Engineering Electromagnetics

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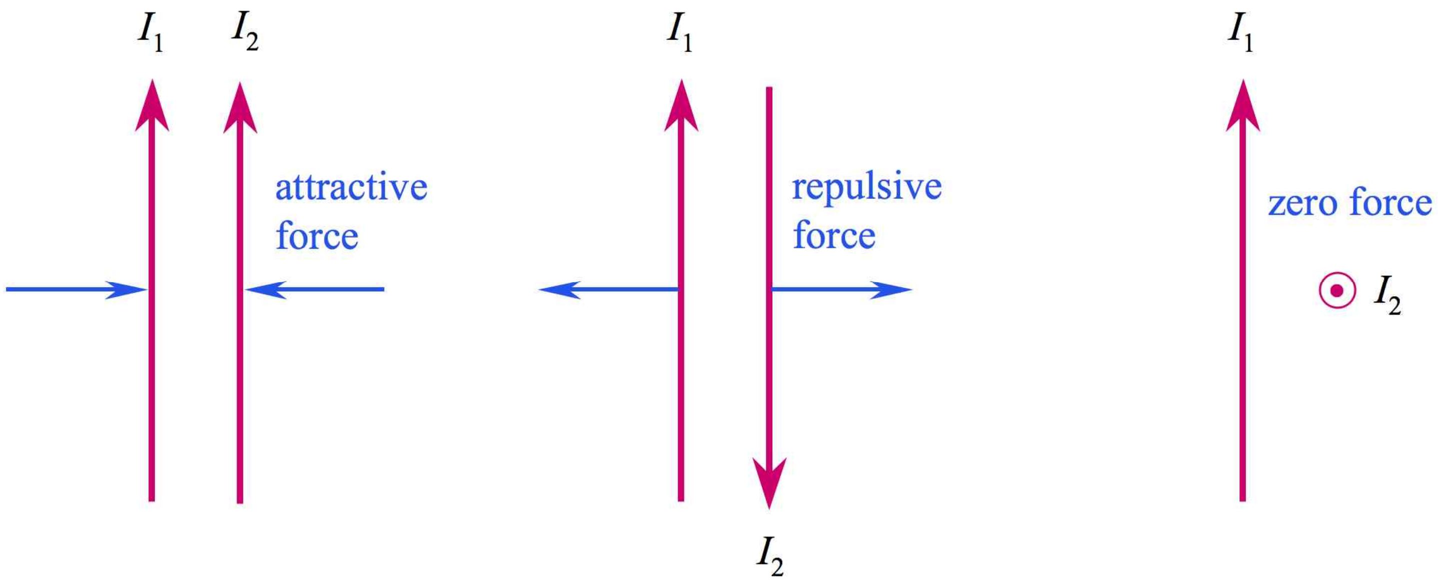
Chapter 7:

The Steady Magnetic Field

Motivating the Magnetic Field Concept: Forces Between Currents

- Magnetic forces due to charge motioning (or current)



- How can we describe a force field around wire 1 that can be used to determine the force on wire 2?

7.1 Biot-Savart Law

- Source of the steady magnetic field:
  1) Permanent magnet
  2) An electric field changing linearly with time
  3) **Direct current (DC)**

- Differential current element: (vanishingly) very small current section of current-carrying filamentary conductor where cross section radius approaches zero.

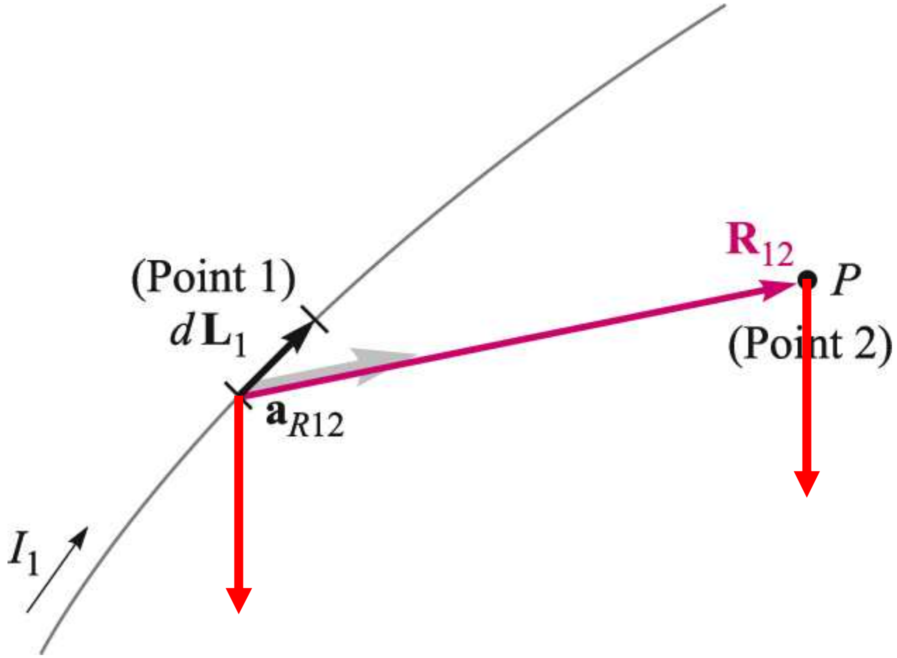
- **H**: magnetic field intensity [A/m]

- Biot-Savart Law (or Ampere's law for current element)

$$d\vec{H} = \frac{Id\vec{L} \times \vec{a}_R}{4\pi R^2} \qquad \leftarrow \vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$$

$$= \frac{Id\vec{L} \times \vec{R}}{4\pi R^3}$$

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$



where P_1 : current element location

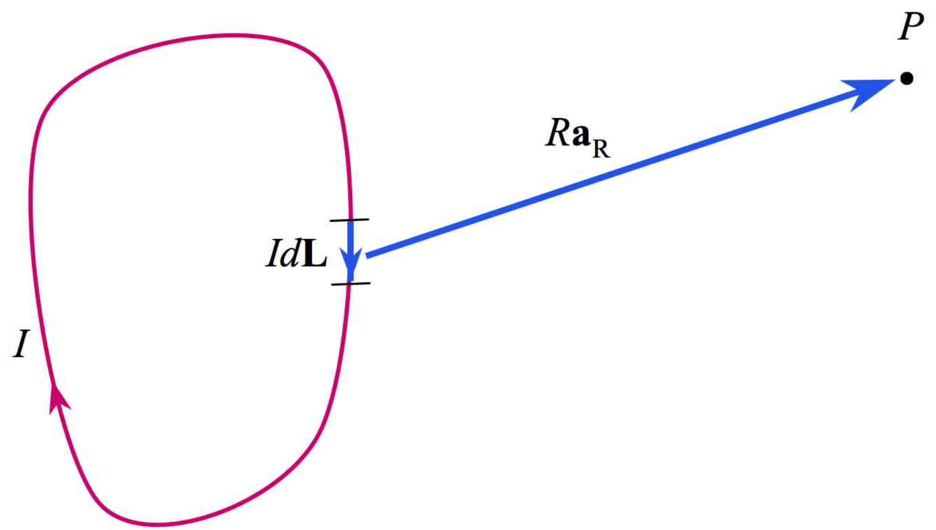
P_2 : (magnetic) field measurement point location

cf.) Coulomb's Law

A point charge of magnitude dQ_1 at point 1 would generate electric field at point 2 as like

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi \epsilon_0 R_{12}^2}$$

- Biot-Savart Law can't be checked experimentally because the differential current element cannot be isolated.



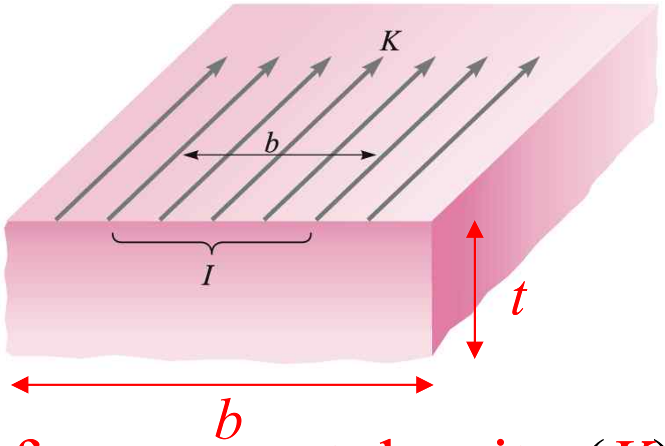
- Since the magnetic field at point *P*, associated with the differential current element $Id\mathbf{L}$

$$d\mathbf{H} = \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2},$$

the total field arising from the closed circuit path is $\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$

- Expressions of Biot-Savart Law on two- and three-dimensional distributed current source ($\vec{J}$: current density, $\vec{K}$: surface current density)
- Current may be expressed in terms of current density (J):

$$I = JS = J(bt) = \text{constant}$$



- If $t \to 0$, (sheet current)

$$I = \lim_{t \to 0} J \cdot (bt) = \text{constant}, \quad \therefore J \to \infty$$

➔ meaningless where J: [A/m^2]

- So current may be expressed in terms of surface current density (K).

Current: $I = Kb$,

where the width b is measured perpendicularly to the direction in which the current is flowing.

- For non-uniform surface current density,

$$I = \int K dN$$

where dN: differential element of path across flowing current

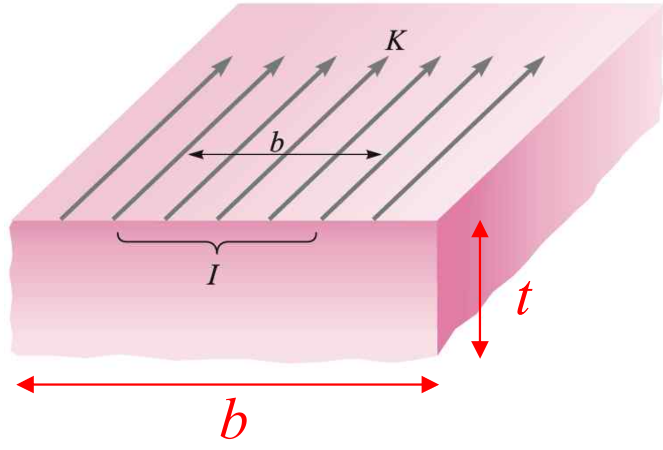
- Current expressions

$$I d\vec{L} = \vec{K} dS = \vec{J} dv$$

$b \times t \times dL = dv$

$b \times dL = dS$

미소 전류소 (differential current element)



where **K** [A/m] : uniform surface current density

- Alternate forms of the Biot-Savart Law

$$\vec{H} = \int_S \frac{\vec{K} dS \times \vec{a}_R}{4\pi R^2} = \int_S \frac{\vec{K} \times \vec{a}_R dS}{4\pi R^2}$$

$$\overset{\text{or}}{=} \int_{vol} \frac{\vec{J} dv \times \vec{a}_R}{4\pi R^2} = \int_{vol} \frac{\vec{J} \times \vec{a}_R dv}{4\pi R^2}$$

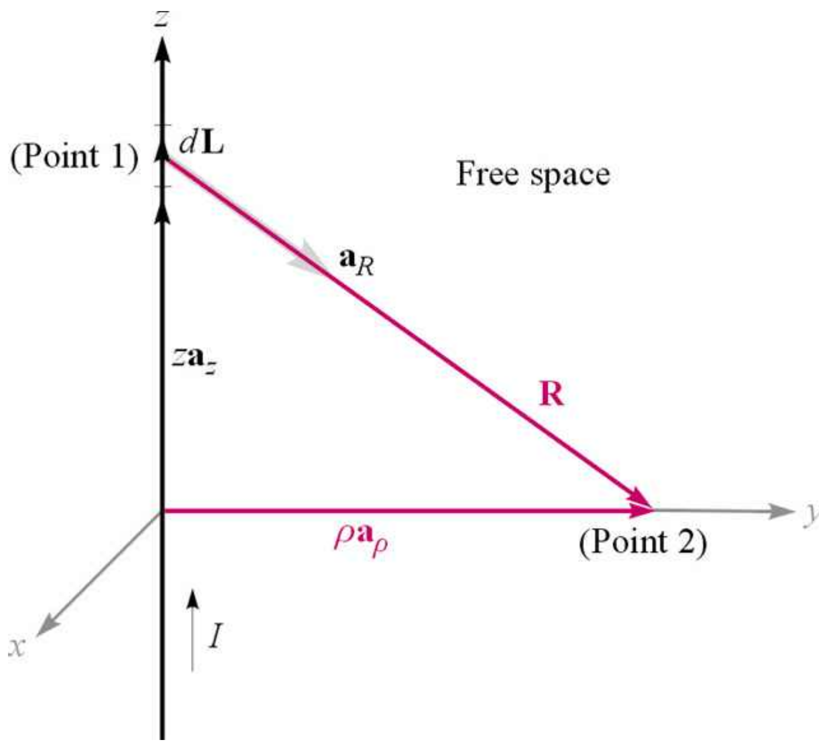
Example of the Biot-Savart Law

- Magnetic field intensity on y axis (equivalently in xy plane) arising from a filament current element of infinite length on z axis

- Source location: $\overrightarrow{r'} = z'\vec{a}_z$
  Measurement field point: $\vec{r} = \rho\vec{a}_\rho$

$$\therefore \vec{R}_{12} = \vec{r} - \vec{r}' = \rho\vec{a}_\rho - z'\vec{a}_z$$

$$\vec{a}_{R12} = \frac{\rho\vec{a}_\rho - z'\vec{a}_z}{\sqrt{\rho^2 + z'^2}}$$



- Since $d\vec{L} = dz'\vec{a}_z$,

$$d\vec{H}_2 = \frac{Idz'\vec{a}_z \times \left(\rho\vec{a}_\rho - z'\vec{a}_z\right)}{4\pi\left(\rho^2 + z'^2\right)^{3/2}}$$

$$\vec{H}_2 = \int_{-\infty}^{\infty} \frac{I dz' \vec{a}_z \times \left(\rho \vec{a}_\rho - z' \vec{a}_z\right)}{4\pi\left(\rho^2 + z'^2\right)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho dz' \vec{a}_\phi}{\left(\rho^2 + z'^2\right)^{3/2}} = \frac{I\rho\vec{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{\left(\rho^2 + z'^2\right)^{3/2}}$$

$$z' = \rho \tan\theta \;\rightarrow\; dz' = \rho \sec^2\theta d\theta$$

$$z' : -\infty \leftrightarrow \infty \qquad \theta : -\frac{\pi}{2} \leftrightarrow \frac{\pi}{2}$$

$$\vec{H}_2 = \frac{I\rho\vec{a}_\phi}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho \sec^2\theta d\theta}{\left(\rho^2 + \rho^2\tan^2\theta\right)^{3/2}} = \frac{I\rho\vec{a}_\phi}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho \sec^2\theta}{\rho^3 \sec^3\theta} d\theta$$

