# Engineering Electromagnetics

W.H. Hayt Jr. and J. A. Buck

## Chapter 6:

## Capacitance

# 6.1 Capacitance Defined

- A simple capacitor consists of two oppositely charged conductors surrounded by a uniform dielectric.
- Charge ($Q$) due to incremental **E** (and in **D**): $Q = \oint_S \mathbf{D} \cdot d\mathbf{S}$
- Potential difference between conductors:

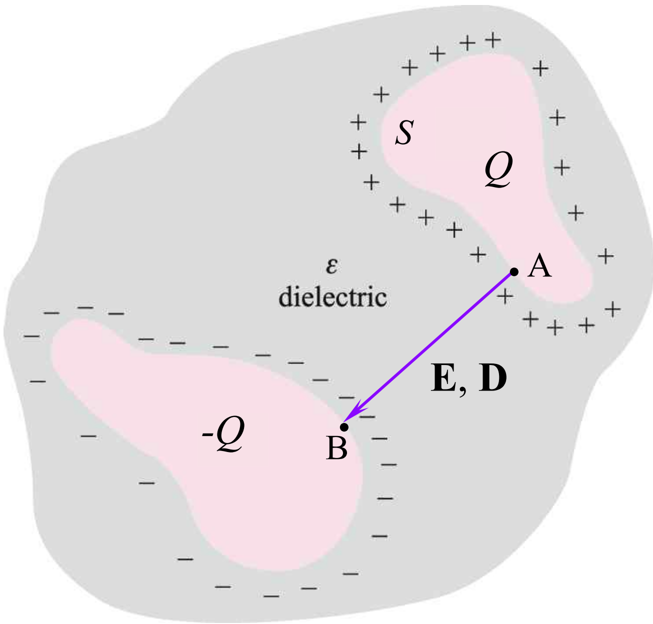
$$V_0 = -\int_B^A \mathbf{E} \cdot d\mathbf{L}$$

- Capacitance: A ratio of magnitudes of total charge on either conductor to the potential difference between conductors.

$$C = \frac{Q}{V_0} = \frac{\oint_s \varepsilon \vec{E} \cdot d\vec{S}}{-\int_-^+ \vec{E} \cdot d\vec{L}}$$

: general form

[F: farad] = [C/V]

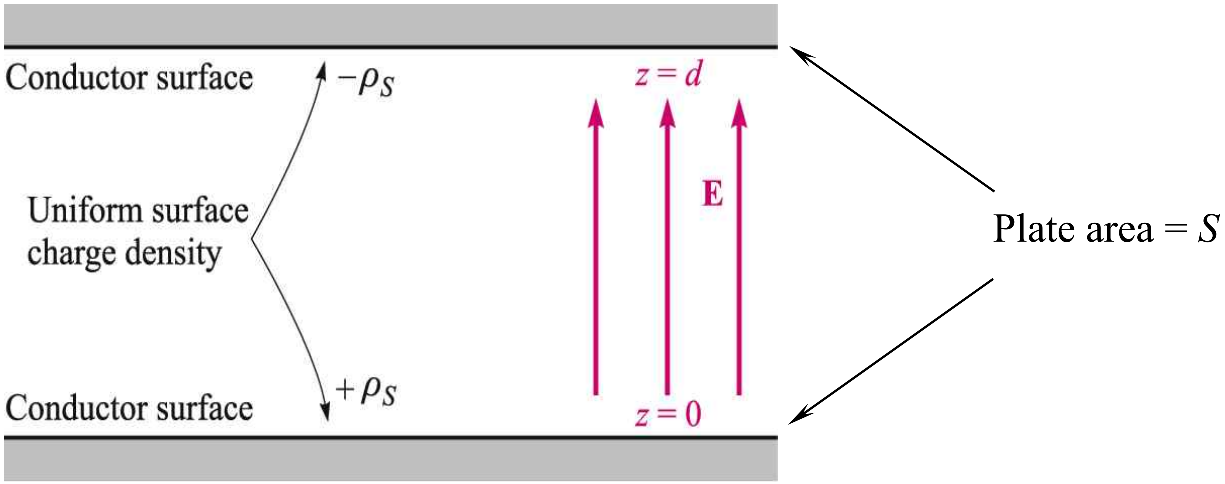



- If $Q \rightarrow NQ$, then $\vec{D}$ or $\vec{E} \rightarrow N\vec{D}$ or $N\vec{E}$ ➔ Meaningless!!!

➔ $C = f$ (physical dimensions of system of conductor and the permittivity of the homogeneous dielectric)

# 6.2 Parallel Plate Capacitor

- Let assume infinite parallel plate conductor with $S >> d$.




- For the region of $0 < z < d$,

$$\vec{E} = \frac{\rho_s}{2\varepsilon}\vec{a}_z + \frac{(-\rho_s)}{2\varepsilon}(-\vec{a}_z) = \frac{\rho_s}{\varepsilon}\vec{a}_z$$

$$\vec{D} = \varepsilon\vec{E} = \rho_S\vec{a}_z \quad \Rightarrow D_N = D_z = \rho_s$$

➔ electric flux density는 표면 전하분포와 같음
(∵ boundary condition between conductor and free space )

- Electric flux density at the lower plane (@ $z = 0$)

$$\vec{D}\cdot\vec{n}_l\Big|_{z=0} = \vec{D}\cdot\vec{a}_z = \rho_S \quad \Rightarrow \vec{D} = \rho_S\vec{a}_z$$

- Electric flux density at the upper plane (@ $z = d$)

$$\vec{D}\cdot\vec{n}_u\Big|_{z=d} = \vec{D}\cdot(-\vec{a}_z) = -\rho_S \quad \Rightarrow \vec{D} = (-\rho_S)(-\vec{a}_z) = \rho_S\vec{a}_z$$

- Potential difference between lower and upper planes:

$$V_0 = -\int_{upper}^{lower}\vec{E}\cdot d\vec{L} = -\int_d^0 \frac{\rho_s}{\varepsilon}dz = \frac{\rho_s}{\varepsilon}d = Ed$$

- With area of $S$ and $d$ (separation of planes, << $S$),

$$Q = \rho_s S \qquad V_0 = Ed = \frac{\rho_s}{\varepsilon}d$$

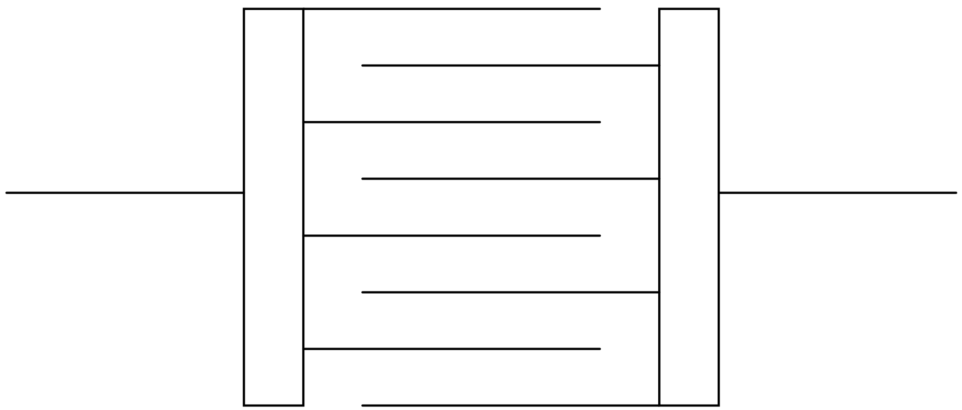
$$C = \frac{Q}{V_0} = \frac{\rho_S S}{\frac{\rho_S}{\varepsilon}d} = \varepsilon\frac{S}{d}$$

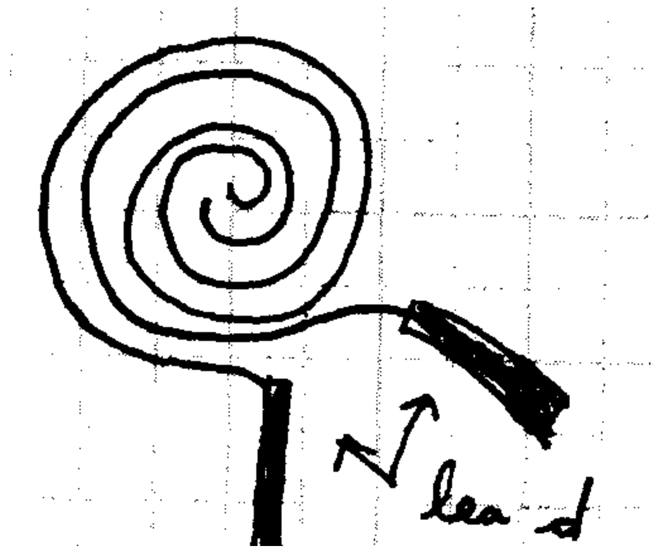
[Ex.] Mica dielectric capacitor: $\varepsilon_r = 6$, $S = 10$ in$^2$, $d = 0.01$ in

$$S = 10 \times 0.0254^2 = 6.45 \times 10^{-3} \ [\text{m}^2]$$

$$d = 0.01 \times 0.0254 = 2.54 \times 10^{-4} \ [\text{m}]$$

$$C = \frac{6 \times 8.854 \times 10^{-12} \times 6.45 \times 10^{-3}}{2.54 \times 10^{-4}} = 1.349 \, [\text{nF}]$$





# Stored Energy in a Capacitor

$$W_E = \frac{1}{2}\int_{vol} \vec{D}\cdot\vec{E}dv = \frac{1}{2}\int_{vol} \varepsilon E^2 dv$$ ← Eq. (45) in Sec.4.8

$$= \frac{1}{2}\int_0^S \int_0^d \varepsilon \frac{\rho_s^2}{\varepsilon^2} dzdS = \frac{1}{2}\frac{\rho_s^2}{\varepsilon}\cdot Sd = \frac{1}{2}\underline{\frac{\varepsilon S}{d}}\cdot\underline{\frac{\rho_s^2 d^2}{\varepsilon^2}}$$

$C$ $\quad V_0^2$

$$= \underline{\frac{1}{2}CV_0^2} = \frac{1}{2}\frac{Q}{V_0}V_0^2 = \underline{\frac{1}{2}QV_0}$$

$$= \frac{1}{2}Q\frac{Q}{C} = \underline{\frac{1}{2}\frac{Q^2}{C}}$$

$\rightarrow \varepsilon\uparrow \rightarrow W_E\uparrow \quad (C = \varepsilon\frac{S}{d})$

$$W_E = \frac{1}{2}C\,V_0^2 = \frac{1}{2}Q\,V_0 = \frac{1}{2}\frac{Q^2}{C}$$

➔ (같은 전하량일 때 유전율이 높으면 많은 energy 저장 가능)

