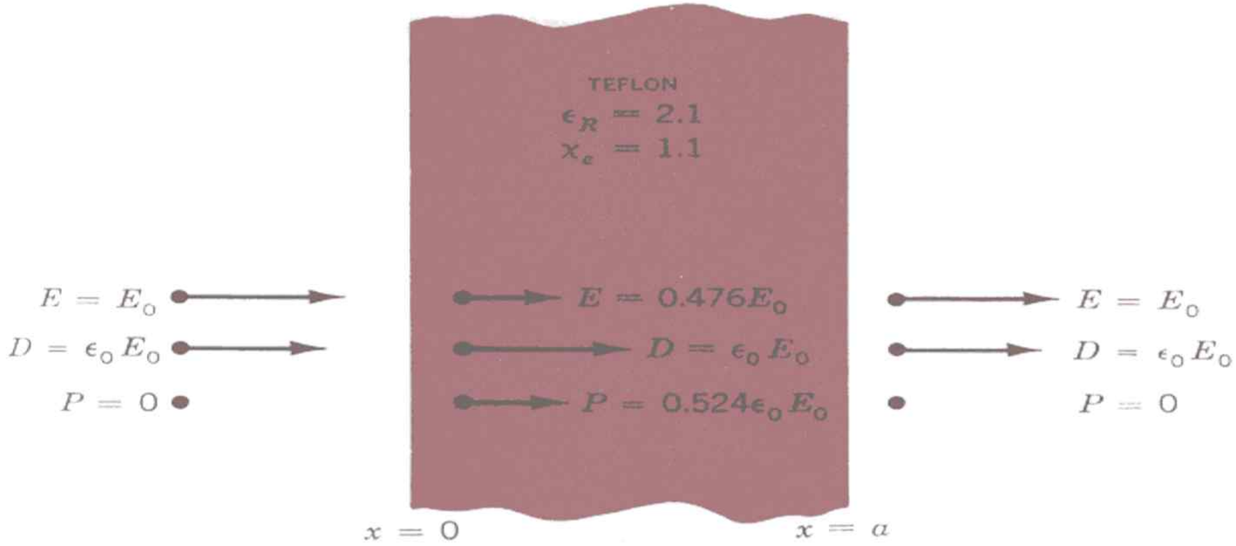
$$E_2 = \frac{\varepsilon_1}{\varepsilon_2} E_1 \sqrt{\cos^2\theta_1 + \left(\frac{\varepsilon_2}{\varepsilon_1}\right)^2 \sin^2\theta_1}$$

$$= E_1 \sqrt{\left(\frac{\varepsilon_1}{\varepsilon_2}\right)^2 \cos^2\theta_1 + \sin^2\theta_1} > E_1$$

(If electric field intensity on one side of dielectric materials boundary is known, then we can know the electric field intensity on other side.)

- If $\varepsilon_1 < \varepsilon_2$, then $\vec{D}_2 > \vec{D}_1$ and $\vec{E}_2 < \vec{E}_1$

[Ex.]



$$D_{\text{in}} = D_{\text{out}} = \varepsilon_0 E_0 \vec{a}_x \quad \left(D_{N1} = D_{N2}\right)$$

$$\vec{E}_{in} = \frac{\vec{D}_{in}}{\varepsilon} = \frac{\varepsilon_0 E_0 \vec{a}_x}{\varepsilon_r \varepsilon_0} = 0.476 E_0 \vec{a}_x \qquad \vec{D} = \varepsilon_0 \vec{E} + \vec{P}$$

$$\vec{P}_{\text{in}} = \vec{D}_{\text{in}} - \varepsilon_0 \vec{E}_{\text{in}} = \vec{D}_{\text{out}} - \varepsilon_0 \vec{E}_{\text{in}} = \varepsilon_0 E_0 \vec{a}_x - 0.476 \varepsilon_0 E_0 \vec{a}_x = 0.524 \varepsilon_0 E_0 \vec{a}_x$$

$$\therefore \begin{cases} \vec{D}_{\text{in}} = \varepsilon_0 E_0 \vec{a}_x & \left(0 \le x \le a\right) \\ \vec{E}_{\text{in}} = 0.476 E_0 \vec{a}_x & \left(0 \le x \le a\right) \\ \vec{P}_{\text{in}} = 0.524 \varepsilon_0 E_0 \vec{a}_x & \left(0 \le x \le a\right) \end{cases}$$

(If we know $\vec{E}$ or $\vec{D}$ in one side of dielectric materials boundary, then we can know $\vec{E}$ or $\vec{D}$ in other side.)