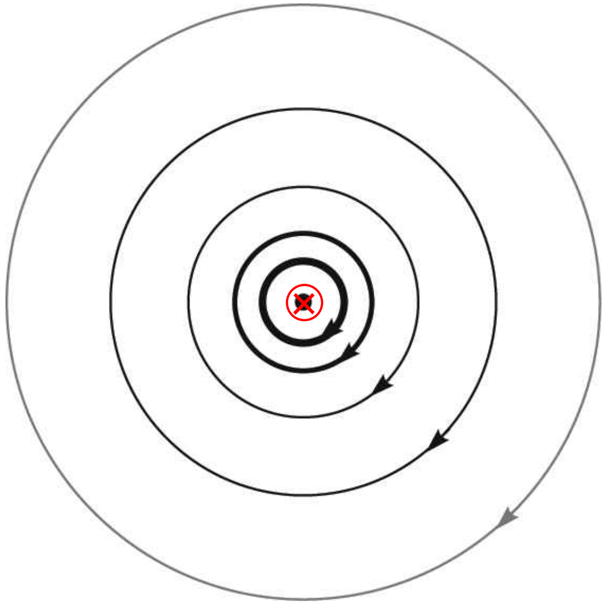
$$= \frac{I\vec{a}_\phi}{4\pi\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = \frac{I\vec{a}_\phi}{4\pi\rho} \left[\sin\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{I}{2\pi\rho}\vec{a}_\phi$$

$$\left(\therefore \left|\vec{H}\right| = f(\rho) \neq f(\phi, z)\right)$$

- Current go into the page.
- Magnetic field streamlines are concentric circles, whose magnitudes decrease as the inverse distance from the z axis

$$\vec{H} \propto \frac{1}{\rho}\vec{a}_\phi$$

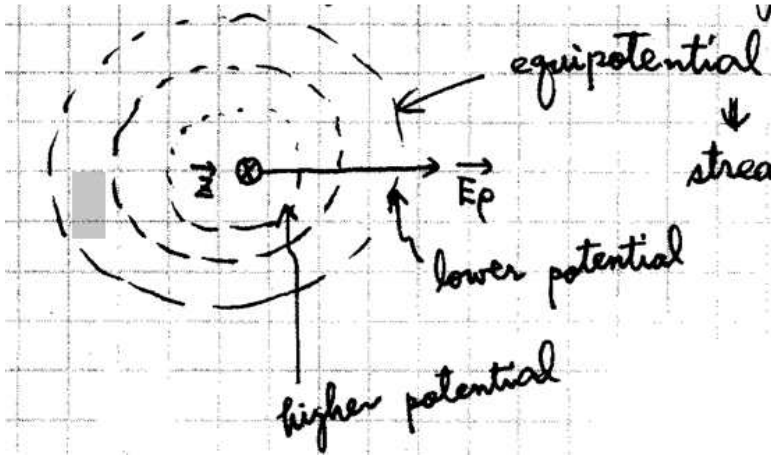
$$\vec{a}_H = \vec{a}_\phi$$



cf.) Electric field intensity for line charge

$$\vec{E} = \frac{\rho_L}{2\pi\varepsilon_o\rho}\vec{a}_\rho$$

$$\vec{E} \propto \frac{1}{\rho}\vec{a}_\rho \quad \text{and} \quad \vec{a}_E = \vec{a}_\phi$$



[Ex.] (Magnetic) Field arising from a finite current segment

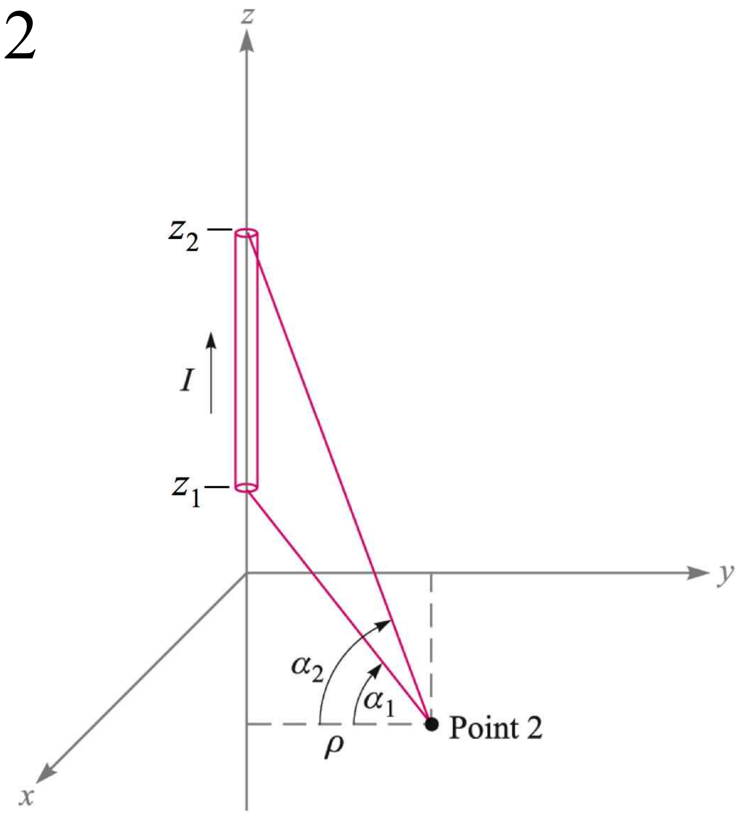
- Field to be found in the *xy* plane at point 2

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

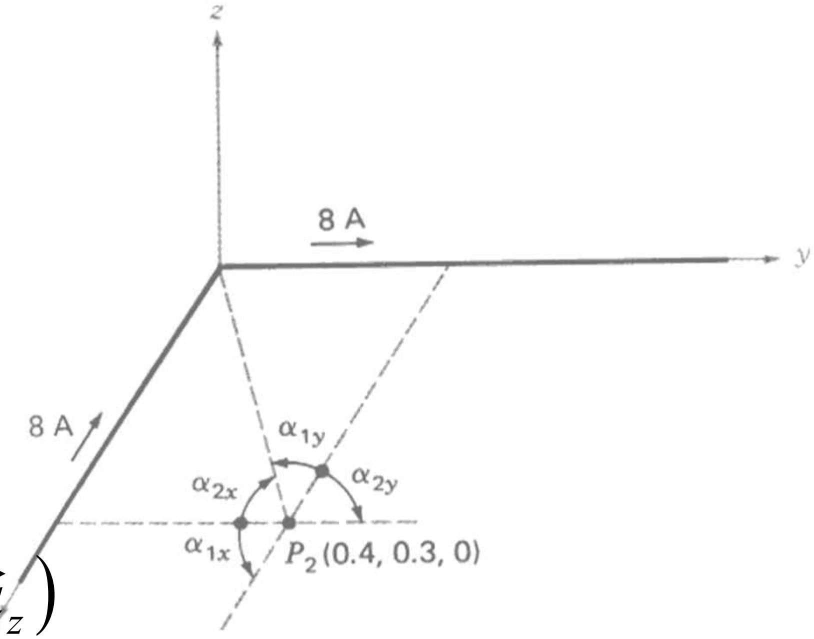
$$\vec{H}_2 = \frac{I\vec{a}_\phi}{4\pi\rho}\left[\sin\theta\right]_{\alpha_1}^{\alpha_2} = \frac{I}{4\pi\rho}\left[\sin\alpha_2 - \sin\alpha_1\right]\vec{a}_\phi$$

where

$$\left(\begin{array}{l} z' = \rho\tan\theta \quad \tan\theta = \dfrac{z'}{\rho} \\ \theta = \tan^{-1}\dfrac{z'}{\rho} \\ \alpha_2 = \tan^{-1}\dfrac{z_2}{\rho} \\ \alpha_1 = \tan^{-1}\dfrac{z_1}{\rho} \end{array} \right)$$



[Ex. 7.1]



$$\alpha_{1x} = -90^{\circ}$$

$$\alpha_{2x} = \tan^{-1}\left(\frac{0.4}{0.3}\right) = 53.1^{\circ}$$

$$\vec{H}_{2(x)} = \frac{8}{4\pi \times (0.3)}\left[\sin 53.1^{\circ} - \sin(-90^{\circ})\right](-\vec{a}_z)$$

$$= -\frac{2}{0.3\pi}(1.8)\vec{a}_z = -\frac{12}{\pi}\vec{a}_z \leftarrow Id\vec{\mathrm{L}} \times \vec{a}_R = -Idx\vec{a}_x \times \vec{a}_y = -Idx\vec{a}_z$$

$$\alpha_{1y} = -\tan^{-1}\left(\frac{0.3}{0.4}\right) = -36.9^{\circ} \qquad \alpha_{2y} = 90^{\circ}$$

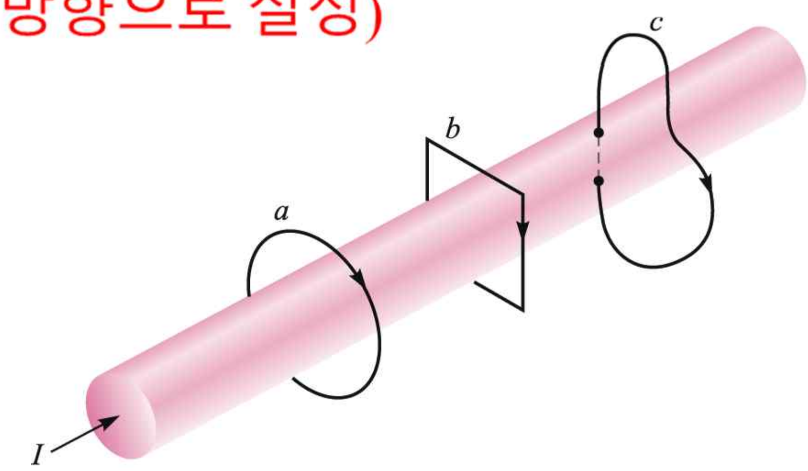
$$\vec{H}_{2(y)} = \frac{8}{4\pi \times (0.4)}\left[\sin 90^{\circ} - \sin(-36.9^{\circ})\right](-\vec{a}_z) = -\frac{8}{\pi}\vec{a}_z \ [\mathrm{A/m}]$$

$$\therefore \vec{H}_2 = \vec{H}_{2(x)} + \vec{H}_{2(y)} = \left(-\frac{12}{\pi} - \frac{8}{\pi}\right)\vec{a}_z = -\frac{20}{\pi}\vec{a}_z = -6.37\vec{a}_z \ [\mathrm{A/m}]$$

7.2 Ampere's Circuital Law

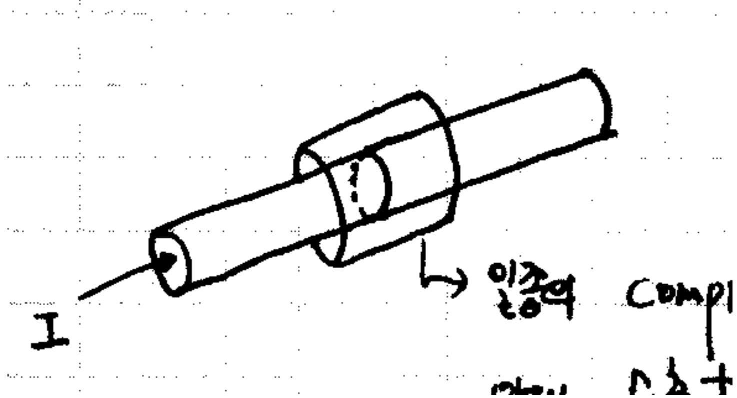
- The line integral of **H** about *any closed path* is exactly equal to the direct current enclosed by that path. (Proof: 7-7)
- Define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed. ($\vec{H}$-field를 오른나사 방향으로 회전하는 방향으로 선적분할 때 나사진행 방향을 (+) 전류 진행방향으로 설정)

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

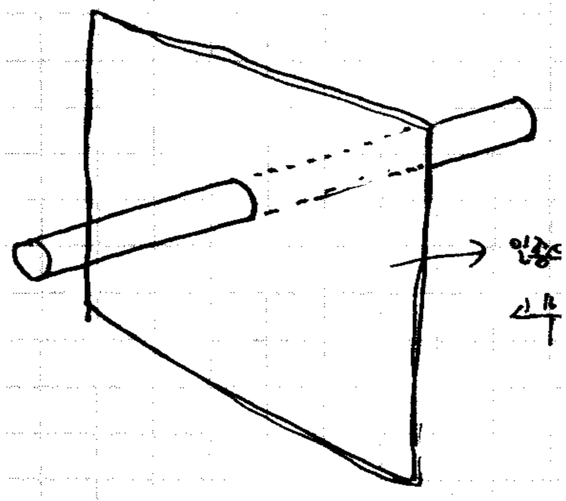


1. Since the closed paths *a* and *b* include current path, $\oint \vec{H} \cdot d\vec{L} = I$
2. Since the integral over path *c* include a portion of current path, $\oint \vec{H} \cdot d\vec{L} \neq I$
3. The direction of current is decided by right-handed screw direction path.
4. Paths *a* and *b* are different integration paths, the current are same.