# 6.3 Several Capacitance Examples

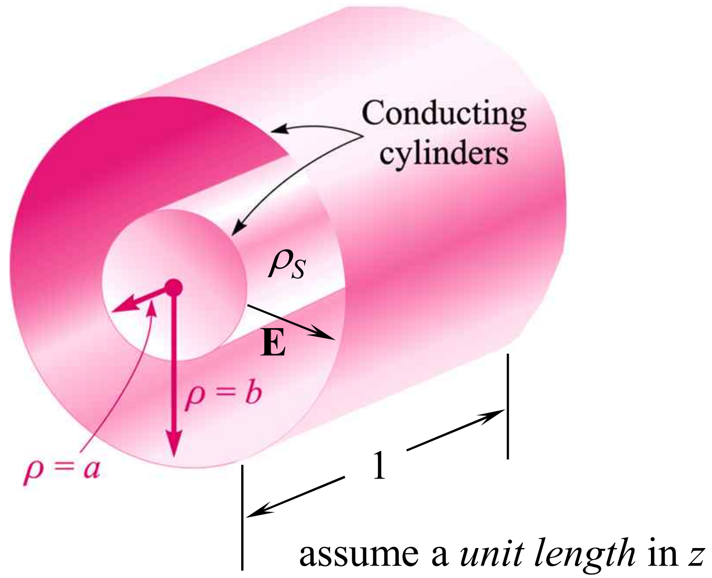
## 6.3.1 Coaxial Cable

- Coaxial capacitor with inner radius *a*, outer radius *b*, and length *L*

- Work for moving a charge *Q* from *a* to *b*:

$$W = -Q\int \vec{E}\cdot d\vec{L} = -Q\int_a^b \frac{\rho_L}{2\pi\varepsilon\rho}\vec{a}_\rho \cdot d\rho\vec{a}_\rho$$

$$= -Q\int_a^b \frac{\rho_L}{2\pi\varepsilon}\frac{d\rho}{\rho} = -\frac{Q\rho_L}{2\pi\varepsilon}\ln\frac{b}{a}$$




- Work for moving a charge *Q* from *b* to *a*:

$$W = \frac{Q\rho_L}{2\pi\varepsilon}\ln\frac{b}{a}$$

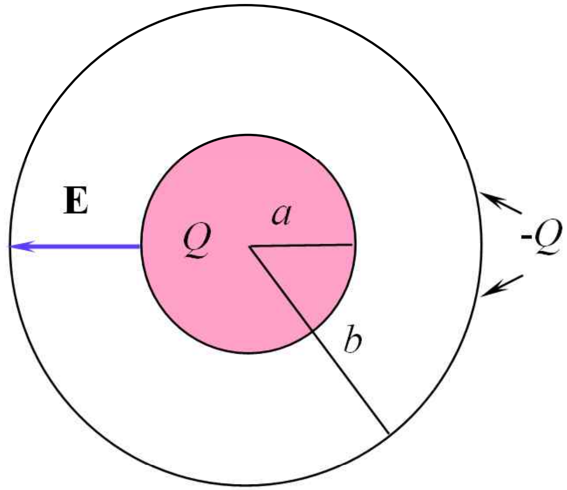
$$V_{ab} = \frac{W}{Q} = \frac{\rho_L}{2\pi\varepsilon}\ln\frac{b}{a}$$

$$C = \frac{Q}{V_{ab}} = \frac{\rho_L L}{\frac{\rho_L}{2\pi\varepsilon}\ln\frac{b}{a}} = \frac{2\pi\varepsilon L}{\ln\frac{b}{a}}$$

# 6.3.2 Spherical Capacitor

- Spherical capacitor formed of two concentric spherical conducting shells of radii *a* and *b* (*a* < *b*)

$$\mathbf{E} = E_r \mathbf{a}_r = \frac{Q}{4\pi\epsilon r^2}\mathbf{a}_r$$




- Potential difference between inner and outer spheres :

$$V_0 = -\int_b^a \mathbf{E}\cdot d\mathbf{L} = -\int_b^a \frac{Q}{4\pi\epsilon r^2}\mathbf{a}_r \cdot \mathbf{a}_r \, dr$$

$$= \frac{Q}{4\pi\varepsilon}\left[\frac{1}{r}\right]_b^a = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$ where $Q$: total charge on inner sphere

- Capacitance: $$C = \frac{Q}{V_0} = \frac{4\pi\epsilon}{(1/a) - (1/b)}$$

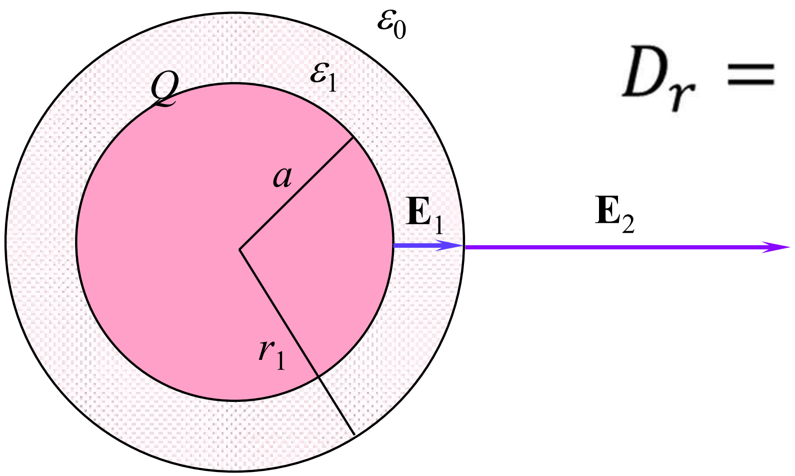
- If $b \to \infty$, $C = 4\pi\varepsilon a$

[Ex.] Marble sphere with diameter of 1 cm in free space

$(a = 0.5 \text{ cm}, \varepsilon_r = 1)$

$C = 4\pi\varepsilon a = 0.556 \text{ pF}$

# Sphere conducting shell consisting of several slabs




$$D_r = \frac{Q}{4\pi r^2}, \quad E_r = \frac{Q}{4\pi\epsilon_1 r^2} \qquad (a < r < r_1)$$

$$= \frac{Q}{4\pi\epsilon_0 r^2} \qquad (r_1 < r)$$

$$V_a - V_\infty = -\int_{r_1}^{a} \frac{Q\,dr}{4\pi\epsilon_1 r^2} - \int_{\infty}^{r_1} \frac{Q\,dr}{4\pi\epsilon_0 r^2} = \frac{Q}{4\pi}\left[\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}\right] = V_0$$

$$C = \frac{4\pi}{\frac{1}{\epsilon_1}\left(\frac{1}{a} - \frac{1}{r_1}\right) + \frac{1}{\epsilon_0 r_1}}$$

# 6.3.3 Capacitors with Multiple Dielectrics

- Assumption: $E_1$ and $E_2$ are both uniform.

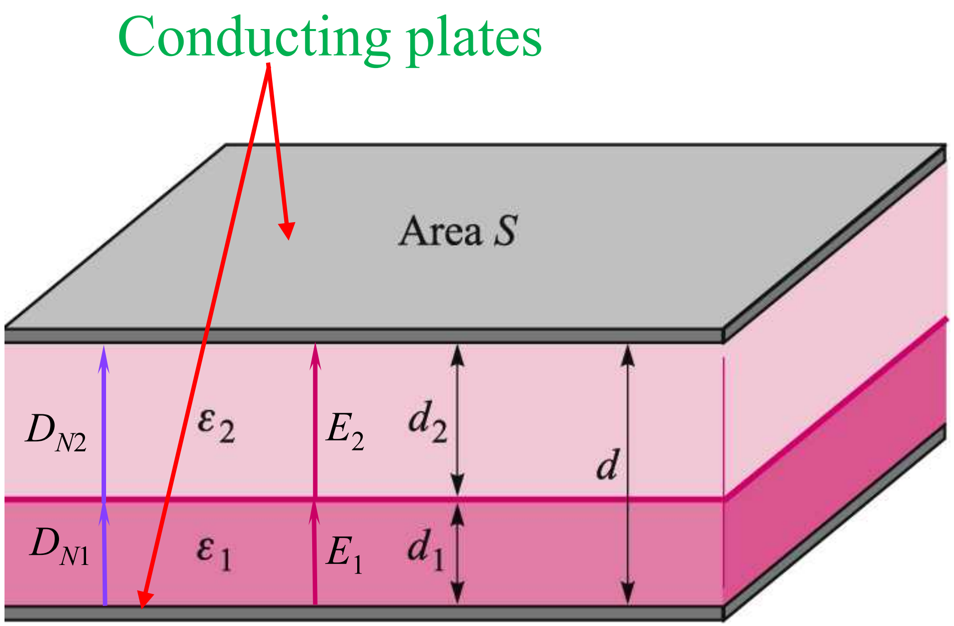
$$V_0 = E_1 d_1 + E_2 d_2$$

- At dielectric interface, $D_{N1} = D_{N2}$ (by boundary condition)

$$\Rightarrow \epsilon_1 E_1 = \epsilon_2 E_2$$

$$V_0 = E_1 d_1 + \frac{\varepsilon_1}{\varepsilon_2} E_1 d_2 = \left( d_1 + \frac{\varepsilon_1}{\varepsilon_2} d_2 \right) E_1$$

$$E_1 = \frac{V_0}{d_1 + d_2(\epsilon_1/\epsilon_2)}$$




- Surface charge density:

$$\rho_{S1} = D_1 = \varepsilon_1 E_1 = \frac{V_0}{\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}}$$

$$D_1 = D_2 \Rightarrow C = \frac{Q}{V_0} = \frac{\rho_s S}{V_0} = \frac{V_0 \cdot S \Big/ \left(\frac{d_1}{\varepsilon_1} + \frac{d_2}{\varepsilon_2}\right)}{V_0} = \frac{1}{\frac{d_1}{\varepsilon_1 S} + \frac{d_2}{\varepsilon_2 S}}$$

$$= \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}} \quad \Leftrightarrow \quad \frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2}$$

cf.) (Other solution) Charge on one plate: $Q$

Charge density: $\frac{Q}{S} = D$ $\qquad D_{N1} = D_{N2}$

$$E_1 = \frac{D}{\varepsilon_1} = \frac{Q}{\varepsilon_1 S} \qquad V_1 = E_1 d_1 = \frac{Q d_1}{\varepsilon_1 S}$$

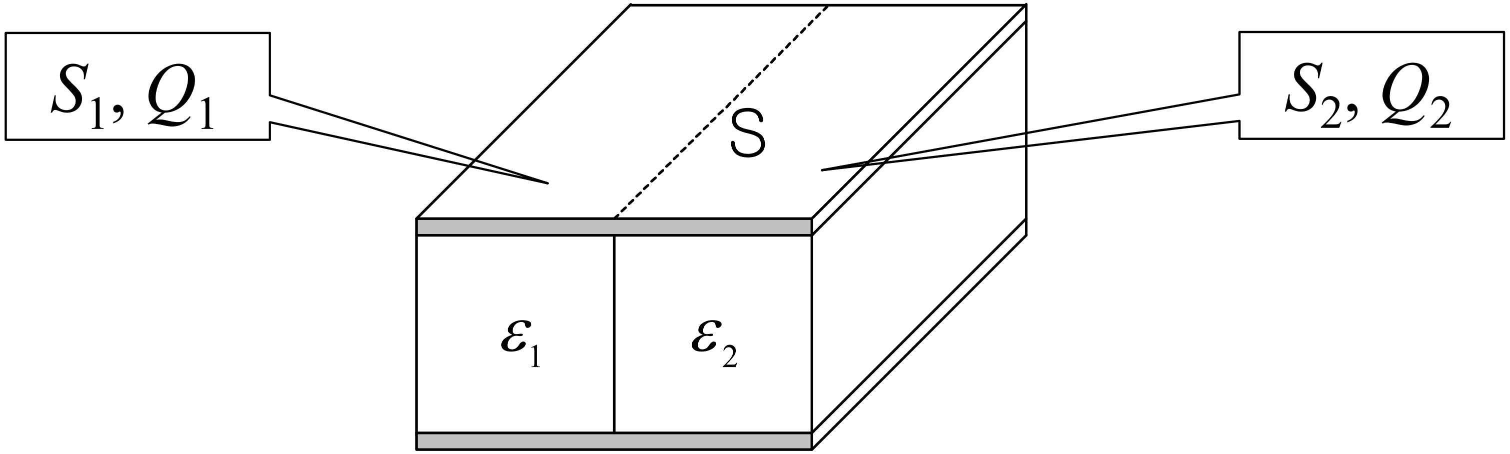
$$\frac{V_1}{Q} = \frac{d_1}{\varepsilon_1 S} = \frac{1}{C_1}$$

$$E_2 = \frac{D}{\varepsilon_2} = \frac{Q}{\varepsilon_2 S} \qquad V_2 = E_2 d_2 = \frac{Q d_2}{\varepsilon_2 S}$$

$$\frac{V_2}{Q} = \frac{d_2}{\varepsilon_2 S} = \frac{1}{C_2}$$

$$\therefore C = \frac{Q}{V} = \frac{Q}{V_1 + V_2} = \frac{1}{\frac{d_1}{\varepsilon_1 S} + \frac{d_2}{\varepsilon_2 S}} = \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}$$

▪ Parallel plate capacitor consisting of several dielectric layers




$$E_1 = E_2 = \frac{V_0}{d} = \frac{D_1}{\varepsilon_1} = \frac{D_2}{\varepsilon_2} = \frac{Q_1}{\varepsilon_1 S_1} = \frac{Q_2}{\varepsilon_2 S_2}$$

$$Q_1 = \frac{\varepsilon_1 S_1}{d} V_0 \qquad Q_2 = \frac{\varepsilon_2 S_2}{d} V_0$$

$$C = \frac{Q}{V_0} = \frac{Q_1 + Q_2}{V_0} = \frac{\varepsilon_1 S_1}{d} + \frac{\varepsilon_2 S_2}{d} = C_1 + C_2$$

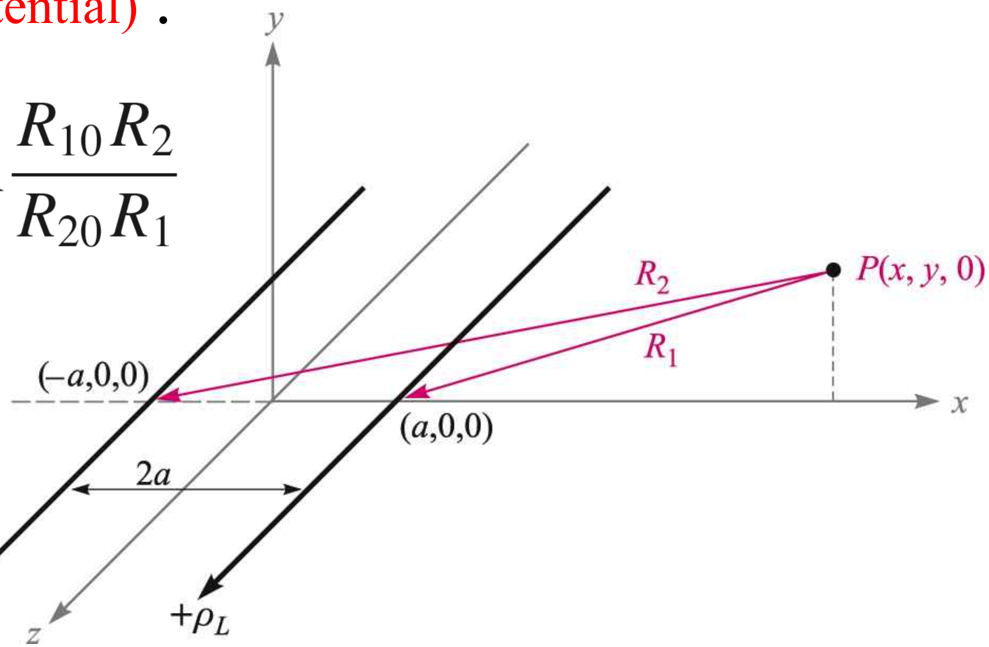
# 6.4 Capacitance of a Two-Wire Transmission Line

- Potential of a single line charge on the *z* axis, with a zero reference at $\rho = R_0$:

$$\vec{E} = E_\rho \vec{a}_\rho = \frac{\rho_L}{2\pi\varepsilon\rho}\vec{a}_\rho \quad \rightarrow \quad V = -\int \vec{E}\cdot d\vec{L} = -\int_{R_0}^{R} \frac{\rho_L}{2\pi\varepsilon\rho} d\rho = \frac{\rho_L}{2\pi\varepsilon}\ln\frac{R_0}{R}$$

- Combined potential field from the positive and negative lines at $R_1$ and $R_2$ (dual line charge가 존재할 때의 potential) :

$$V = \frac{\rho_L}{2\pi\epsilon}\left(\ln\frac{R_{10}}{R_1} - \ln\frac{R_{20}}{R_2}\right) = \frac{\rho_L}{2\pi\epsilon}\ln\frac{R_{10}R_2}{R_{20}R_1}$$



- Choose $R_{10} = R_{20}$ ($x = 0$ plane: zero potential plane).
- Potential at $P(x, y, 0)$

$$V = \frac{\rho_L}{2\pi\varepsilon}\ln\frac{R_2}{R_1}$$

$$= \frac{\rho_L}{2\pi\varepsilon}\ln\sqrt{\frac{(x+a)^2+y^2}{(x-a)^2+y^2}} = \frac{\rho_L}{4\pi\varepsilon}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

: $R_{10} = R_{20}$ 일 때 임의의 위치에서의 potential.

- $K_1$: dimensionless parameter that is a function of the potential $V_0$ where is an equipotential surface $V = V_0$

$$\boldsymbol{V_0 = \frac{\rho_L}{4\pi\varepsilon}\ln K_1}$$

$$K_1 = e^{4\pi\varepsilon V_0/\rho_L} \qquad \text{where} \qquad \boldsymbol{K_1 = \frac{(x+a)^2+y^2}{(x-a)^2+y^2}} \quad \text{---- (1)}$$

- Solving equation (1)

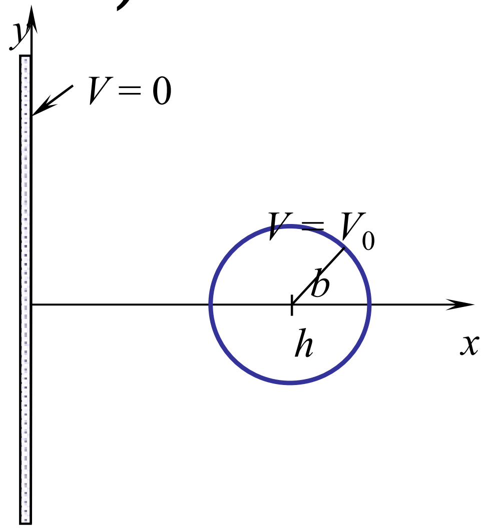
$$K_1x^2 - 2aK_1x + a^2K_1 + K_1y^2 = x^2 + 2ax + a^2 + y^2$$

$$x^2(K_1-1) - 2a(K_1+1)x + a^2(K_1-1) + y^2(K_1-1) = 0$$

$$x^2 - 2ax\frac{K_1+1}{K_1-1} + y^2 + a^2 = 0$$

$$\left(x - a\frac{K_1+1}{K_1-1}\right)^2 + y^2 = a^2\frac{(K_1+1)^2}{(K_1-1)^2} - a^2$$

$$= \frac{\left(K_1^2+2K_1+1-K_1^2+2K_1-1\right)}{(K_1-1)^2}a^2 = \frac{4a^2K_1}{(K_1-1)^2}$$




$$\left(x - a\frac{K_1+1}{K_1-1}\right)^2 + y^2 = \left(\frac{2a\sqrt{K_1}}{K_1-1}\right)^2$$ → Center : $(h, 0)$ and Radius : $b$

where $h = a\dfrac{K_1+1}{K_1-1}$, $b = \dfrac{2a\sqrt{K_1}}{K_1-1}$ ----(2) : equipotential 궤적

➔ $h, b = f[a, \rho_L, K_1 \text{ (or } V_0)]$

- Solving the last two equations for $a$ and $K_1$

$$\boldsymbol{h(K_1 - 1) = a(K_1 + 1)} \qquad \boldsymbol{K_1(a - h) = -a - h}$$

$$\boldsymbol{K_1 = \frac{a+h}{h-a} = \frac{h+a}{h-a}}$$

$$b = \frac{2a\sqrt{K_1}}{K_1 - 1} = \frac{2a\sqrt{\dfrac{h+a}{h-a}}}{\dfrac{h+a}{h-a} - 1} = \sqrt{h^2 - a^2} \quad \Rightarrow \quad \therefore \underline{\boldsymbol{a = \sqrt{h^2 - b^2}}}$$

$$\boldsymbol{bK_1 - b = 2a\sqrt{K_1}}, \quad bK_1 - 2a\sqrt{K_1} - b = 0, \quad b(\sqrt{K_1})^2 - 2a\sqrt{K_1} - b = 0$$

$$\sqrt{K_1} = \frac{a + \sqrt{a^2 + b^2}}{b} = \frac{\sqrt{h^2 - b^2} + \sqrt{h^2 - b^2 + b^2}}{b} = \boldsymbol{\frac{h + \sqrt{h^2 - b^2}}{b}}$$

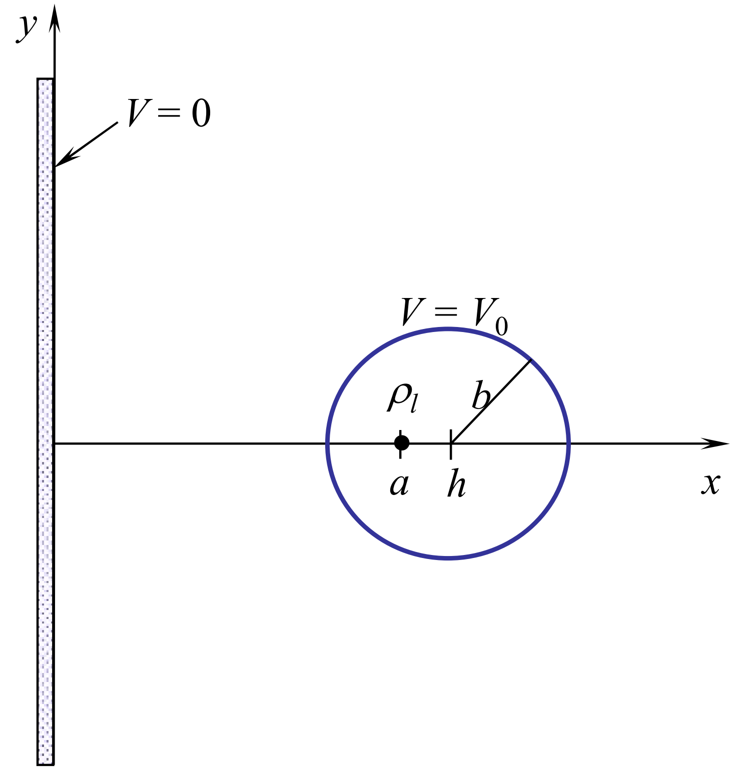
- Since $K_1 = e^{4\pi\varepsilon V_0/\rho_L}$ (or $\sqrt{K_1} = e^{2\pi\varepsilon V_0/\rho_L}$)

$$\boldsymbol{\rho_L = \frac{4\pi\varepsilon V_0}{\ln K_1}} \qquad \Rightarrow \; a,\ \rho_L,\ K_1 = g\,[h,\ b,\ V_0]$$

- Solving Given ($h$, $b$, $V_0$), then ($a$, $\rho_L$, $K_1$) can be determined.