- Compression tube connecting two current wires



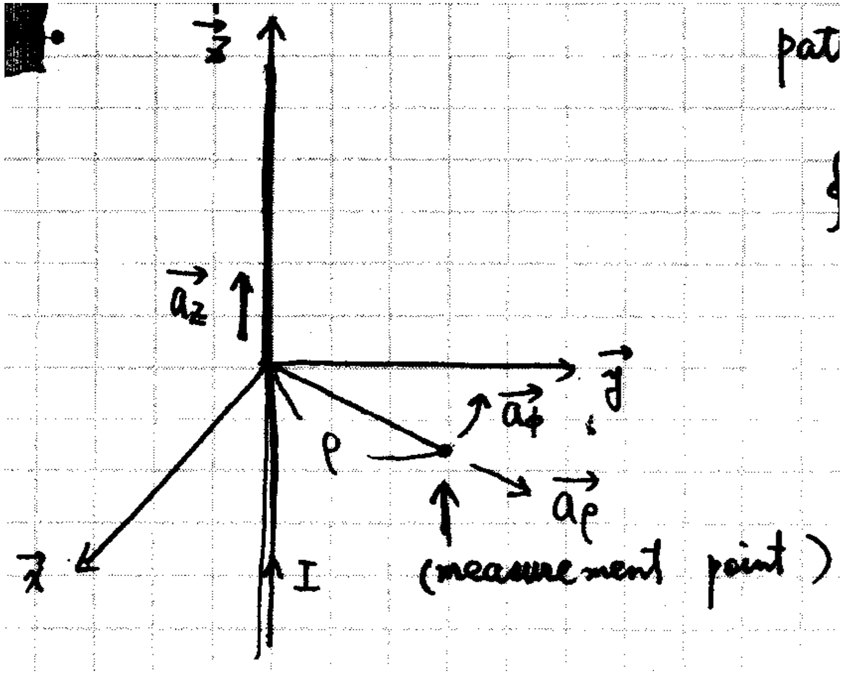
- Rubber plate passing through current wire

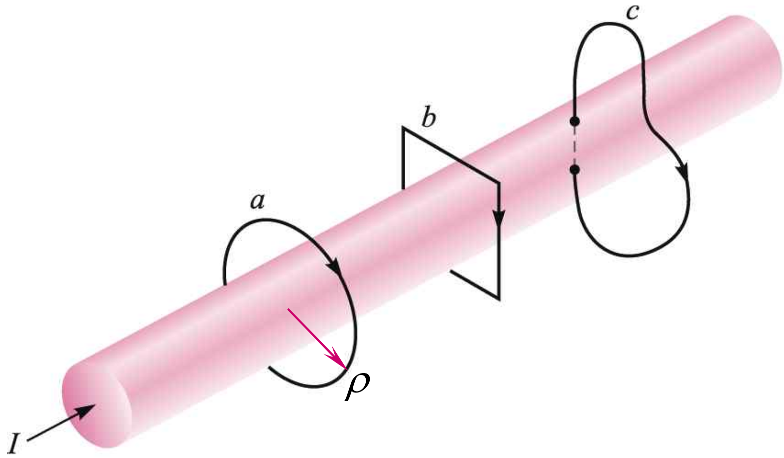


Total currents passing through compression tube and rubber plate are same.

cf.) Gauss' law: 폐곡면 내부의 전하량 유도
Ampere's: 폐곡선 내부의 전류량 유도

[Ex. 1] Path is a circle of radius (ρ).



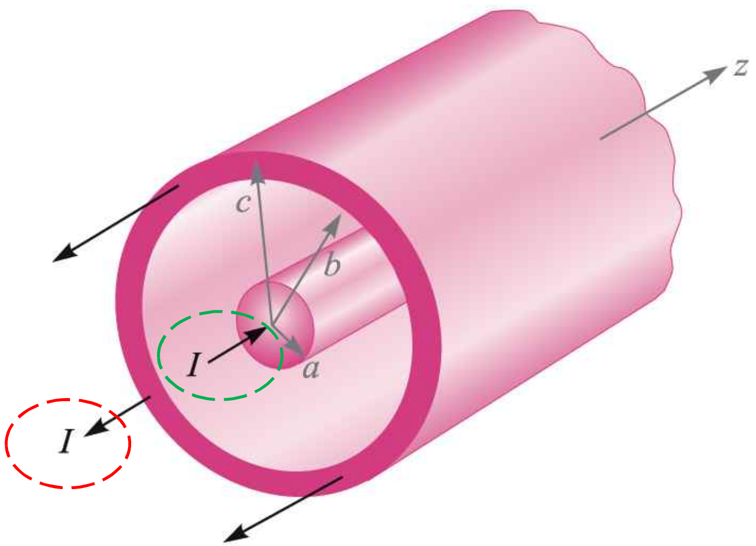


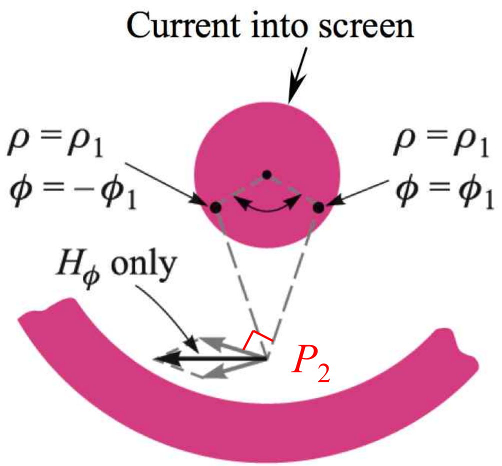
$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

$$\therefore H_\phi = \frac{I}{2\pi\rho} \quad \rightarrow \quad \vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi$$

[Ex. 2] Infinitely long coaxial transmission line.

- In the coax line, we have two concentric *solid* conductors that carry equal and opposite currents, I.

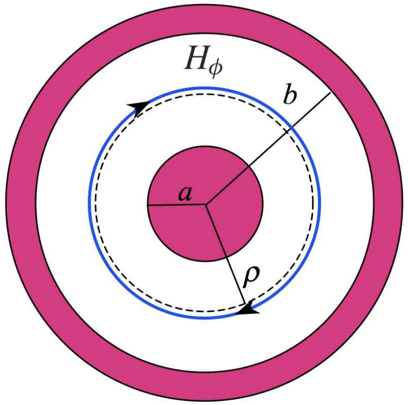




(coaxial cable은 단면적을 가지고 있으므로 하나의 current filament 가 아니고, 무한개의 current filament의 합으로 생각할 수 있음.)
(임의의 전류 filament에 의한 P_2점에서 느끼는 $\vec{H}_2$ 는 단순히 $\vec{a}_\rho$ 및 $\vec{a}_\phi$의 함수가 아님. 그러나 전류 filament의 symmetric 특성에 의해 최종적으로 생긴 $\vec{H}_2$ 는 $\vec{a}_\phi$의 함수가 됨.)

- Case 1: $a \le \rho \le b$

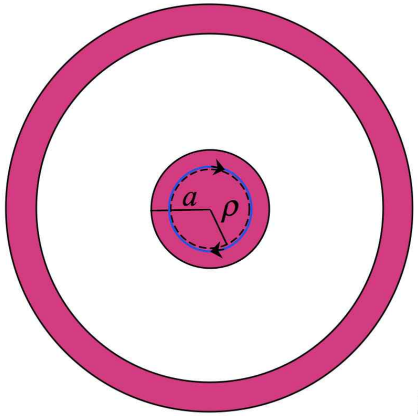
$$H_\phi = \frac{I}{2\pi\rho}$$ ($\rho > a$ 이므로 전류 I를 전부 고려)



- Case 2: $\rho \le a$

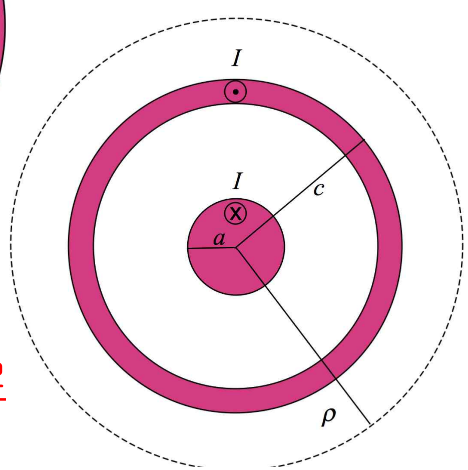
$$I_{encl} = I\frac{\pi\rho^2}{\pi a^2} = I\frac{\rho^2}{a^2}$$ (closed current)

$$\therefore 2\pi\rho H_\phi = I\frac{\rho^2}{a^2} \qquad H_\phi = \frac{I\rho}{2\pi a^2}$$



- Case 3: $\rho \ge c$

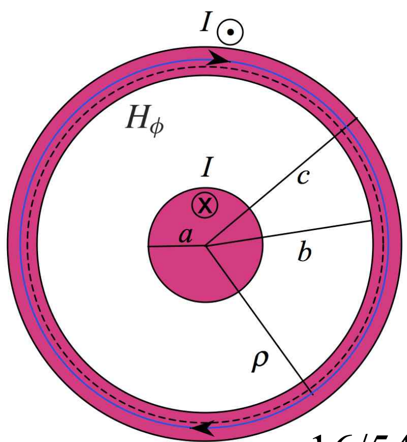
$I_{\mathrm{encl}} = 0$ ➔ $H_\varphi = 0$ (내부에 포함된 유효 전류합이 "0"이므로



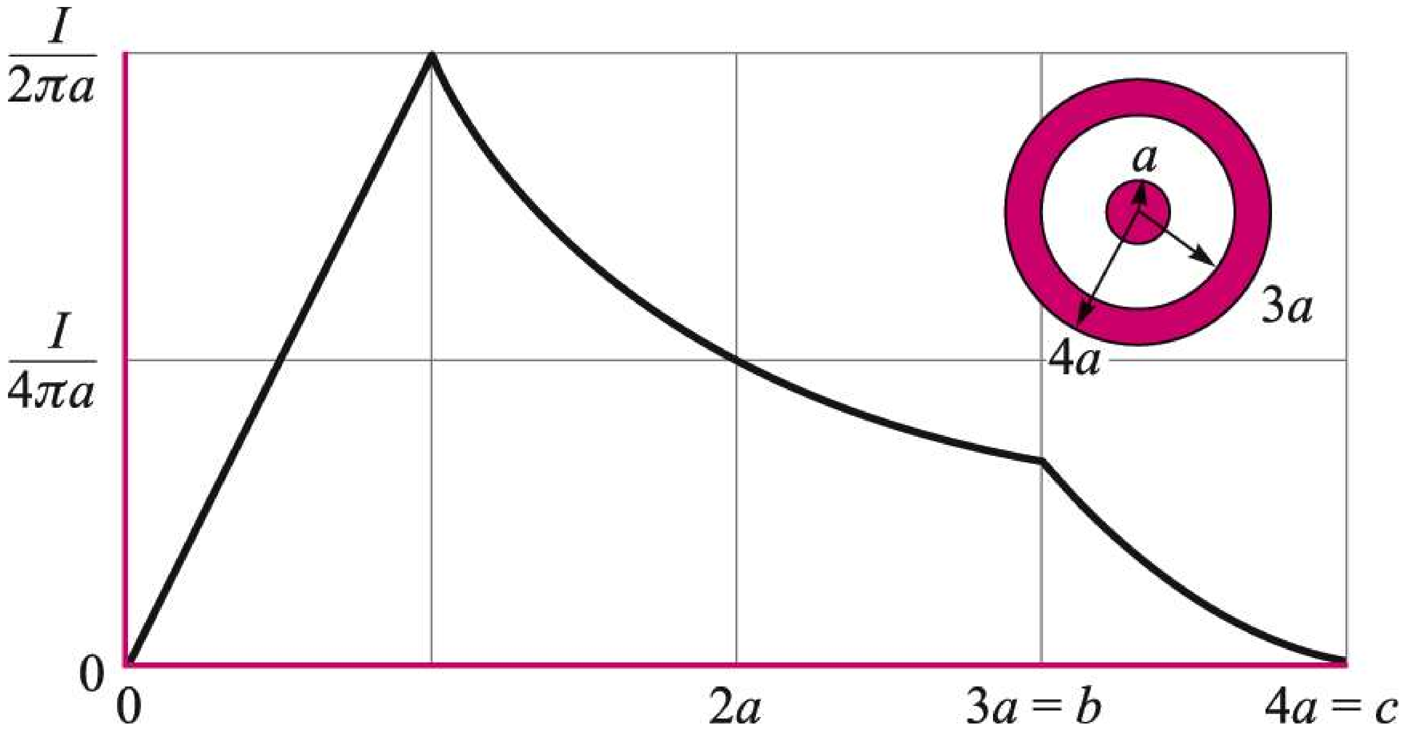
- Case 4: $b \le \rho \le c$

$$2\pi\rho H_\phi = I - I\frac{\rho^2\pi - b^2\pi}{c^2\pi - b^2\pi} = I - I\frac{(\rho^2 - b^2)\pi}{(c^2 - b^2)\pi} = I\frac{c^2 - \rho^2}{c^2 - b^2}$$

$$H_\phi = \frac{I}{2\pi\rho}\frac{c^2 - \rho^2}{c^2 - b^2}$$



- $\vec{H}$-field variation for a coaxial cable with $b = 3a$, $c = 4a$



1) H –filed is continuous at all the conductor boundaries.

2) The external H-field at outer side of outer conductor is zero.

➔ Equal positive and negative currents would not produce any noticeable effect in an adjacent circuit.

7.2.4 Magnetic Field Arising from a Current Sheet

- Current sheet $\vec{K} = K_y \vec{a}_y \quad @ z = 0$

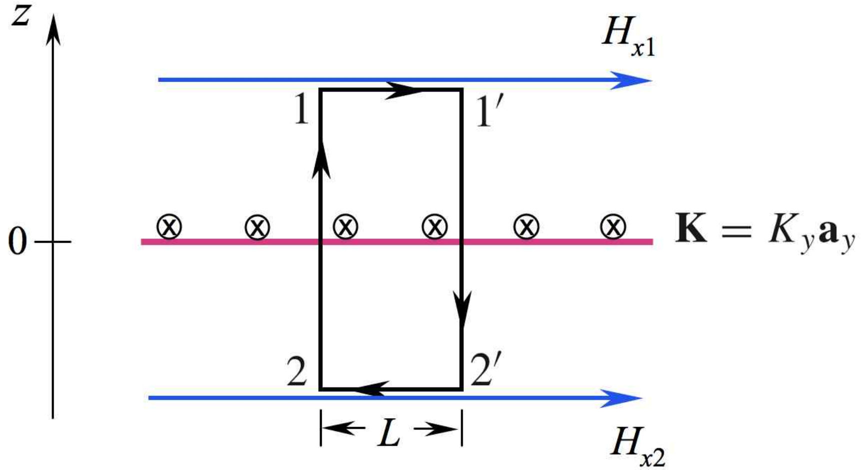
$$\Rightarrow H_y = 0 \quad \leftarrow (d\vec{H} = \frac{\vec{K}dS \times \vec{a}_R}{4\pi R^2}, \quad \vec{H} \perp \vec{K})$$

- Current sheet is subdivided into a number of filaments.