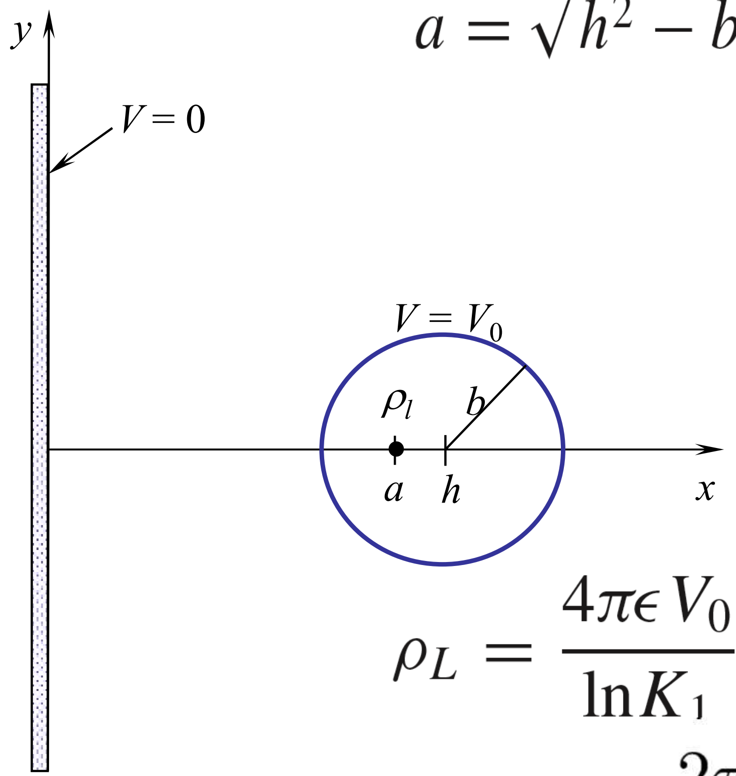
$$C = \frac{Q}{V} = \frac{\rho_L L}{V_0} = \frac{\rho_L L}{\dfrac{\rho_L \ln K_1}{4\pi\varepsilon}} = \frac{4\pi\varepsilon L}{\ln K_1} = \frac{2\pi\varepsilon L}{\ln \sqrt{K_1}}$$

$$= \frac{2\pi\varepsilon L}{\ln\left[\dfrac{h + \sqrt{h^2 - b^2}}{b}\right]} = \frac{2\pi\varepsilon L}{\cosh^{-1}\left(\dfrac{h}{b}\right)}$$



# Numerical Example

- Choose $b$ = 5 m, $h$ = 13 m, and $V_0$ = 100 V. Find $\rho_L$ and $C$.




$$a = \sqrt{h^2 - b^2} = \sqrt{13^2 - 5^2} = 12 \quad \text{m}$$

$$\sqrt{K_1} = \frac{h + \sqrt{h^2 - b^2}}{b} = \frac{13 + 12}{5} = 5$$

$$\rho_L = \frac{4\pi\epsilon V_0}{\ln K_1} = \frac{4\pi \times 8.854 \times 10^{-12} \times 100}{\ln 25} = 3.46 \text{ nC/m}$$

$$C = \frac{2\pi\epsilon}{\cosh^{-1}(h/b)} = \frac{2\pi \times 8.854 \times 10^{-12}}{\cosh^{-1}(13/5)} = 34.6 \text{ pF/m}$$

($x$ = $a$ = 12, $y$ = 0에서 단위 선전하 밀도 3.46 nC/m이며,
단위 길이 당 34.6 pF/m의 용량을 갖음을 유추해 낼 수 있음)

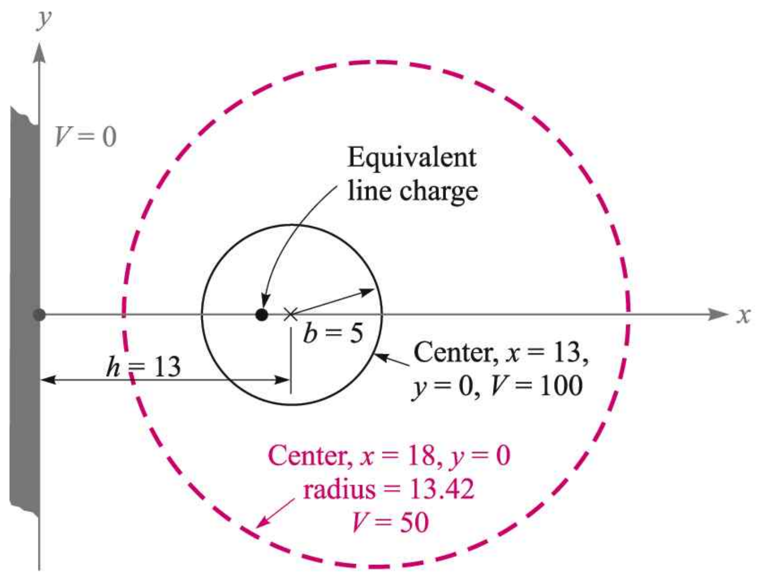
# Inverse Numerical Example

- Identify the cylinder representing the ($V_1 =$ ) 50 [V] equipotential surface.

$$K_1 = e^{4\pi\epsilon V_1/\rho_L} = e^{4\pi \times 8.854\times 10^{-12}\times 50/3.46\times 10^{-9}} = 5.00$$

$$b = \frac{2a\sqrt{K_1}}{K_1 - 1} = \frac{2\times 12\times\sqrt{5}}{5-1} = 13.42\ [\text{m}]$$

$$h = a\frac{K_1+1}{K_1-1} = 12\times\frac{5+1}{5-1} = 18\ \ [\text{m}]$$




$h = 13, b = 5, \therefore K_1 = 25; \therefore \rho_L = 3.46 \times 10^{-9}$ C/m, $\therefore a = 12$

If $V_1 = 50, K_1 = 5, h = 18, b = 13.42, \rho_L$ unchanged

$$C = \frac{2\pi\varepsilon_0 L}{\ln 5} = 34.6 \text{ pF/m}$$

▪ Electric field intensity

$$\vec{E} = -\nabla V = -\nabla\left[\frac{\rho_L}{4\pi\varepsilon}\ln\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right] \leftarrow \frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx} \qquad \frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= -\frac{\rho_L}{4\pi\varepsilon}\left[\frac{(x-a)^2+y^2}{(x+a)^2+y^2}\right]\left[\frac{\left\{(x-a)^2+y^2\right\}\cdot 2(x+a) - \left\{(x+a)^2+y^2\right\}\cdot 2(x-a)}{\left\{(x-a)^2+y^2\right\}^2}\vec{a}_x\right.$$

$$\left. + \frac{\left\{(x-a)^2+y^2\right\}\cdot 2y - \left\{(x+a)^2+y^2\right\}\cdot 2y}{\left\{(x-a)^2+y^2\right\}^2}\vec{a}_y\right]$$

$$= -\frac{\rho_L}{4\pi\varepsilon}\left[\frac{2(x+a)\vec{a}_x + 2y\vec{a}_y}{(x+a)^2+y^2} - \frac{2(x-a)\vec{a}_x + 2y\vec{a}_y}{(x-a)^2+y^2}\right]$$

$$= -\frac{\rho_L}{2\pi\varepsilon}\left[\frac{(x+a)\vec{a}_x + y\vec{a}_y}{(x+a)^2+y^2} - \frac{(x-a)\vec{a}_x + y\vec{a}_y}{(x-a)^2+y^2}\right]$$

- Electric flux density

$$\vec{D} = \varepsilon\vec{E} = -\frac{\rho_L}{2\pi}\left[\frac{(x+a)\vec{a}_x + y\vec{a}_y}{(x+a)^2 + y^2} - \frac{(x-a)\vec{a}_x + y\vec{a}_y}{(x-a)^2 + y^2}\right]$$

where $\rho_{s.\max}$ at ($x = h - b$, $y = 0$) and $\rho_{s.\min}$ at ($x = h + b$, $y = 0$)

(∵ $+\rho_L$ 에서 $-\rho_L$ 로 가는 최단 거리이므로 electric flux가 최대가 됨)

$$\rho_{s.\max} = -D_{x,\, x=h-b,\, y=0} = \frac{\rho_L}{2\pi}\left[\frac{h-b+a}{(h-b+a)^2} - \frac{h-b-a}{(h-b-a)^2}\right] \quad (∵ -\vec{a}_x \text{ 방향이므로})$$

$$\rho_{s.\min} = -D_{x,\, x=h+b,\, y=0} = -\frac{\rho_L}{2\pi}\left[\frac{h+b+a}{(h+b+a)^2} - \frac{h+b-a}{(h+b-a)^2}\right] \quad (∵ \vec{a}_x \text{ 방향이므로})$$

- In case of $h = 13$, $b = 5$, and $a = 12$,

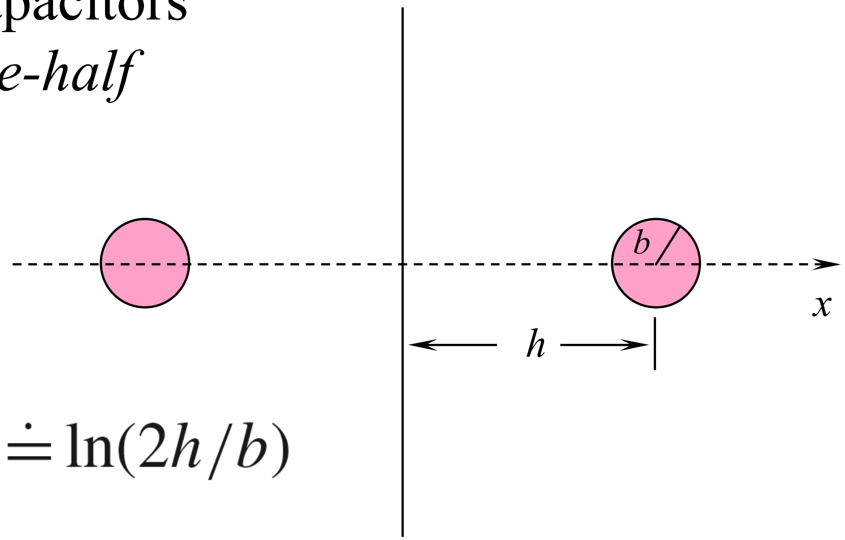
$$\rho_{s.\max} = \frac{3.46\times10^{-9}}{2\pi}\left[\frac{13-5+12}{(13-5+12)^2} - \frac{13-5-12}{(13-5-12)^2}\right] = 0.1650\ [\mathrm{nC/m^2}]$$

$$\rho_{s.\min} = -\frac{3.46\times10^{-9}}{2\pi}\left[\frac{13+5+12}{(13+5+12)^2} - \frac{13+5-12}{(13+5-12)^2}\right] = 0.0734\ [\mathrm{nC/m^2}]$$

$$\rho_{s.\max} = 2.25\rho_{s.\min}$$

# Capacitance of a Two-Wire Line

▪ The line geometry is shown here. With two cylinders (and a plane of zero potential between them), the structure represents two cylinder/plane capacitors in *series*, and so the overall capacitance is *one-half* the result derived previously.