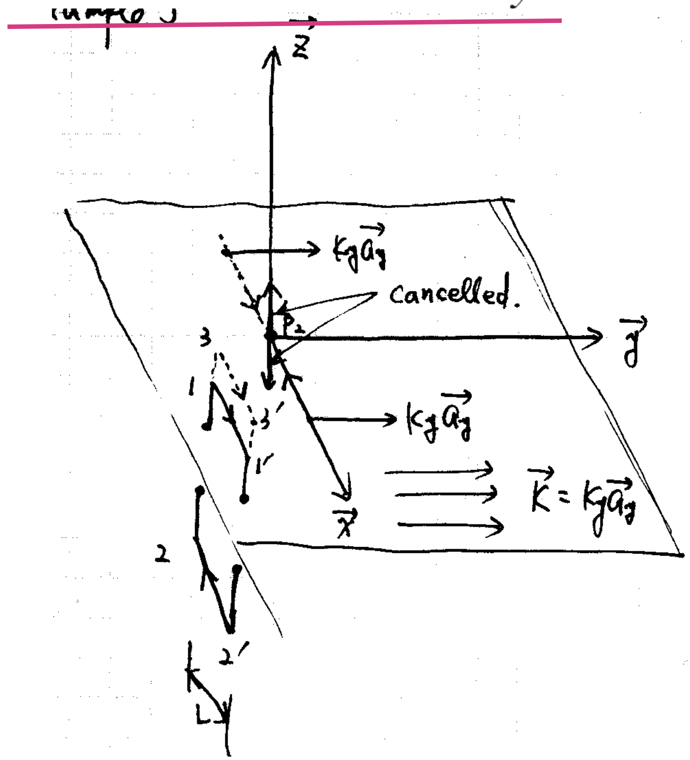
$H_z = 0$ ($\because$ By a symmetrically located pair of filaments) ➔ $H_x \neq 0$

- Choose 1-1'-2'-2-1 path

$$H_{x1}L + H_{x2}(-L) = K_y L \quad \text{or} \quad H_{x1} - H_{x2} = K_y$$



edge view

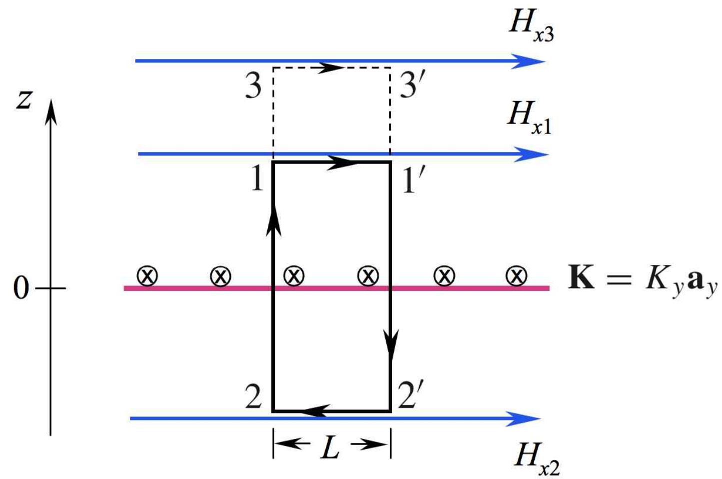


- Choose 3-3'-2'-2-3 path

$$H_{x3}L + H_{x2}(-L) = K_y L \qquad \Leftarrow \; H_{x1} - H_{x2} = K_y$$

$$H_{x3} - H_{x2} = K_y \quad \Rightarrow \quad H_{x3} = H_{x1}$$

($\therefore H_x$ is the same for all positive z.)



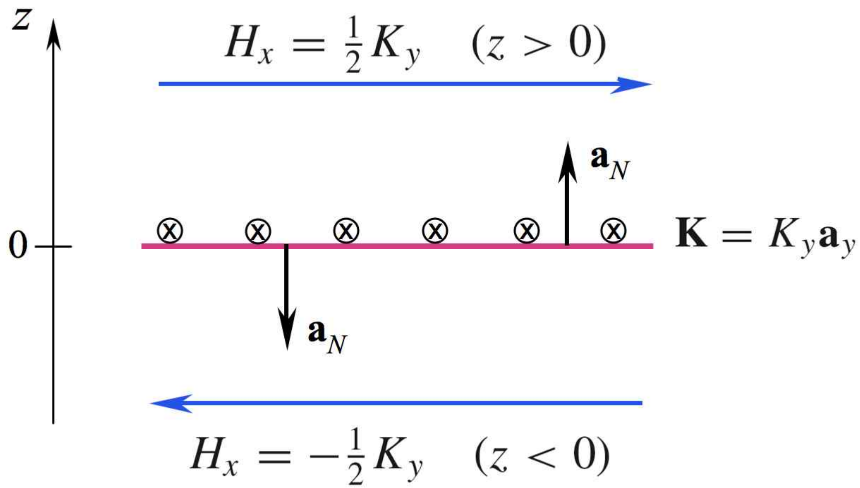
- By symmetric property,

$$H_x = \frac{1}{2}K_y \;\; (@\, z > 0)$$

(H-field on paths 1-1' and 3-3')

$$H_x = -\frac{1}{2}K_y \;\; (@\, z < 0)$$

(H-field on path 2-2')



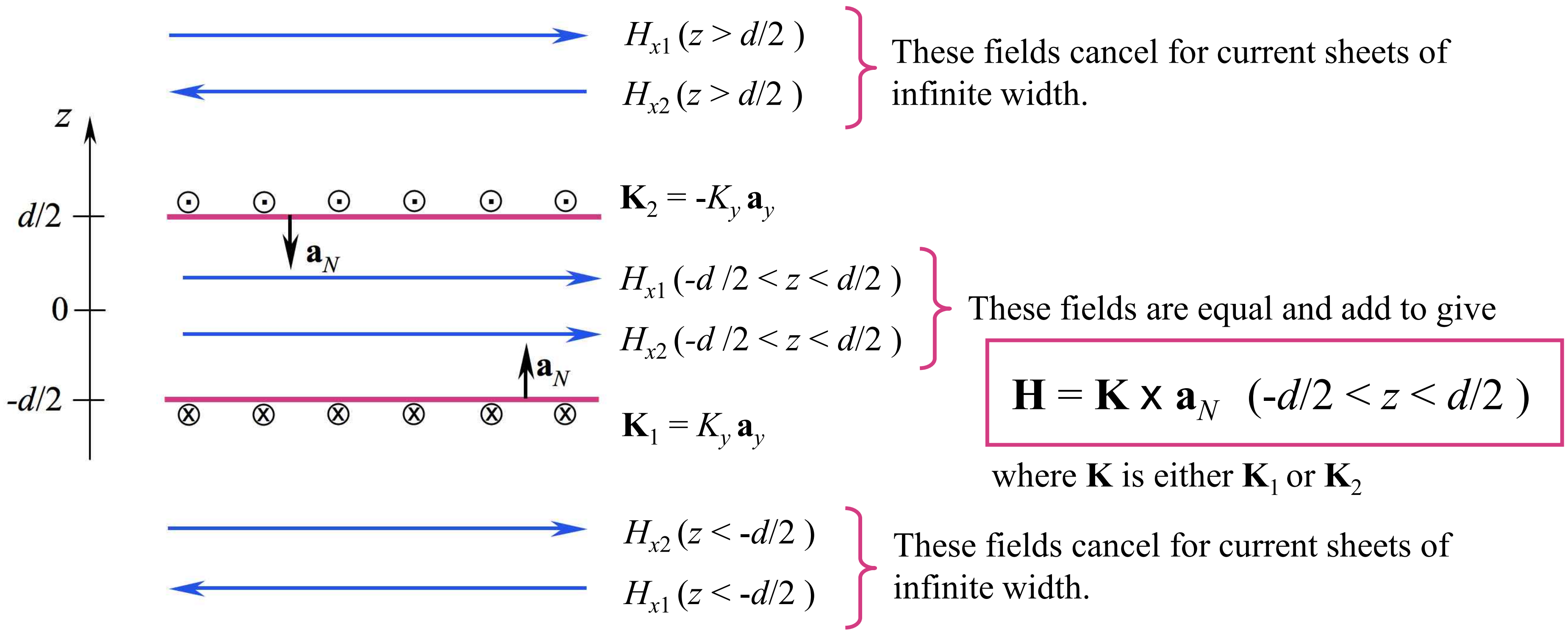
$$\Rightarrow \vec{H} = \frac{1}{2}\vec{K} \times \vec{a}_N \Leftrightarrow \vec{H} \neq f(\text{distance}) \quad \text{cf.)}\,\vec{E} = \frac{\rho_s}{2\varepsilon_o}\vec{a}_N$$

where $\vec{a}_N$: unit vector normal to current sheet

- Magnetic field intensity in case two current sheets are located at

$\vec{K}_1 = K_y \vec{a}_y$ (@ z = $-d/2$) and $\vec{K}_2 = -K_y \vec{a}_y$ (@ $z = d/2$)

➔ $\vec{H} = K_y \vec{a}_x$ (@ $-d/2 < z < d/2$) and $\vec{H} = 0$ (@ $z < -d/2$, $z > d/2$)

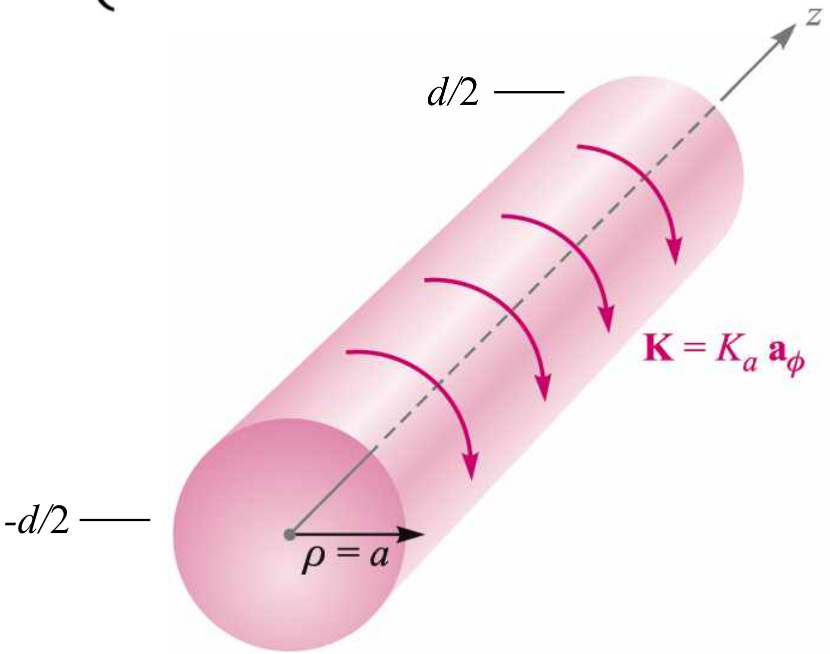


Magnetic Fields within Solenoids and Toroids

- Infinite long solenoid for radius a and uniform current density $K_a \vec{a}_\phi$ [A/m]. (Exercise #7-13]

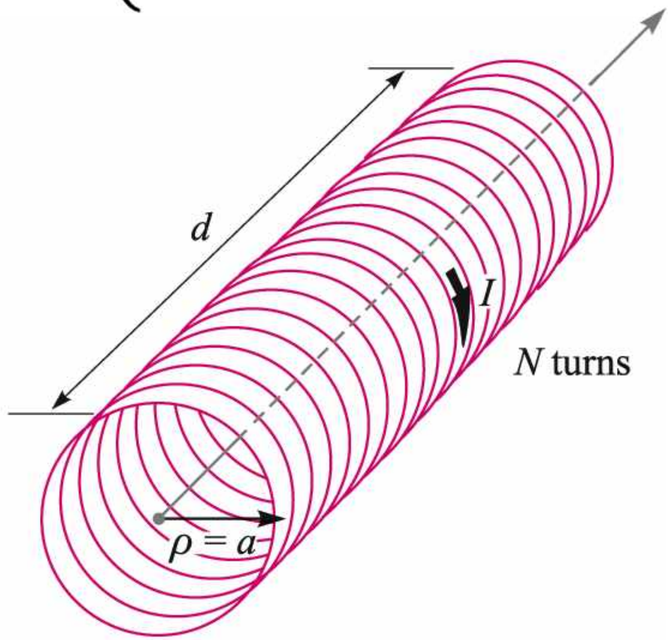
$$\begin{cases} \vec{H} = K_a \vec{a}_z & (@\, \rho < a) \\ \vec{H} = 0 & (@\, \rho > a) \end{cases}$$

$$\begin{cases} \vec{H} = \dfrac{NI}{d} \vec{a}_z & (@\, \rho < a) \\ \vec{H} = 0 & (@\, \rho > a) \end{cases}$$



$\mathbf{H} = K_a \mathbf{a}_z$, $\rho < a$
$\mathbf{H} = 0$, $\rho > a$

(a)



$\mathbf{H} = \dfrac{NI}{d} \mathbf{a}_z$
(well inside coil)

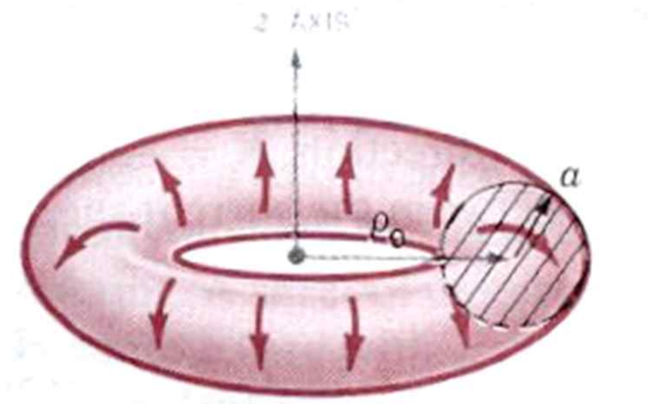
(b)