$$C = \frac{2\pi\epsilon L}{\ln[(h+\sqrt{h^2-b^2})/b]} = \frac{2\pi\epsilon L}{\cosh^{-1}(h/b)}$$

$$C = \frac{\pi\epsilon L}{\cosh^{-1}(h/b)}$$

▪ If $b << h$,

$$\ln\left[\left(h+\sqrt{h^2-b^2}\right)/b\right] \doteq \ln[(h+h)/b] \doteq \ln(2h/b)$$

$$\therefore C = \frac{2\pi\varepsilon L}{\ln\left[\frac{2h}{b}\right]} \leftarrow C = \frac{2\pi\varepsilon L}{\ln\left[\frac{h+\sqrt{h^2-b^2}}{b}\right]}$$

$$C_{total} = \frac{C}{2}$$

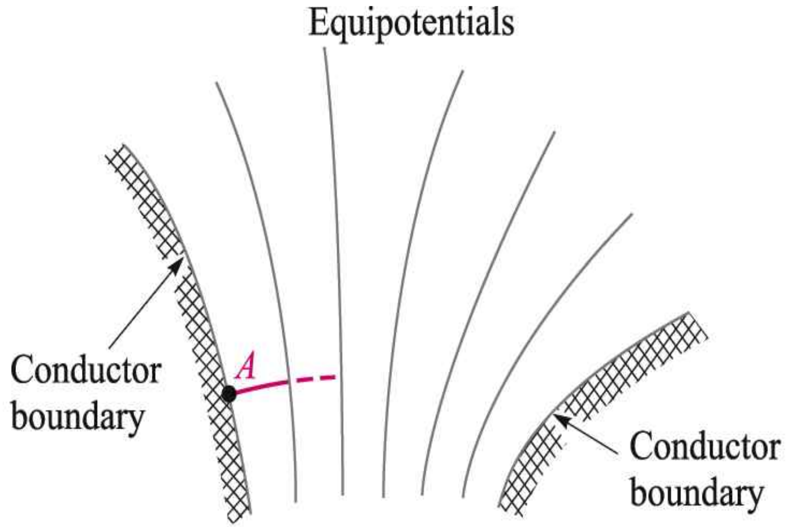
# 6.5 Using Field Sketches to Estimate Capacitance

This method employs these properties of conductors and fields:

1. A conductor boundary is an equipotential surface.
2. The electric field intensity and electric flux density are both perpendicular to the equipotential surfaces.
3. **E** and **D** are therefore perpendicular to the conductor boundaries and possess zero tangential values.
4. The lines of electric flux, or streamlines, begin and terminate on charge and hence, in a charge-free, homogeneous dielectric, begin and terminate only on the conductor boundaries.

# Sketching Equipotentials

Given the conductor boundaries, equipotentials may be sketched in. An attempt is made to establish approximately equal potential differences between them.




A line of electric flux density, **D**, is then started (at point *A*), and then drawn such that it crosses equipotential lines at right-angles.

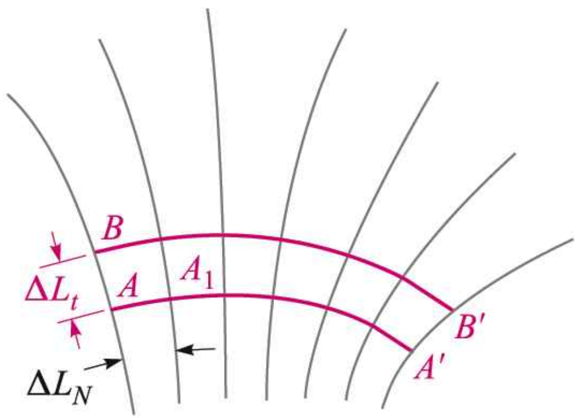
# Sketching Lines of Electric Flux

Two Lines of **D** are shown (sketched in red), and spacings between adjacent field lines and between adjacent equipotential surfaces are noted. All field lines *must* intersect equipotentials at $90^0$.

The volume between the two red field lines (having unit depth into the screen) forms a "tube" of flux, of amount $\Delta\psi$. This is the same as the charge on the conductor that is bounded by the tube.

The electric field can now be written as:

$$E = \frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta L_t}$$




Then, assuming potential difference $\Delta V$ between adjacent equipotentials, we may also write the electric field as:

$$E = \frac{\Delta V}{\Delta L_N}$$

# Graphical Relationship Between Incremental Potential Difference and Electric Flux

We now have two expressions for electric field, based on the line spacings in the sketch:

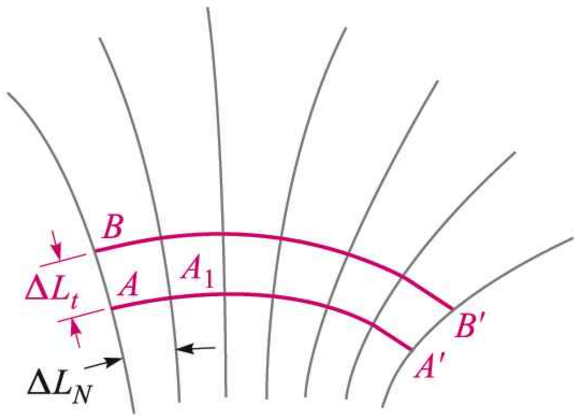
$$E = \frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta L_t} \qquad\qquad E = \frac{\Delta V}{\Delta L_N}$$

Setting the two expressions equal results in

$$\frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta L_t} = \frac{\Delta V}{\Delta L_N}$$

or:

$$\frac{\Delta L_t}{\Delta L_N} = \text{constant} = \frac{1}{\epsilon}\frac{\Delta\Psi}{\Delta V}$$




In other words, the sketch is done such that the ratio $\Delta L_t / \Delta L_N$ is fixed. The easiest way to do this is to make

$$\Delta L_t = \Delta L_N$$

So draw the sketch such that *each grid segment is approximately square*

# Determining Capacitance from the Field Sketch

Here we see the completed sketch, drawn such that the region is divided into curvilinear squares. Each segment along the conductor walls (again of unit depth) contains flux $\Delta\psi$, (or charge $\Delta Q$). Each distance increment between conductors represents a change in potential of $\Delta V$.

The number of segments along the conductor wall is $N_Q$

The total charge on either conductor is therefore:

$$Q = N_Q \Delta Q = N_Q \Delta \Psi$$

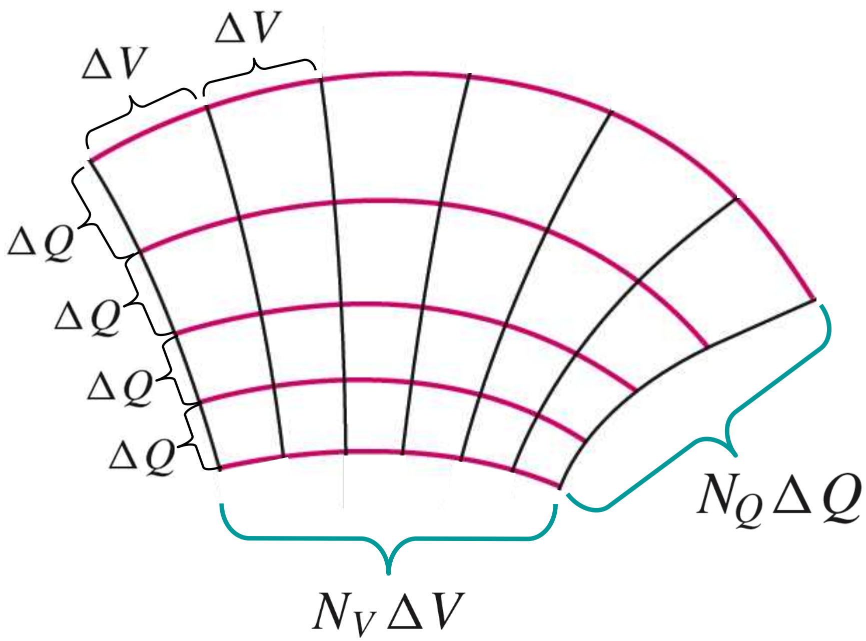
The number of segments between conductors is $N_V$

The potential difference between conductor is therefore:

$$V_0 = N_V \Delta V$$

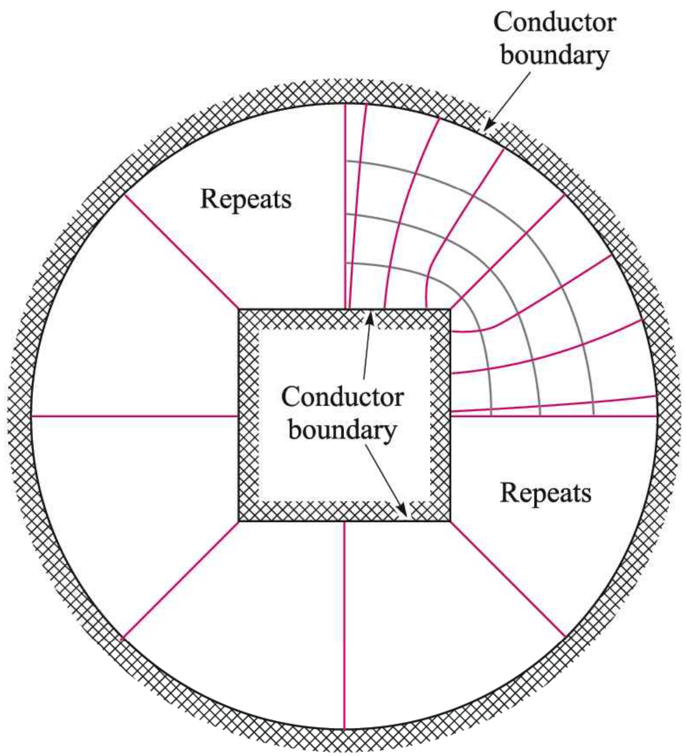
So that the capacitance is now: $$C = \frac{N_Q \Delta Q}{N_V \Delta V}$$

or finally: $$C = \frac{N_Q}{N_V} \epsilon \frac{\Delta L_t}{\Delta L_N} = \epsilon \frac{N_Q}{N_V}$$



# Example

In this transmission line, we wish to estimate the capacitance per unit length. The field and equipotential lines in one quadrant are shown.




In this case, division of the range parallel to the conductors into an integral number of squares was not achieved. Instead, over one-eighth of the distance around the perimeter, we have 3.25 divisions.

Between conductors there are exactly four squares.

Therefore:

$$C = \epsilon \frac{N_Q}{N_V} = \epsilon_0 \frac{8 \times 3.25}{4} = \underline{57.6\,\mathrm{pF/m}}$$

# 6.6 Poisson's and Laplace's Equations

- Gauss's law as Maxwell's first equation:

$$\nabla \cdot \mathbf{D} = \rho_v$$

(임의의 폐곡면을 통해 밖으로 나오는 electric flux density의 합은 폐곡면 내부에 존재하는 전하량과 같다.)