Surface Current Model of a Toroid

$$\vec{K} = K_a \vec{a}_z \ \ @\rho = \rho_0 - a, z = 0$$

$$\begin{cases} \vec{H} = K_a \dfrac{\rho_0 - a}{\rho} \vec{a}_\phi & \text{(inside toroid)} \\ \vec{H} = 0 & \text{(outside)} \end{cases}$$

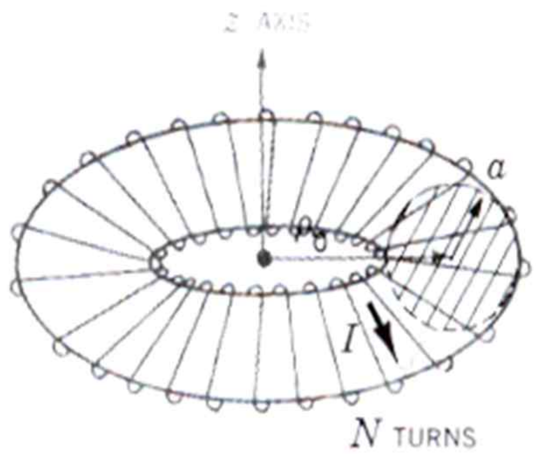
$$\begin{cases} \vec{H} = \dfrac{NI}{2\pi\rho} \vec{a}_\phi & \text{(inside toroid)} \\ \vec{H} = 0 & \text{(outside)} \end{cases}$$



$\mathbf{K} = K_a \mathbf{a}_z$ at $\rho = \rho_0 - a,\ z = 0$

$\mathbf{H} = K_a \dfrac{\rho_0 - a}{\rho} \mathbf{a}_\phi$ (INSIDE TOROID)

$\mathbf{H} = 0$ (OUTSIDE)

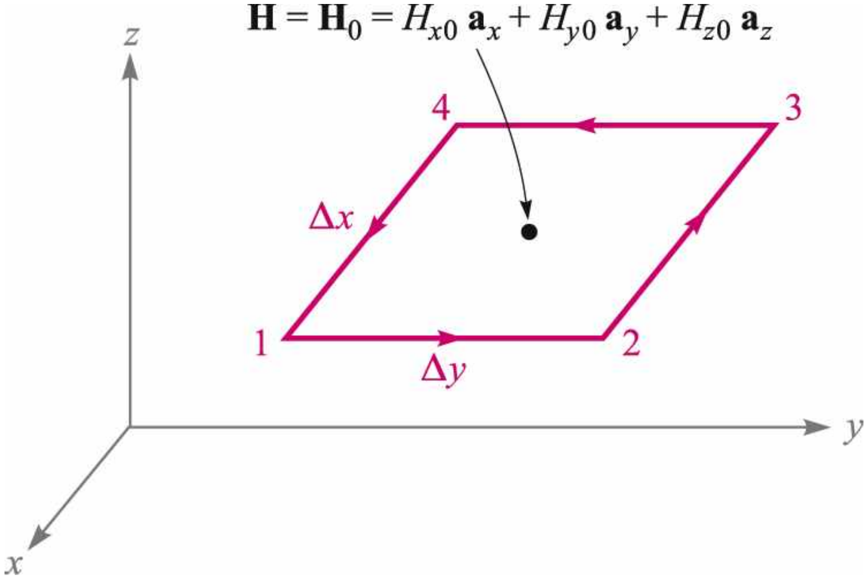


$\mathbf{H} = \dfrac{NI}{2\pi\rho} \mathbf{a}_\phi$ (WELL INSIDE TOROID)

7.3 Curl

- Incremental closed magnetic field closed path (1-2-3-4-1) of sides with Δx and Δy
- By the some current, the magnetic field at center

$$\vec{H}_0 = H_{x0}\vec{a}_x + H_{y0}\vec{a}_y + H_{z0}\vec{a}_z$$



- Overall magnetic field intensity over specific closed path (1-2-3-4-1)

$$\oint \vec{H}\cdot d\vec{L} = (\vec{H}\cdot\Delta\vec{L})_{1-2} + (\vec{H}\cdot\Delta\vec{L})_{2-3} + (\vec{H}\cdot\Delta\vec{L})_{3-4} + (\vec{H}\cdot\Delta\vec{L})_{4-1}$$

where $(\vec{H}\cdot\Delta\vec{L})_{1-2} = (H_y\vec{a}_y\cdot\Delta y\vec{a}_y)_{1-2} = H_{y,1-2}\Delta y$

$$\approx \left[H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)\right]\Delta y \quad \text{(by Taylor series)}$$

▪ By the same way,

$$(\vec{H}\cdot\Delta\vec{L})_{2-3} \cong H_{x,2-3}(-\Delta x) \cong -[H_{x0} + \frac{\partial H_x}{\partial y}(\frac{1}{2}\Delta y)]\Delta x$$

$$(\vec{H}\cdot\Delta\vec{L})_{3-4} \cong H_{y,3-4}(-\Delta y) \cong [H_{y0} + \frac{\partial H_y}{\partial x}(-\frac{1}{2}\Delta x)](-\Delta y)$$

$$(\vec{H}\cdot\Delta\vec{L})_{4-1} \cong H_{x,4-1}(\Delta x) \cong [H_{x0} + \frac{\partial H_x}{\partial y}(-\frac{1}{2}\Delta y)](\Delta x)$$

$$\therefore \quad \oint \vec{H}\cdot d\vec{L} \cong \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x\Delta y$$

$$= I = J_z \Delta x \Delta y$$: current enclosed by the path

or $$\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta x\Delta y} \cong \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \qquad \lim_{\Delta x,\Delta y\to 0}\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta x\Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

: unit area에 대한 $\vec{H}$ -field의 외곽선 적분과 unit area 면적을 지나는 전류와의 관계

cf.) Gauss's Law $$\mathrm{div}\,\vec{D} = \lim_{\Delta v\to 0}\frac{\oint \vec{D}\cdot d\vec{S}}{\Delta v} = \rho_v$$

- By analogous process,

$$\lim_{\Delta y,\Delta z\to 0}\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta y\Delta z}=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}=J_x \qquad \lim_{\Delta z,\Delta x\to 0}\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta z\Delta x}=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}=J_y$$

- Generally $(\text{curl }\vec{H})_N=\lim_{\Delta S_N\to 0}\frac{\oint \vec{H}\cdot d\vec{L}}{\Delta S_N}=J_N$

where ΔS_N: planar area enclosed by the closed integral

$$\text{curl }\mathbf{H}=\nabla\times\mathbf{H}=\begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ H_x & H_y & H_z\end{vmatrix}$$

$$=\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right)\mathbf{a}_x+\left(\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x}\right)\mathbf{a}_y+\left(\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y}\right)\mathbf{a}_z$$

$$=\vec{J}=J_x\vec{a}_x+J_y\vec{a}_y+J_z\vec{a}_z \quad \text{(Cartesian coordinate)}$$

$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)}$$

$$\nabla \times \mathbf{H} = \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)}$$

$\oint$: circulation

$\oint \vec{E}\cdot d\vec{L} = 0 \quad \Rightarrow$ Work required to carry a charge around a closed path is zero.

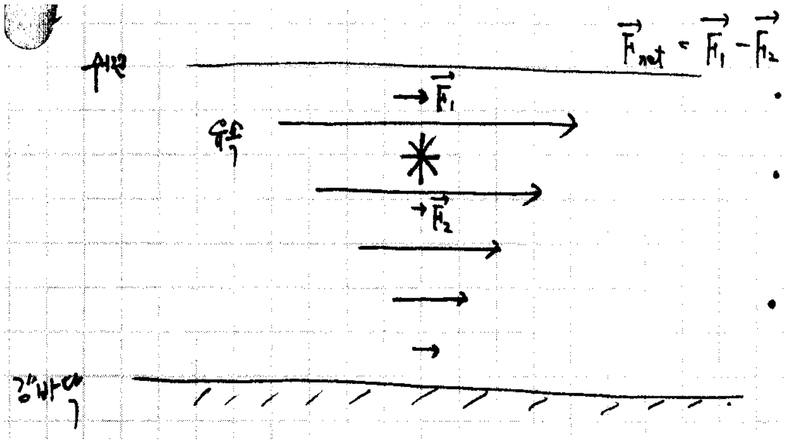
$$\nabla \times \vec{E} = \lim_{\Delta S_N \to 0} \frac{\oint \vec{E}\cdot d\vec{L}}{\Delta S_N} = 0 \qquad (\because \oint \vec{E}\cdot d\vec{L} = 0) \qquad \text{(for electrostatic)}$$

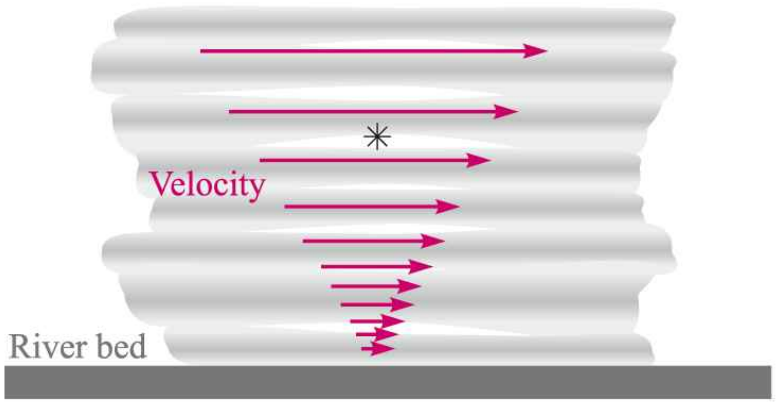
$$\leftrightarrow \quad \nabla \times \vec{H} = \lim_{\Delta S_N \to 0} \frac{\oint \vec{H}\cdot d\vec{L}}{\Delta S_N} = \lim_{\Delta S_N \to 0} \frac{I_N}{\Delta S_N} \neq 0$$

Visualization of Curl

- Curl meter: "paddle wheel" in a flowing stream of water (wheel axis points into the screen.)

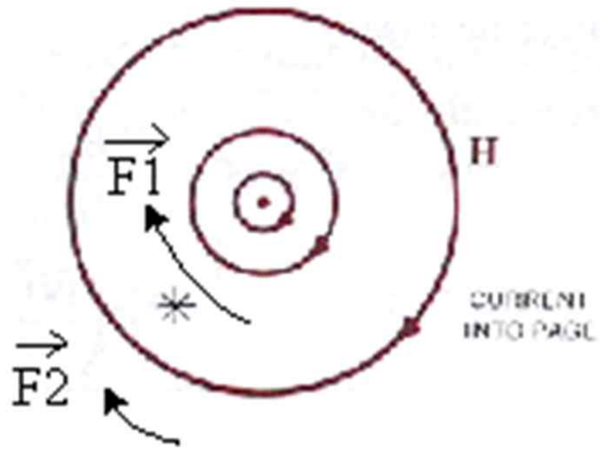
- Since $\vec{F}_1 > \vec{F}_2$, paddle wheel rotates clockwise.



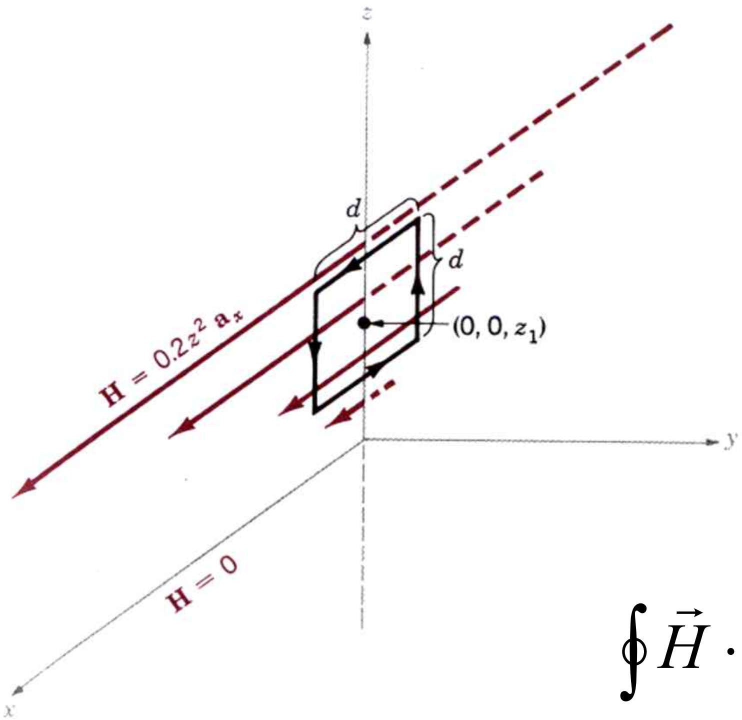


(*a*)

- Current go through page.
- Since $\vec{F}_1 > \vec{F}_2$, paddle wheel rotates count-clockwise.