- Electric flux density: $\mathbf{D} = \epsilon \mathbf{E}$

- Gradient relationship: $\mathbf{E} = -\nabla V$

- As a result,

$$\nabla \cdot \vec{D} = \nabla \cdot (\varepsilon \vec{E}) = \varepsilon \nabla \cdot \vec{E} = \varepsilon \nabla \cdot (-\nabla V) = -\varepsilon \nabla \cdot (\nabla V) = \rho_v$$

$$\therefore \nabla \cdot (\nabla V) = -\frac{\rho_v}{\varepsilon} \quad \text{: Poisson's equation}$$

where $\varepsilon$ : constant for homogeneous material

$$\begin{cases} \nabla \cdot \vec{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z} & : \text{Divergence} \\ \nabla V = \dfrac{\partial V}{\partial x}\vec{a}_x + \dfrac{\partial V}{\partial y}\vec{a}_y + \dfrac{\partial V}{\partial z}\vec{a}_z & : \text{Gradient} \end{cases}$$

$$= A_x\vec{a}_x + A_y\vec{a}_y + A_z\vec{a}_z$$

$$\therefore \nabla \cdot (\nabla V) = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

$$= \nabla^2 V$$

$$= -\frac{\rho_v}{\varepsilon}$$

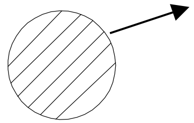
$\nabla^2$ : "Laplacian" or "del squared"

- If $\rho_v = 0$ (zero volume charge density),

$$\nabla^2 V = 0 \quad : \text{Laplace equation}$$

전계의 원천이 되는 점전하, 선전하, 면전하들이 경계면에 존재한다고 하여도, volume 내부에 전하가 없으면 $\rho_v = 0$ 이 됨

# Laplacian Operator in the Three Coordinate Systems



(특정 volume 내부에는 전하가 없고 경계면(선, 점)에만 전하가 있다면 Laplace eq.을 풀고 그 경계면 (선, 점)에 대한 경계조건을 알면 그것으로 인한 각종 정보 유도 가능)

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(rectangular)}$$

$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}$$

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}$$

- Laplace equations are also solved by certain boundary conditions (V, $\vec{E}$, etc).

# 6-7 Examples of the solution of Laplace's equation ($\nabla^2 V = 0$)

- Direct integration: simple method
  (3-dimensional problem: indirect integration, more difficult).
- Direct integration method is applicable only to problems that are "one-dimensional".

$$\frac{\partial^2 V}{\partial x^2}, \frac{\partial^2 V}{\partial y^2}, \frac{\partial^2 V}{\partial z^2} \quad \text{(cartensian coord.)}$$

$$\frac{\partial^2 V}{\partial z^2} \quad \text{(cylindrical coord.)}$$

$$\frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \quad \text{(spherical coord.)}$$

➔ Same problem

Laplace 방정식에서 각 좌표계에
방향축별 로 9개가 있지만,
그 중에 5개 축은 같은 형태이므로
5가지 축방향 문제만 풀면 해결됨.

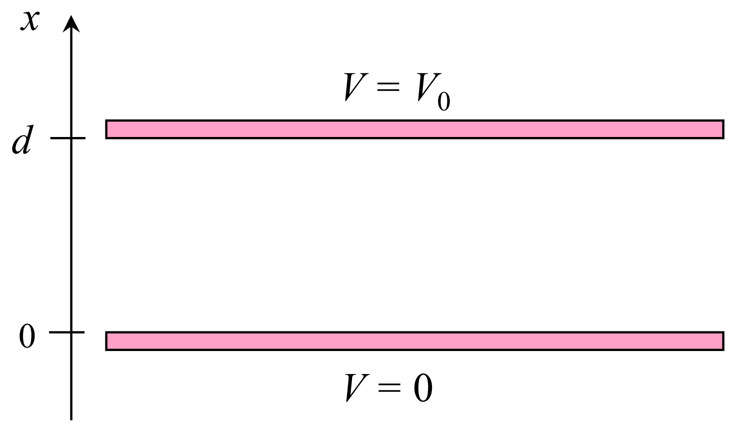
# [Ex. 1] Parallel Plate Capacitor

- Assumption: $V$ is a function of only $x$.

$$\frac{\partial^2 V}{\partial x^2} = 0 \quad \Rightarrow \quad \frac{d^2 V}{dx^2} = 0$$

$$\frac{dV}{dx} = A$$

$V = Ax + B$ : general solution
where $A$ and $B$: constant




Boundary conditions:

1. $V = 0$ at $x = 0$
2. $V = V_0$ at $x = d$

$$\left.\begin{aligned} V_1 &= Ax_1 + B \\ V_2 &= Ax_2 + B \end{aligned}\right\} \Rightarrow V = \underbrace{\frac{V_1 - V_2}{x_1 - x_2}}_{A} x + \underbrace{\frac{V_2 x_1 - V_1 x_2}{x_1 - x_2}}_{B} \quad \begin{cases} V = 0 = V_1 & \text{at} \quad x = 0 = x_1 \\ V = V_0 = V_2 & \text{at} \quad x = d = x_2 \end{cases}$$

# Interior Potential Field

Apply boundary condition 1:

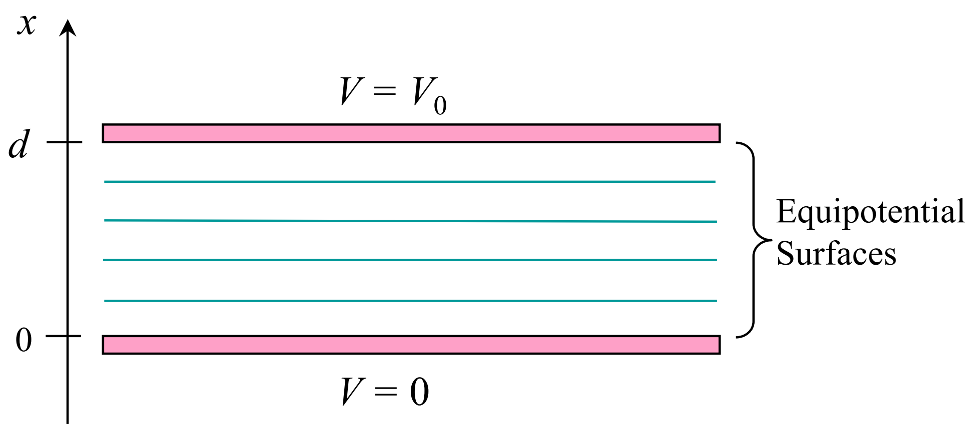
$$0 = A(0) + B \longrightarrow B = 0$$

Apply boundary condition 2:

$$V_0 = Ad \longrightarrow A = \frac{V_0}{d}$$

Finally:

$$V = \frac{V_0 x}{d}$$




Boundary conditions:

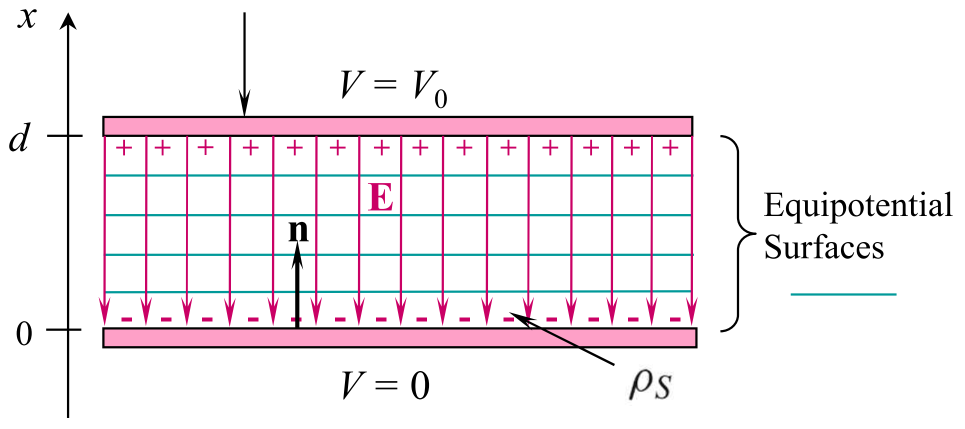
1. $V = 0$ at $x = 0$
2. $V = V_0$ at $x = d$

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\nabla V = -\frac{V_0}{d}\mathbf{a}_x$$

(∵ 전위가 큰 쪽에서 낮은 쪽으로 향하므로)

$$\mathbf{D} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$




At the lower plate surface ($z = 0$):

$$\mathbf{D}_S = \mathbf{D}\Big|_{x=0} = -\epsilon \frac{V_0}{d}\mathbf{a}_x \qquad \text{and} \qquad \mathbf{n} = \mathbf{a}_x$$ (∵ 전위가 증가하는 방향)

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S \qquad Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$$

$$C = \frac{|Q|}{V_0} = \frac{\varepsilon \frac{V_0 S}{d}}{V_0} = \varepsilon \frac{S}{d}$$

# [Ex. 2] Coaxial Transmission Line or Capacitor

- Assumption: $V$ is a function of only $\rho$.

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right)=0 \rightarrow \frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dV}{d\rho}\right)=0$$

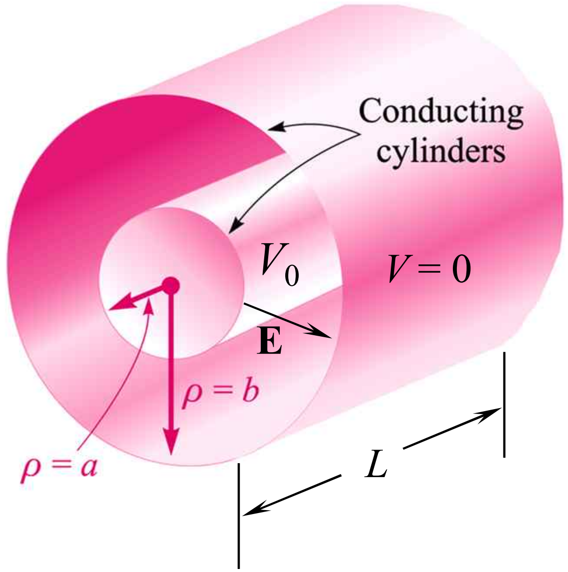
$$\rho\frac{dV}{d\rho}=A \qquad V=A\ln\rho+B$$

- Our goal is to evaluate the potential function in the region ($a < \rho < b$)

$$V_0 = A\ln a + B$$
$$-)\,0 = A\ln b + B$$

$$V_0 = A\ln\frac{a}{b} \qquad A=\frac{V_0}{\ln\frac{a}{b}}$$

$$B=-A\ln b=-\frac{V_0}{\ln\frac{a}{b}}\ln b$$




Boundary conditions:

1. $V = 0$ at $\rho = b$
2. $V = V_0$ at $\rho = a$

$$\therefore V = A\ln\rho + B = \frac{V_0}{\ln\frac{a}{b}}\ln\rho - \frac{V_0}{\ln\frac{a}{b}}\ln b = \frac{V_0}{\ln\frac{a}{b}}\ln\frac{\rho}{b} = \frac{V_0}{\ln\frac{b}{a}}\ln(\frac{b}{\rho})$$

$$V(\rho) = V_0\frac{\ln(b/\rho)}{\ln(b/a)}$$

$$\vec{E} = -\nabla V = -\frac{V_0}{\ln(\frac{b}{a})}\cdot\frac{1}{\frac{b}{\rho}}\cdot(-b\frac{1}{\rho^2})\vec{a}_\rho = \frac{V_0}{\rho}\frac{1}{\ln(\frac{b}{a})}\vec{a}_\rho$$

$$D_{N(\rho=a)} = \frac{\varepsilon V_0}{a\ln(\frac{b}{a})} \quad : \text{scalar}$$

$$Q = \frac{\varepsilon V_0}{a\ln(\frac{b}{a})}S = \frac{\varepsilon V_0 2\pi a L}{a\ln(\frac{b}{a})} = \frac{\varepsilon V_0 2\pi L}{\ln(\frac{b}{a})}$$

$$C = \frac{Q}{V} = \frac{\varepsilon V_0 2\pi L/\ln(b/a)}{V_0\ln(b/\rho)/\ln(b/a)} = \frac{2\pi\varepsilon L}{\ln(\frac{b}{a})} \qquad \leftarrow \rho = a$$

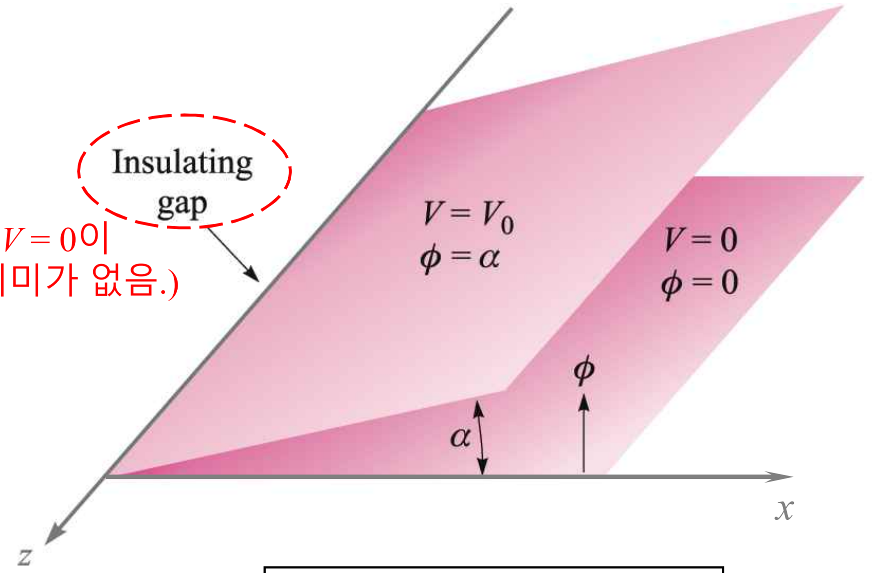
# [Ex. 3] Angled Plate Geometry

- Assumption: $V$ is a function of only $\phi$.

$$\frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0$$

$$\frac{d^2V}{d\phi^2} = 0$$

$$V = A\phi + B$$




- By applying B.C, $V(\phi) = V_0 \dfrac{\phi}{\alpha}$

Boundary Conditions:

1. $V = 0$ at $\phi = 0$
2. $V = V_0$ at $\phi = \alpha$

$$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{dV}{d\phi}\mathbf{a}_\phi = -\frac{V_0}{\alpha\rho}\mathbf{a}_\phi$$

$$\vec{E} = f(\rho) \neq f(\phi), \qquad \text{but} \quad V = f(\phi)$$

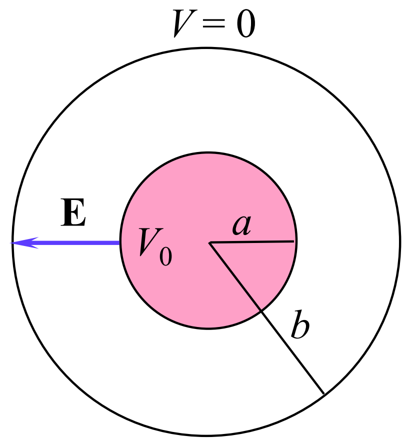
# [Ex. 4] Concentric Sphere Geometry

- Assumption: $V$ is a function of only $r$.

$$V = V_0 \frac{\frac{1}{r} - \frac{1}{b}}{\frac{1}{a} - \frac{1}{b}}$$

$$C = \frac{4\pi\varepsilon}{\frac{1}{a} - \frac{1}{b}}$$

Left in the exercise.




Boundary Conditions:

1. $V = 0$ at $r = b$
2. $V = V_0$ at $r = a$

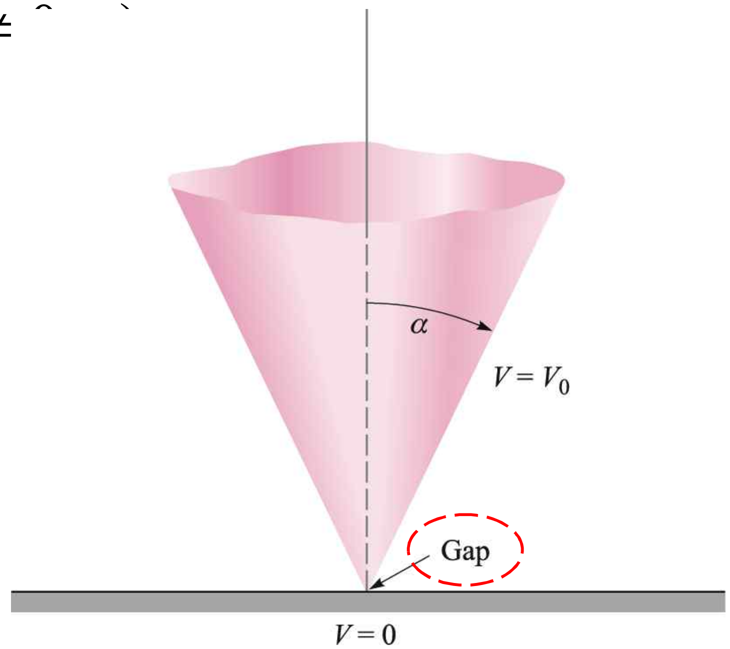
# [Ex. 5] $\theta$ – Dependent Potential Field

- Assumption: $V$ is a function of only $\theta$ in spherical coordinate.

$$\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right) = 0 \qquad (r \neq 0 \quad , \ \theta \neq$$

$$\sin\theta \frac{dV}{d\theta} = A$$

$$V = \int \frac{A\,d\theta}{\sin\theta} + B = A\ln\left(\tan\frac{\theta}{2}\right) + B$$




- Boundary condition 1:

$$0 = A \ln\tan\left(\frac{\pi}{4}\right) + B = B \quad \Rightarrow \quad B = 0$$

- Boundary condition 2:

$$V_0 = A \ln\tan\left(\frac{\alpha}{2}\right) \quad \Rightarrow \quad A = \frac{V_0}{\ln\tan(\alpha/2)}$$

$$V(\theta) = V_0 \frac{\ln\tan(\theta/2)}{\ln\tan(\alpha/2)}$$

Boundary Conditions:

1. $V = 0$ at $\theta = \pi/2$
2. $V = V_0$ at $\theta = \alpha$

$$\vec{E} = -\nabla V = -\frac{1}{r}\frac{\partial V}{\partial \theta}\vec{a}_\theta = -\frac{1}{r}\frac{V_0}{\ln(\tan\frac{\alpha}{2})}\cdot\frac{1}{\tan\frac{\theta}{2}}\cdot\sec^2\frac{\theta}{2}\cdot\frac{1}{2}\vec{a}_\theta$$

$$= -\frac{1}{r}\frac{V_0}{\ln(\tan\frac{\alpha}{2})}\cdot\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\cdot\frac{1}{\cos^2\frac{\theta}{2}}\cdot\frac{1}{2}\vec{a}_\theta = -\frac{V_0}{r\sin\theta\ln(\tan\frac{\alpha}{2})}\vec{a}_\theta$$

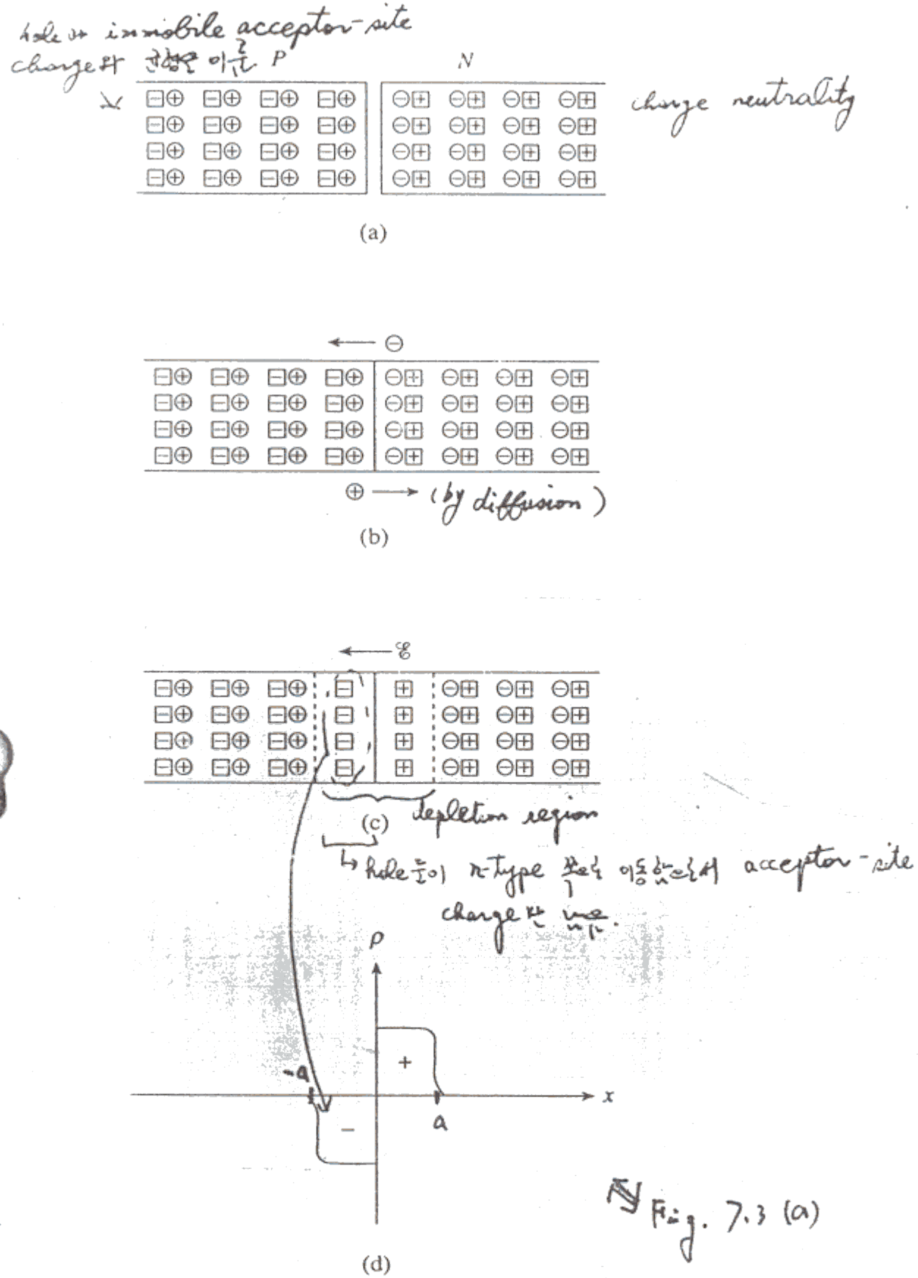
$$\rho_s = -\frac{\varepsilon V_0}{r\sin\alpha\ln(\tan\frac{\alpha}{2})}$$ : surface charge density on the cone

$$Q = -\frac{\varepsilon V_0}{\sin\alpha\ln(\tan\frac{\alpha}{2})}\int_0^\infty\int_0^{2\pi}\frac{1}{r}r\sin\alpha\, d\phi\, dr \qquad \theta = \alpha : \text{constant}$$

$$= -\frac{2\pi\varepsilon V_0}{\ln(\tan\frac{\alpha}{2})}\int_0^\infty dr \approx -\frac{2\pi\varepsilon V_0}{\ln(\tan\frac{\alpha}{2})}\int_0^{r_1} dr = \frac{2\pi\varepsilon V_O}{\ln\left(\frac{1}{\tan\frac{\alpha}{2}}\right)}r_1 = \frac{2\pi\varepsilon V_0 r_1}{\ln\left(\cot\frac{\alpha}{2}\right)}$$

$$C = \frac{Q}{V_0} \approx \frac{2\pi\varepsilon r_1}{\ln(\cot\frac{\alpha}{2})}$$