[Example] $\vec{H} = \begin{cases} 0.2z^2\vec{a}_x & @z > 0 \\ 0 & @\text{elsewhere.} \end{cases}$



Square path: length = d

center $(0, 0, z_1)$ @$y = 0$ plane

$z_1 > d/2$

Solution 1)

$$\oint \vec{H}\cdot d\vec{L} = 0.2\left(z_1 + \frac{1}{2}d\right)^2 d + 0 + 0.2\left(z_1 - \frac{1}{2}d\right)^2 (-d) + 0$$

$$= 0.4z_1 d^2$$

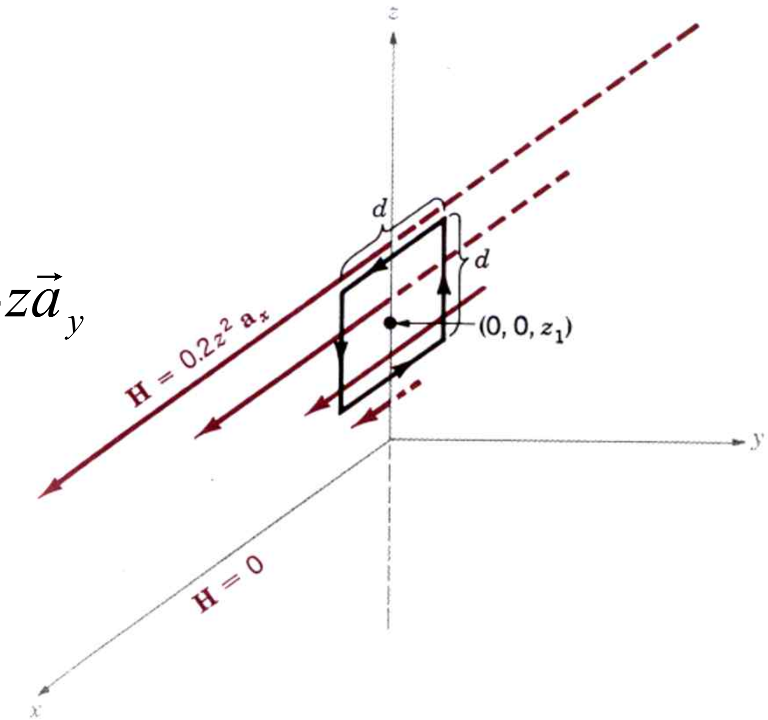
$$\left(\nabla\times\vec{H}\right)_y = \lim_{d\to 0}\frac{\oint \vec{H}\cdot d\vec{L}}{d^2} = \lim_{d\to 0}\frac{0.4z_1 d^2}{d^2} = 0.4z_1$$

: 사각형 변의 길이가 d인 loop를 관통하여 지나는 전류

$\therefore \nabla\times\vec{H} = 0.4z_1\vec{a}_y$ (∵ 선적분 면에 수직(normal) 방향은 $\vec{a}_y$ 방향)

Solution 2)

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}\left(0.2z^2\right)\vec{a}_y = 0.4z\vec{a}_y$$



$$@\, z = z_1, \quad \nabla \times \vec{H} = 0.4z_1\vec{a}_y \qquad ///$$

- Curl $\vec{H} = \nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\vec{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\vec{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\vec{a}_z$

$$= \vec{J}$$

: point form of Ampere's circuital law (time-invariant condition)

➔ The second equation of Maxwell's four equations

cf.) $\nabla \times \vec{E} = 0 \quad \left(\because \oint \vec{E} \cdot d\vec{L} = 0\right)$

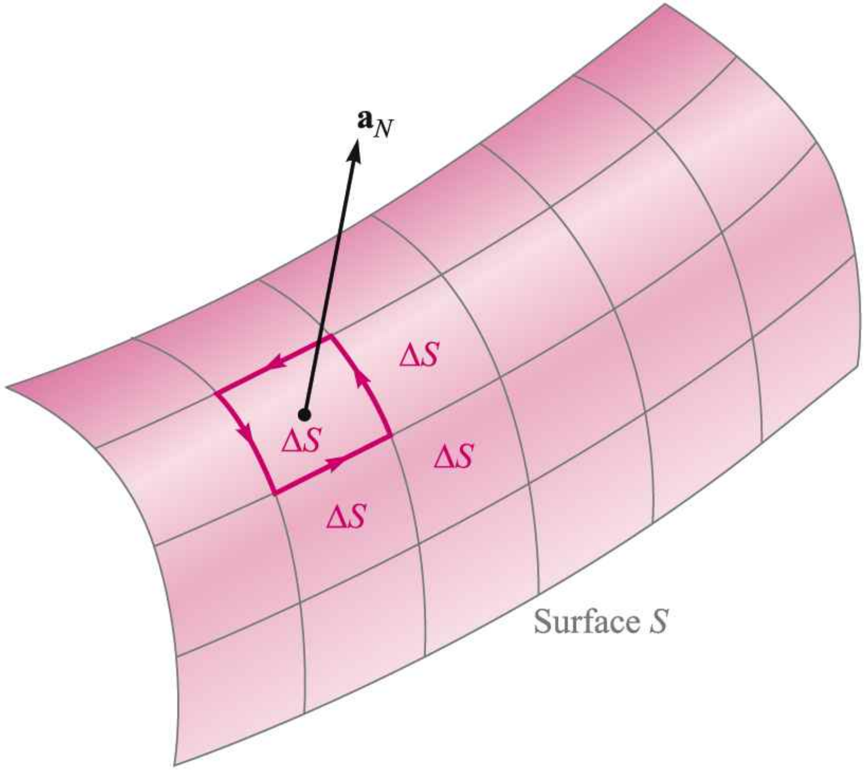
➔ The fourth equation of Maxwell's four equations
in case of time-invariant condition

7.4 Stokes' Theorem

- The surface S is broken up into incremental surfaces of areas ΔS.

$$\frac{\oint \vec{H} \cdot d\vec{L}_{\Delta S}}{\Delta S} \cong \left(\nabla \times \vec{H}\right)_N$$

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$



where N: right-handed direction normal to surface

$d\vec{L}_{\Delta S}$: closed path vector of perimeter of ΔS

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S} \quad \leftarrow d\vec{S} = dS\vec{a}_N$$

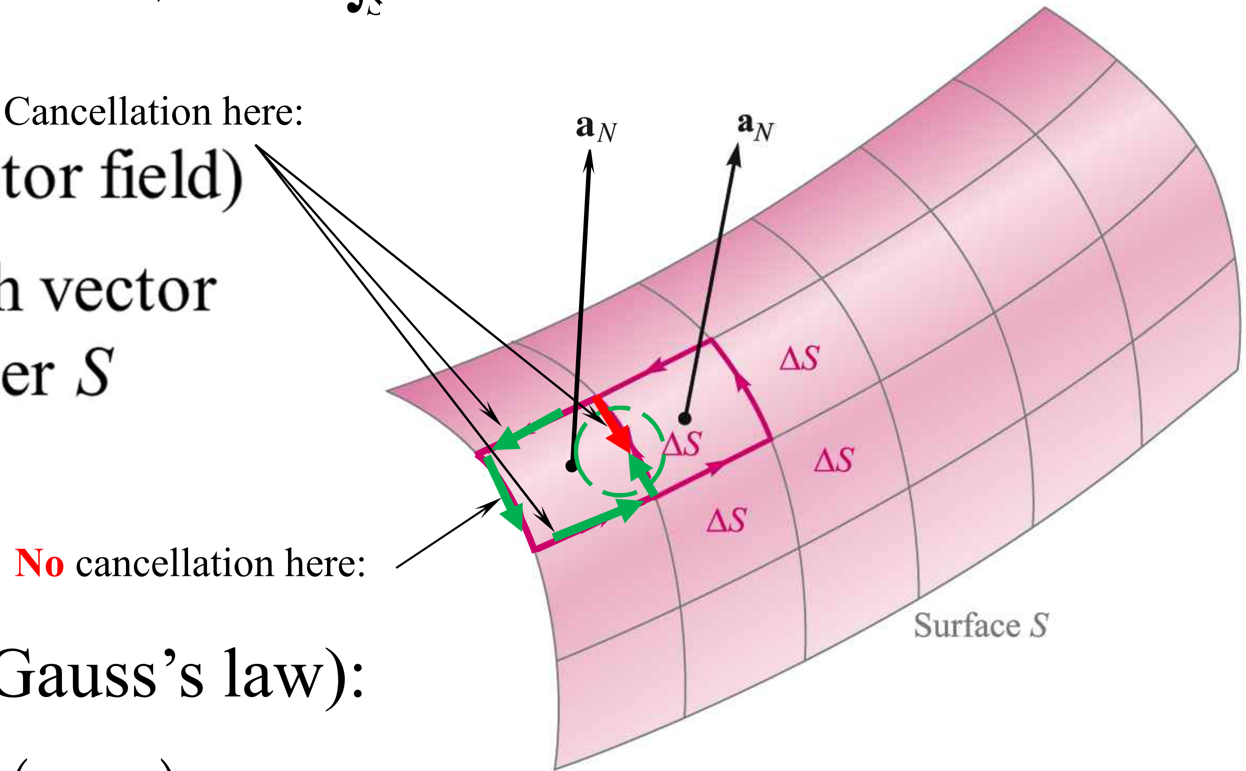
where $\vec{a}_N$: normal unit vector in right-handed direction normal to ΔS.

▪ Let us comprise S for every ΔS.

$$\oint \vec{H}\cdot d\vec{L}_{\Delta S} \doteq \oint \vec{H}\cdot d\vec{L} = \int \left(\nabla\times\vec{H}\right)\cdot\Delta\vec{S} = \int_S \vec{J}_N\cdot\Delta\vec{S} = I$$

: Stokes' theorm
(holding for any vector field)

where $d\vec{L}$: closed path vector of perimeter S



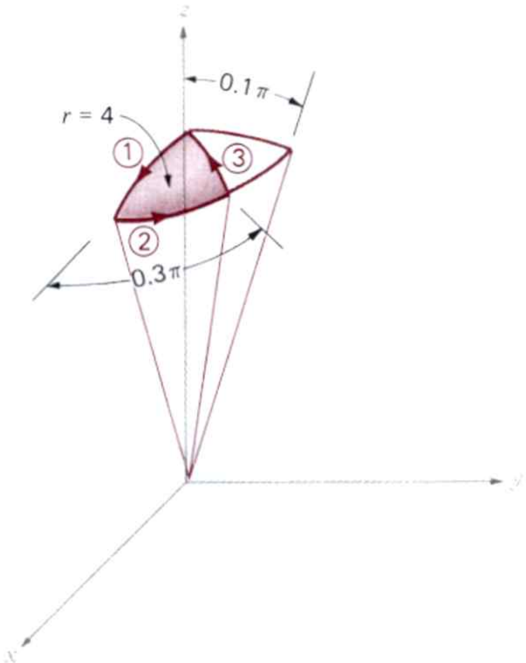
cf.) Divergence theorm (Gauss's law):

$$\oint \vec{D}_s\cdot d\vec{S} = \int \rho_v dv = \int \left(\nabla\cdot\vec{D}\right)dv$$ (closed 면적분 ↔ 체적 적분)

▪ Stokes' theorem: $\oint \vec{H}\cdot d\vec{L} = I = \oint_S \vec{J}\cdot d\vec{S} = \int (\nabla\times\vec{H})\cdot d\vec{S}$

(선적분 ↔ open 면적분)

[Ex. 7.3] Portion of sphere on $r = 4$, $0 \le \theta \le 0.1\pi$, $0 \le \phi \le 0.3\pi$



Path segment 1)
$r = 4,\ 0 \le \theta \le 0.1\pi,\ \phi = 0$

Path segment 2)
$r = 4,\ \theta = 0.1\pi,\ 0 \le \phi \le 0.3\pi$

Path segment 3)
$r = 4,\ 0 \le \theta \le 0.1\pi,\ \phi = 0.3\pi$

$$\vec{H} = 6r\sin\phi\vec{a}_r + 18r\sin\theta\cos\phi\vec{a}_\phi = H_r\vec{a}_r + H_\phi\vec{a}_\phi$$

- Solution 1) $d\vec{L} = dr\vec{a}_r + rd\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi = rd\theta\,\vec{a}_\theta + r\sin\theta\,d\phi\,\vec{a}_\phi$

: in spherical coordinate ($\because$ $r = 4$: constant)

$$\therefore \oint \vec{H}\cdot d\vec{L} = \int_1 H_\theta r d\theta + \int_2 H_\phi r\sin\theta d\phi + \int_3 H_\theta r d\theta$$

r, ϕ : constant
➔ $dr = 0 = d\phi$

r, θ : constant
➔ $dr = 0 = d\theta$

r, ϕ : constant
➔ $dr = 0 = d\phi$

$$\therefore \oint \vec{H}\cdot d\vec{L} = \int_2 H_\phi r\sin\theta d\phi \qquad (\because H_\theta = 0)$$

$$= \int_0^{0.3\pi}\left[18\cdot(4)\cdot\sin(0.1\pi)\cdot\cos\phi\right]\cdot(4)\sin(0.1\pi)d\phi$$

$$= 288\sin^2(0.1\pi)\sin(0.3\pi) = 22.2 \text{ [A]}$$

- Solution 2)

$$\nabla\times\vec{H} = \frac{1}{r\sin\theta}\left[\frac{\partial(H_\phi\sin\theta)}{\partial\theta} - \frac{\partial H_\theta}{\partial\phi}\right]\vec{a}_r + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial H_r}{\partial\phi} - \frac{\partial(rH_\phi)}{\partial r}\right]\vec{a}_\theta + \frac{1}{r}\left[\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial\theta}\right]\vec{a}_\phi$$

$$= \frac{1}{r\sin\theta}\left[36r\sin\theta\cos\theta\cos\phi\right]\vec{a}_r + \frac{1}{r}\left[\frac{6r\cos\phi}{\sin\theta} - 36r\sin\theta\cos\phi\right]\vec{a}_\theta + \frac{1}{r}[0]\vec{a}_\phi$$

$$d\vec{S} = r^2\sin\theta d\theta\, d\phi\, \vec{a}_r$$

$$\therefore \int_s(\nabla\times\vec{H})\cdot d\vec{S} = \int_0^{0.3\pi}\int_0^{0.1\pi}(36\cos\theta\cos\phi)(16\sin\theta)d\theta d\phi \quad \leftarrow \vec{a}_r \text{ 성분}$$

$$= \int_0^{0.3\pi}576\left[\frac{1}{2}\sin^2\theta\right]_0^{0.1\pi}\cos\phi d\phi = 288(\sin^2 0.1\pi)\times(\sin 0.3\pi)$$