# 6.8 Example of the Solution of Poisson's Equation: The P-N Junction Capacitance




- P 및 N-type semiconductor 가 서로 격리되어 있음.
- P-type에서는 hole (⊕) 과 움직일 수 없는 acceptor-site charge (⊟)가 전하균형을 이루고 있음.
- N-type에서는 전자 (⊖) 와 움직일 수 없는 이온화된 donor (⊞ )들이 전하균형을 이루고 있음.
- P 및 N-type 반도체가 접합됨
- P-type에서는 N-type에 비해 hole 이, N-type에서는 P-type에 비해 전자가 많으므로 상대 영역으로 확신이 발생함.
- P-type에서는 N-type에 비해 hole 이, N-type에서는 P-type에 비해 전자가 많으므로 상대 영역으로 확신이 발생함.
- 결과적으로 접합(junction) 근처로부터 전하들이 확산해 나감에 따라 접합 근처에서는 불균형적으로 dopant들이 그림 (c)와 같이 남게 됨.

- Simple expression of charge distribution

$$\rho_v = 2\rho_{v0}\operatorname{sech}\frac{x}{a}\tanh\frac{x}{a} \qquad \text{--- (1)}$$

$a$ → Almost a half length of the depletion region

$$\rho_{v.\max} = \rho_{v0} \quad @\, x = 0.881a$$

- Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

$$\frac{d^2V}{dx^2} = -\frac{2\rho_{v0}}{\varepsilon}\operatorname{sech}\frac{x}{a}\tanh\frac{x}{a} \quad \leftarrow \quad \frac{d}{dx}\operatorname{sech}u = -\operatorname{sech}u\,\tanh u\frac{du}{dx}$$

- Assumption: one-dimensional problem

$$\frac{dV}{dx} = \frac{2\rho_{v0}a}{\varepsilon}\operatorname{sech}\frac{x}{a} + C_1 = -E_x$$

$$E_x = -\frac{2\rho_{v0}a}{\varepsilon}\operatorname{sech}\frac{x}{a} - C_1 \quad \leftarrow \quad \operatorname{sech}x = \frac{2}{e^x + e^{-x}}$$

- As $x \rightarrow \pm\infty$, $E_x \approx 0$ ➔ $C_1 = 0$

$$\therefore E_x = -\frac{2\rho_{v0} a}{\varepsilon} \operatorname{sech} \frac{x}{a} = -\frac{dV}{dx} \quad \text{--- (2)}$$

$$\int \operatorname{sech} u\, du = \sin^{-1}(\tanh u)$$

$$= 2 \tan^{-1} e^{u}$$

$$V = \frac{4\rho_{v0} a^2}{\varepsilon} \tan^{-1}[e^{\frac{x}{a}}] + C_2$$

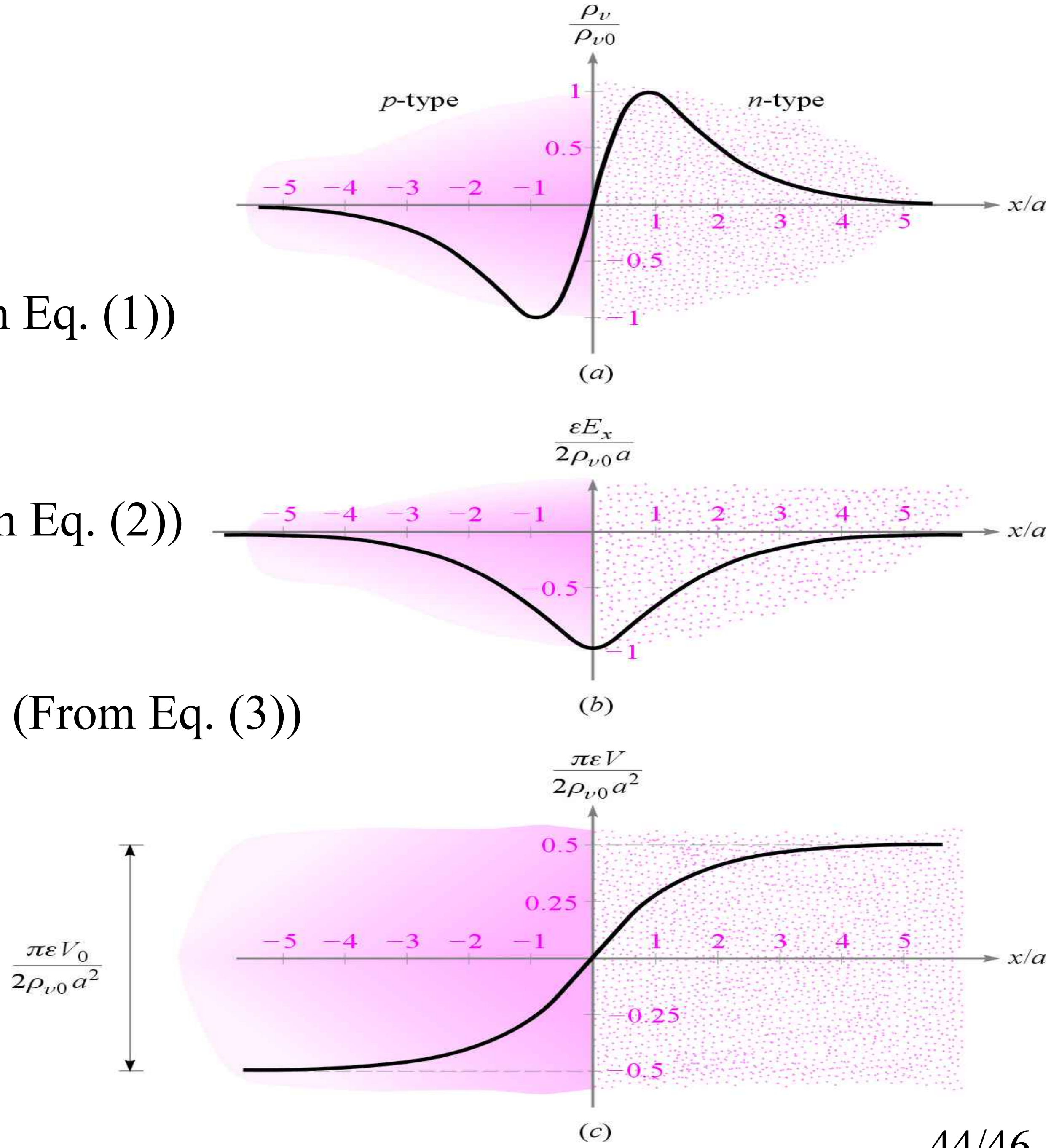
- Since $V = 0$ @ $x = 0$ (arbitrarily zero reference plane),

$$0 = \frac{4\rho_{v0} a^2}{\varepsilon} \cdot \frac{\pi}{4} + C_2$$

$$C_2 = -\frac{4\rho_{v0} a^2}{\varepsilon} \cdot \frac{\pi}{4}$$

$$\therefore V = \frac{4\rho_{v0} a^2}{\varepsilon} [\tan^{-1}(e^{\frac{x}{a}}) - \frac{\pi}{4}] \quad \text{--- (3)}$$

$$\begin{cases} \dfrac{\rho_v}{\rho_{v0}} = 2\operatorname{sech}\dfrac{x}{a}\tanh\dfrac{x}{a} & \text{(From Eq. (1))} \\[2ex] \dfrac{\varepsilon E_x}{2\rho_{v0}a} = -\operatorname{sech}\dfrac{x}{a} & \text{(From Eq. (2))} \\[2ex] \dfrac{\pi\varepsilon V}{2\rho_{v0}a^2} = 2\pi[\tan^{-1}(e^{\frac{x}{a}}) - \dfrac{\pi}{4}] & \text{(From Eq. (3))} \end{cases}$$



- Total potential difference $V_0$ across the junction:

$$V_0 = V_{x\to\infty} - V_{x\to-\infty} = \frac{4\rho_{vo}a^2}{\varepsilon}\left\{\left[\frac{\pi}{2}-\frac{\pi}{4}\right]-\left[0-\frac{\pi}{4}\right]\right\} = \frac{4\rho_{v0}a^2}{\varepsilon}\frac{\pi}{2}$$

$$= \frac{2\pi\rho_{v0}a^2}{\varepsilon} \quad \Rightarrow \quad a = \sqrt{\frac{\varepsilon V_0}{2\pi\rho_{v0}}}$$

- Total positive charge:

$$Q = S\int_0^\infty \rho_v dx$$

$\leftarrow$ $Q = \int \rho_v dv = S\int \rho_v dx$

단면적 $S$ =constant

$$= S\int_0^\infty 2\rho_{v0}\operatorname{sech}\frac{x}{a}\tanh\frac{x}{a}dx$$

$$= 2S\rho_{v0}a[-\operatorname{sech}\frac{x}{a}]_0^\infty = 2\rho_{v0}aS$$

$$= 2\rho_{v0}S\sqrt{\frac{\varepsilon V_0}{2\pi\rho_{vo}}} = S\sqrt{\frac{2\rho_{v0}\varepsilon V_0}{\pi}}$$

$$I = \frac{dQ}{dt} = C\frac{dV_0}{dt}$$

$$C = \frac{dQ}{dV_0} = S\sqrt{\frac{2\rho_{v0}\varepsilon}{\pi}} \cdot \frac{1}{2}\sqrt{\frac{1}{V_0}}$$

$$= \frac{S}{2}\sqrt{\frac{2\rho_{v0}\varepsilon}{\pi}} \cdot \sqrt{\frac{\varepsilon}{2\pi\rho_{v0}a^2}}$$

$$= \frac{\varepsilon S}{2\pi a} \qquad \leftarrow C = \varepsilon\frac{S}{d}$$

$$C = f\left(\frac{1}{\sqrt{V_0}}\right)$$

← $C$ = constant 인 경우
high $V_0$는 high $a$ (or high separation) 필요