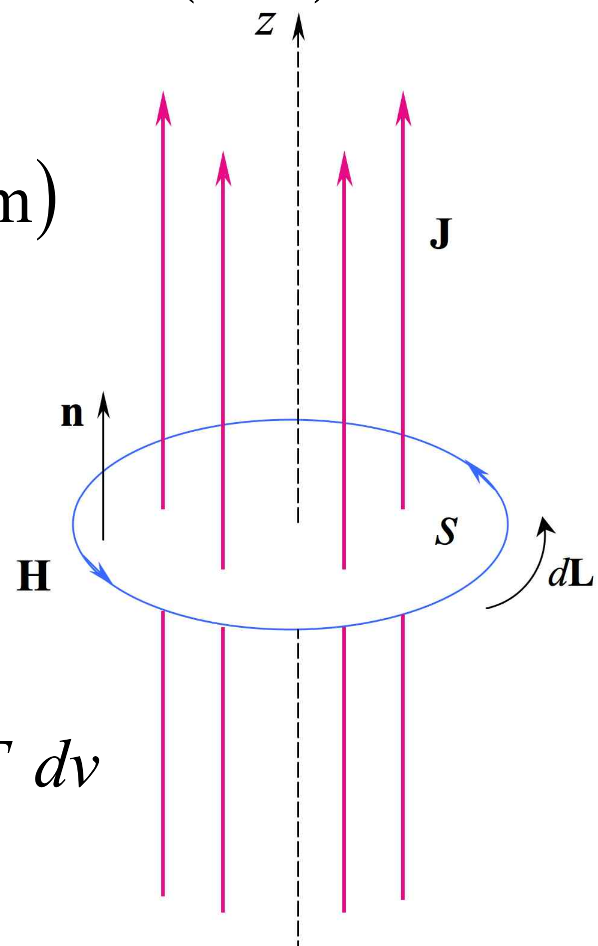
$$= 22.2 \quad \text{[A]}$$: Stokes' theorem is satisfied. ///

- Let us obtain Ampere's circuital law from $\nabla \times \vec{H} = \vec{J}$ (curl)

$$\int_S (\nabla \times \vec{H}) \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I$$

$$= \oint \vec{H} \cdot d\vec{L} \quad \text{(By Stoke's theorem)}$$

$$\therefore \oint \vec{H} \cdot d\vec{L} = I$$



- Vector identity

$\nabla \cdot \nabla \times \vec{A} = T$ $\quad$ $\vec{A}$: arbitrary vector

vector

scalar

$$\int_{vol} \nabla \cdot (\nabla \times \vec{A})\, dv = \int_{vol} T dv \qquad \oint_s (\nabla \times \vec{A}) \cdot d\vec{S} = \int_{vol} T\, dv$$

s → closed surface

(- Gauss 법칙은 체적(volume) v를 감싸고 있는 폐곡면(closed surface)에 대한 면적분임.
- Stokes' theorem은 임의의 폐경로에 의한 개곡면(open surface)에 대한 면적분임.
- 폐곡면의 경우 개곡면적 = 0, 개곡면의 경우 폐곡면적 = 0
- Ex.) 펼친 보자기(wrapping cloth): 개곡면, 묶은 보자기: 폐곡면)

$$\int_{vol} T\, dv = 0 \;\rightarrow\; T\, dv = 0 \;\rightarrow\; T = 0 \;\; (\because v \neq 0) \;\rightarrow\; \nabla \cdot \nabla \times \vec{A} = 0$$

open surface

closed surface

7.5 Magnetic Flux and Flux Density

- Magnetic flux density: $\vec{B}$

$$\vec{B} = \mu_0 \vec{H} \quad [\mathrm{Wb/m^2}] \quad \text{or} \quad [\mathrm{T: telsa}] \quad \text{or} \quad [\mathrm{G: gauss}]$$

where $\mu_0 = 4\pi \times 10^{-7}$ [H/m] : free space *permeability* (isotropic material)

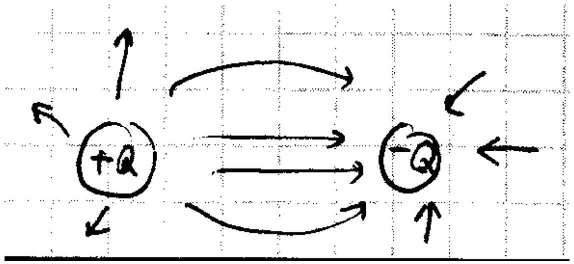
cf.) $\vec{D} = \varepsilon_0 \vec{E}$

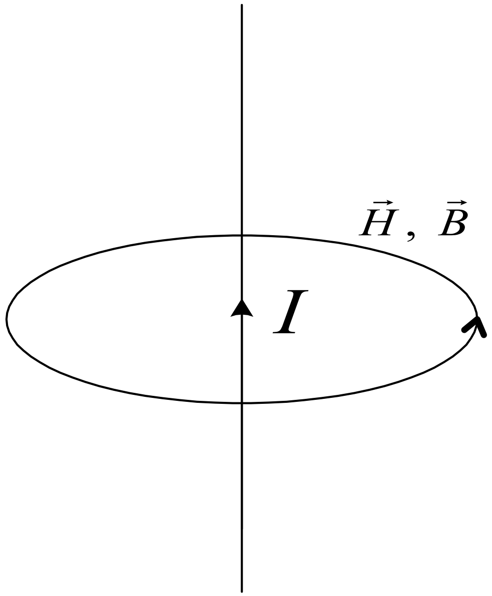
- Magnetic flux: Φ [phi]

$$\Phi = \int_s \vec{B} \cdot d\vec{S} \quad [\mathrm{wb}]$$

cf.) Electric flus: Ψ [psi] (Gauss's law: The total flux passing through any closed surface is equal to the charge enclosed.)

$$\Psi = \oint_S \vec{D} \cdot d\vec{S} = Q$$





$\vec{B} = \mu_0 \vec{H}$: relation between magnetic flux and magnetic field intensity

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} \, dv = 0$$

(Since the magnetic flux line forms the open surface, the magnetic flux line is closed and do not terminate on a magnetic charge(s). ➔ No magnetic flux source

- Differential form: $\nabla \cdot \vec{B} = 0$
- (Differential) Maxwell's equations for static electric field and steady magnetic field:

$\nabla \cdot \mathbf{D} = \rho_v$	Gauss' Law for the electric field
$\nabla \times \mathbf{E} = 0$	Conservative property of the static electric field
$\nabla \times \mathbf{H} = \mathbf{J}$	Ampere's Circuital Law
$\nabla \cdot \mathbf{B} = 0$	Gauss' Law for the Magnetic Field

where, in free space

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

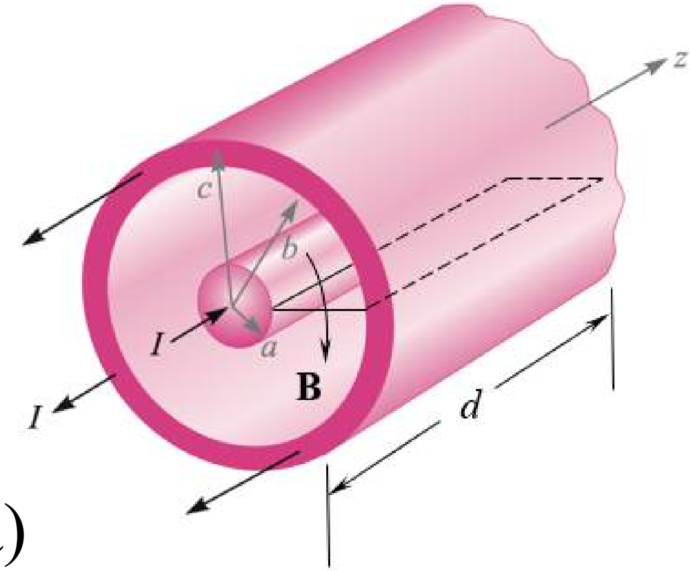
▪ Maxwell's equations in large scale (or integral) form using divergence theorem and Stokes' theorem

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_\nu d\nu$$

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$



[Ex.] $H_\phi = \dfrac{I}{2\pi\rho} \qquad (a < \rho < b)$ from Fig. 7.8(a)

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi$$

Magnetic flux crossing any radial plane on $a \le \rho \le b$ and $0 \le z \le d$:

$$\Phi = \int_s \vec{B} \cdot d\vec{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi \cdot (d\rho \, dz \, \vec{a}_\phi)$$

$$= \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a}$$

7.6 The Scalar and Vector Magnetic Potentials

- For electric field, Q or charge distribution $\longrightarrow \vec{E} \longrightarrow V \longrightarrow C$

  $\longrightarrow V \longrightarrow \vec{E} \longrightarrow C$

- Let us introduce scalar magnetic potential (V_m) with similarity of electric potential

$$\mathbf{H} = -\nabla V_m \quad \Leftrightarrow \quad \mathbf{E} = -\nabla V$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla V_m)$$

: (1) Curl 적용

"zero"

"zero"

Scalar (not slope): current is constant

$$\therefore \quad \mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)$$

(Ex. Region $a < \rho < b$ in coaxial cable)

- In free space,

$$\nabla \cdot \vec{B} = \mu_0 \nabla \cdot \vec{H} = 0$$: (2) Divergence 적용

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

$$\therefore \nabla^2 V_m = 0 \quad (\text{for } \vec{J} = 0)$$: (Magnetic) Laplace's eq.

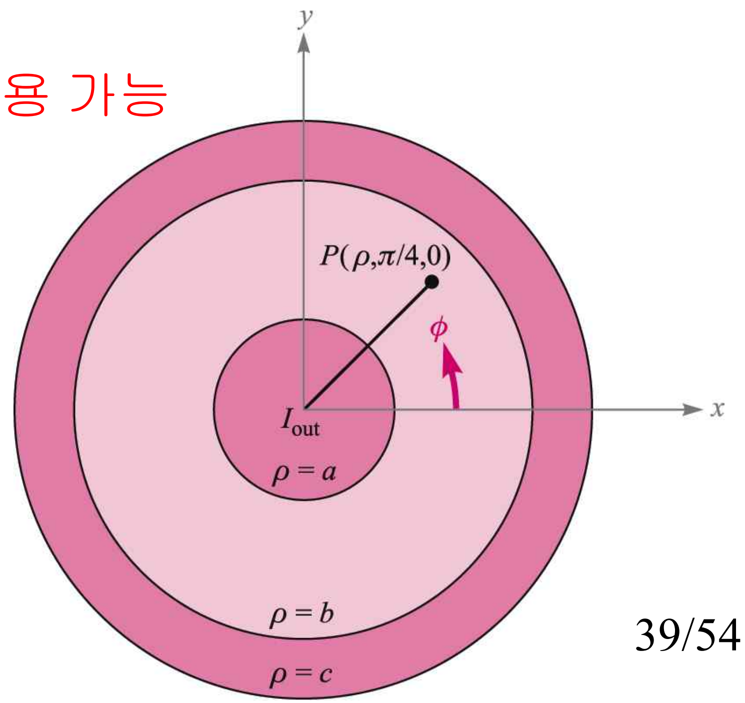
(특정면으로 전류가 흐르고, 다른 특정면에 전류가 흐르지 않을 때에 (magnetic) Laplace's eq. 을 적용 가능)

- Cross section of coaxial line

At $a < \rho < b$, $\vec{J} = 0$ ➔ magnetic potential 적용 가능

$$\Rightarrow \vec{H} = \frac{I}{2\pi\rho} \vec{a}_\phi$$

where I : total current flowing in inner conductor directional to $\vec{a}_z$



$$\vec{H} = \frac{I}{2\pi\rho}\vec{a}_\phi = -\nabla V_m\Big|_\phi = -\frac{1}{\rho}\frac{\partial V_m}{\partial \phi}\vec{a}_\phi$$

$$\therefore \frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

$$V_m = -\frac{I}{2\pi}\phi\ (+k) = \frac{I}{2\pi}(-\phi + \underline{k'}) = -\frac{I}{2\pi}\phi$$

Integral constant = 0 (in this text)

- If $V_m = 0$ at $\phi = 0$ and proceed countclockwise around the circle, magnetic potential at point P (@ $\phi = \pi/4$) is

$$V_{mp} = \frac{I}{2\pi}(2n - \frac{1}{4})\pi \qquad (n = 0,\ \pm 1,\ \pm 2,\ \cdots\cdots)$$

$$= I(n - \frac{1}{8}) \qquad (\qquad \text{"} \qquad)$$

➔Magnetic scalar potential has a **multivaluedness** property.

cf.) Electrostatic case

$$\nabla \times \vec{E} = 0 \quad \text{(point form)}$$

$$\oint \vec{E} \cdot d\vec{L} = 0 \quad \text{(integral form)}$$

$$V_{ab} = -\int_b^a \vec{E} \cdot d\vec{L} \quad \text{(independant of the path)}$$

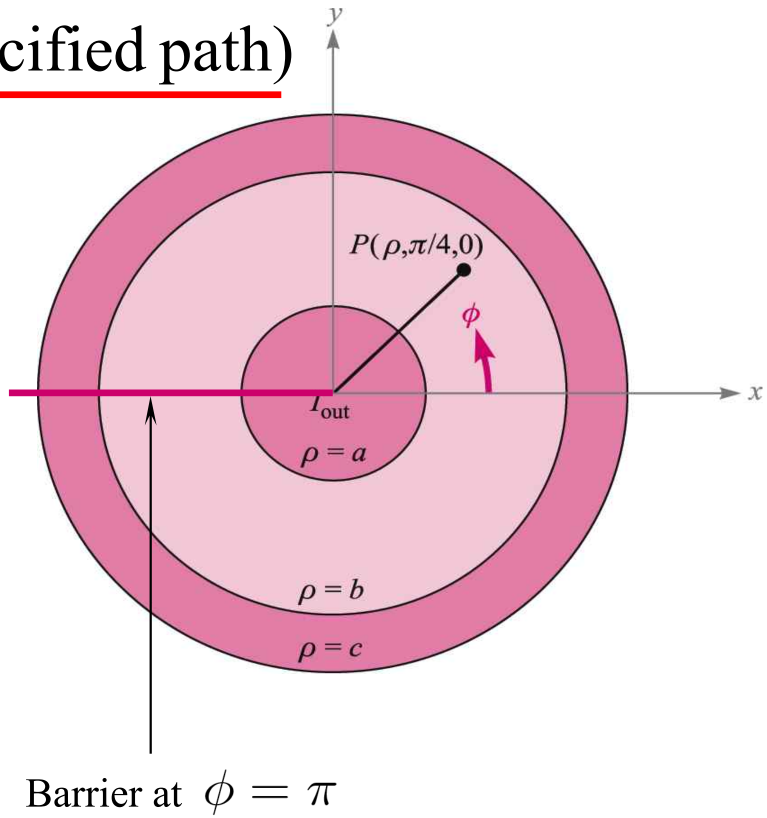
: conservative (or singular) //

$$V_{m,ab} = -\int_a^b \vec{H} \cdot d\vec{L} \quad \text{(depend on the specified path)}$$

- In the above example, we restrict ϕ variation range as $-\pi \sim \pi$.

$$V_m = -\frac{I}{2\pi}\phi \, ,$$

then $V_{mp} = -\dfrac{I}{8}$ @ $\phi = \dfrac{\pi}{4}$



7.6.2 Vector (Magnetic) Potential

- Time varying case
- Useful in studying a electromagnetic wave radiation from antennas

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(Maxwell equation)}$$

- Let $\mathbf{B} = \nabla \times \mathbf{A}$ $\quad \leftarrow \vec{A}$: vector magnetic potential [wb/m]

(since $\nabla \cdot \nabla \times \vec{A} = 0$ is proven in the previous section, $\nabla \cdot \vec{B} = 0$ is satisfied.)

cf.) $\vec{H} = -\nabla V_m$: scalar magnetic potential

$$\nabla \cdot \vec{B} = \nabla \cdot \left(\mu_0 \vec{H}\right) = \nabla \cdot \left(\nabla \times \vec{A}\right)$$

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}$$

$$\Rightarrow \vec{A} = \oint \frac{\mu_0 I d\vec{L}}{4\pi R}$$

$$\nabla \times \vec{H} = \vec{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}$$

(This eq. will be proven in the next section)

(If we know $\vec{A}$, then $\vec{B}$ can be know naturally.)

where I : DC current along a filamentary conductor

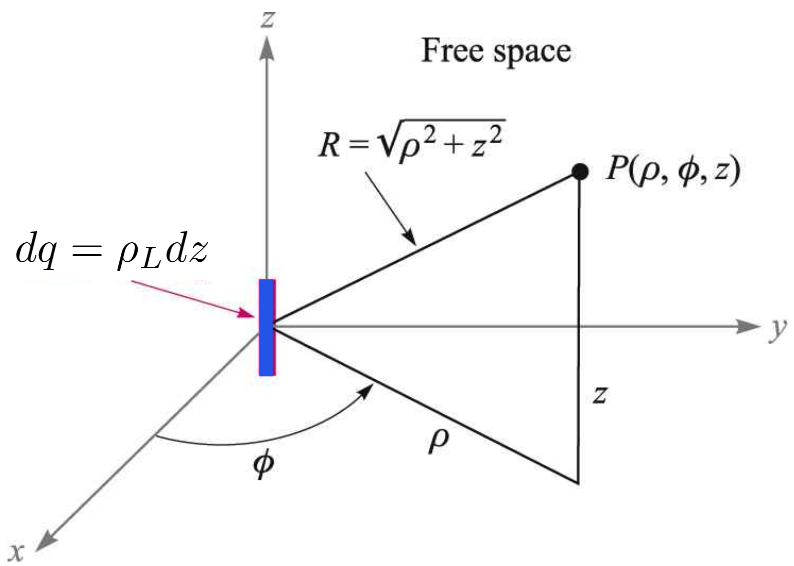
R: distance from differential current length $d\vec{L}$ to a point where $\vec{A}$ is to be found.

- Electrostatic potential

$$V = \int \frac{\rho_L dl}{4\pi\varepsilon_0 R} \quad \Leftarrow V = \frac{Q}{4\pi\varepsilon_0 R} \quad (4.15)$$

- When compare with the electrostatic potential, differential vector magnetic ($\vec{A}$) is $d\vec{A} = \frac{\mu_0 I d\vec{L}}{4\pi R}$ $\rightarrow$ $\vec{A} \mathbin{//} d\vec{L}$ and $|\vec{A}| \propto \frac{1}{R}$

Line Charge

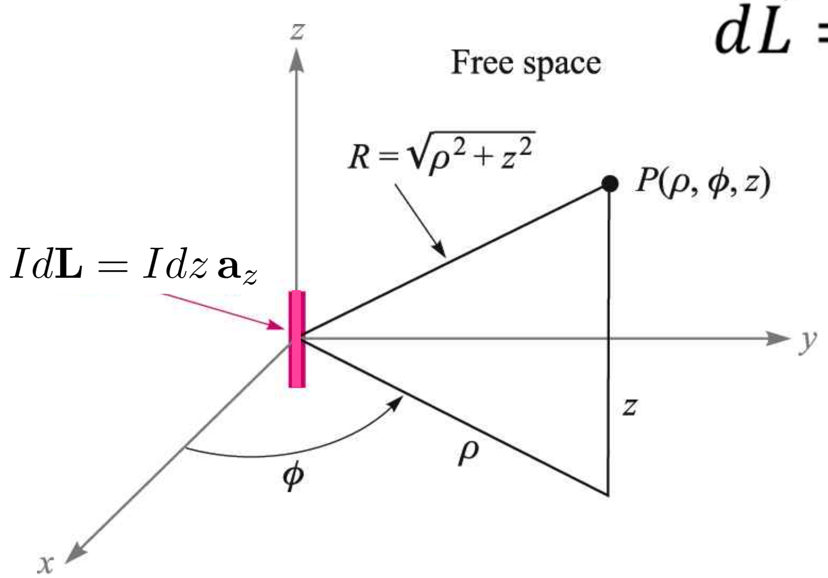


Scalar Electrostatic Potential

$$dV = \frac{dq}{4\pi\epsilon_0 R} = \frac{\rho_L dL}{4\pi\epsilon_0 R}$$

Line Current

$$d\vec{L} = dz\vec{a}_z$$



Vector Magnetic Potential

$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R} = \frac{\mu_0 I dz\,\mathbf{a}_z}{4\pi R} = \frac{\mu_0 I dz \vec{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

▪ In cylindrical coordinate at $P(\rho, \phi, z)$

$$dA_z = \frac{\mu_0 I dz}{4\pi\sqrt{\rho^2 + z^2}} = f(\rho, z)$$

$$dA_\phi = 0$$

$$dA_\rho = 0$$

▪ Since $\nabla \times \vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{a}_\rho + \left(\frac{1}{\rho}\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\vec{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)\vec{a}_z,$

$$d\vec{H} = \frac{1}{\mu_0}\nabla \times d\vec{A} = \frac{1}{\mu_0}\left(-\frac{\partial dA_z}{\partial \rho}\right)\vec{a}_\phi \qquad \leftarrow A_z = f(\rho, z) \neq f(\phi)$$

$$= -\frac{Idz}{4\pi}\left(-\frac{1}{2}\right)\frac{2\rho}{\left(\rho^2 + z^2\right)^{3/2}}\vec{a}_\phi = \frac{Idz\,\vec{a}_z}{4\pi(\rho^2 + z^2)} \times \frac{\rho\vec{a}_\rho + z\vec{a}_z}{\sqrt{\rho^2 + z^2}}$$

$$d\vec{H} = \frac{Id\vec{L} \times \vec{a}_r}{4\pi R^2} = \frac{Id\vec{L}}{4\pi R^2} \times \vec{a}_r$$ 과 같은 형태

- For a current sheet $\vec{K}$,

$$Id\vec{\mathrm{L}} = \vec{\mathrm{K}}dS$$

For a current throughout a volume with a density $\vec{J}$,

$$Id\vec{\mathrm{L}} = \vec{J}dv$$

- Alternative expressions for $\vec{A}$,

$$\vec{A} = \int_s \frac{\mu_0 \vec{\mathrm{K}}dS}{4\pi R}$$

$$\vec{A} = \int_{vol} \frac{\mu_0 \vec{J}dv}{4\pi R}$$

(magnetic potential = 0 (@$R = \infty$) as like electrostatic potential = 0 (@$R = \infty$)
($\because |\vec{A}| \propto \frac{1}{R}$ $\leftarrow$ current location(@$r = 0$) & measurement point (@$r = \infty$))

7.7 Derivation of Steady-Magnetic-Field Law

- Relationships among the magnetic field quantities

$$\vec{H} = \oint \frac{Id\vec{\mathrm{L}} \times \vec{a}_R}{4\pi R^2}$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\vec{B} = \nabla \times \vec{A}$$

<u>* Proof of Biot-Savart Law</u>

- Let $\vec{A} = \int_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R}$

  where current element location: (x_1, y_1, z_1)

  $\vec{A}$ measurement location: (x_2, y_2, z_2)

  differential volume element location: $dv_1 (= dx_1 dy_1 dz_1)$

▪ So $\vec{A}_2 = \int_{vol} \frac{\mu_0 \vec{J}_1 dv_1}{4\pi R_{12}}$

▪ Since $\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{\nabla \times \vec{A}}{\mu_0}$, $\vec{H}_2 = \frac{\nabla_2 \times \vec{A}_2}{\mu_0} = \frac{1}{\mu_0} \nabla_2 \times \int_{vol} \frac{\mu_0 \vec{J}_1 dv_1}{4\pi R_{12}}$

$$= \frac{1}{4\pi} \int_{vol} \nabla_2 \times \frac{\vec{J}_1 dv_1}{R_{12}}$$

$$= \frac{1}{4\pi} \int_{vol} \left(\nabla_2 \times \frac{1}{R_{12}} \vec{J}_1 \right) dv_1$$

▪ By using vector identity, $\nabla \times \left(S\vec{V} \right) = (\nabla S) \times \vec{V} + S\left(\nabla \times \vec{V} \right)$

$$\vec{H}_2 = \frac{1}{4\pi} \int_{vol} \left[\left(\nabla_2 \frac{1}{R_{12}} \right) \times \vec{J}_1 + \frac{1}{R_{12}} \left(\nabla_2 \times \vec{J}_1 \right) \right] dv_1$$

$=0$ $\because$ ($\vec{J}_1$, (x_1, y_1, z_1)) 에 대한 함수를 ∇_2, (x_2, y_2, z_2)) 에 관하여 미분하므로.

▪ Because of $R_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ and

$$\nabla_2 \frac{1}{R_{12}} = -\frac{\vec{R}_{12}}{R_{12}^3} = -\frac{\vec{a}_{R_{12}}}{R_{12}^2} \qquad (\leftarrow \#7\text{-}42)$$

$$\vec{H}_2 = -\frac{1}{4\pi}\int_{vol} \frac{\vec{a}_{R_{12}} \times \vec{J}_1}{R_{12}^2} dv_1$$

$$= \int_{vol} \frac{\vec{J}_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2} dv_1 = \int_{vol} \frac{\vec{J}_1 dv_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2} \qquad \text{(proven)}$$

▪ Replacing $\vec{J} dv_1 = \vec{I}_1 d\vec{L}_1$,

$$\vec{H}_2 = \int \frac{I_1 d\vec{L}_1 \times \vec{a}_{R_{12}}}{4\pi R}$$

$$\Rightarrow \vec{A} = \int_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R} \quad \text{is correct.}$$

* Proof of Ampere's Circuital Law

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}$$

이하 ◯ 이 '0'임을 증명하는 과정

- By using vector identities,

$$\nabla \times \nabla \times \vec{A} = \nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}$$

where $\nabla^2 \vec{A} \equiv \nabla^2 A_x \vec{a}_x + \nabla^2 A_y \vec{a}_y + \nabla^2 A_z \vec{a}_z$

$$\nabla \times \vec{H} = \frac{1}{\mu_0}\left[\nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}\right]$$

- Since $\nabla \cdot \left(V\vec{D}\right) \equiv V\left(\nabla \cdot \vec{D}\right) + \vec{D} \cdot (\nabla V)$,

$$\nabla_2 \cdot \vec{A}_2 = \nabla_2 \cdot \int_{vol} \frac{\mu_o \vec{J}_1 dv_1}{4\pi R_{12}} = \frac{\mu_o}{4\pi} \int_{vol} \nabla_2 \cdot \frac{1}{R_{12}} \vec{J}_1 dv_1$$

$$= \frac{\mu_o}{4\pi} \int_{vol} \left[\frac{1}{R_{12}} \left(\nabla_2 \cdot \vec{J}_1\right) + \vec{J}_1 \cdot \left(\nabla_2 \frac{1}{R_{12}}\right) \right] dv_1 \qquad (62)$$

$= 0$

- Because of $\nabla_2 \frac{1}{R_{12}} = -\frac{\vec{R}_{12}}{R_{12}^3}, \quad \nabla_1 \frac{1}{R_{12}} = \frac{\vec{R}_{12}}{R_{12}^3}$

$$\nabla_1 \frac{1}{R_{12}} = -\nabla_2 \frac{1}{R_{12}}$$

- So, $\nabla_2 \cdot \vec{A}_2 = \frac{\mu_0}{4\pi} \int_{vol} \left[-\vec{J}_1 \cdot \left(\nabla_1 \frac{1}{R_{12}} \right) \right] dv_1 \quad \leftarrow -\vec{D} \cdot (\nabla V) = V(\nabla \cdot \vec{D}) - \nabla \cdot (V\vec{D})$

$$= \frac{\mu_0}{4\pi} \int_{vol} \left[\frac{1}{R_{12}} \left(\nabla_1 \cdot \vec{J}_1 \right) - \nabla_1 \cdot \left(\frac{\vec{J}_1}{R_{12}} \right) \right] dv_1$$

$$= -\frac{\mu_0}{4\pi} \oint \frac{\vec{J}_1}{R_{12}} \cdot d\vec{S}_1$$

$$= 0$$

=0 (왜냐하면 steady magnetic fields)

$\vec{J}_1 = \text{constant} \Rightarrow \frac{\partial J_{1x}}{\partial x} = 0, \frac{\partial J_{1y}}{\partial y} = 0, \frac{\partial J_{1z}}{\partial z} = 0$

(S_1은 모든 체적을 둘러싼 폐곡면. S_1을 모든 전류가 포함되도록 체적을 포함한 폐곡면으로 설정하면, 체적 표면의 밖으로 나가는 전류밀도

$\vec{J}_1 = 0 \Rightarrow \oint \frac{\vec{J}_1}{R_{12}} \cdot d\vec{S}_1 = 0$)

▪ Comparison of magnetic vector potential with electric potential

$$A_x = \int_{vol} \frac{\mu_0 J_x dv}{4\pi R} \quad \leftrightarrow \quad V = \int_{vol} \frac{\rho dv}{4\pi\varepsilon_0 R}$$

$$\Rightarrow \left(J_x \leftrightarrow \rho, \quad \mu_0 \leftrightarrow \frac{1}{\varepsilon_0}, \quad A_x \leftrightarrow V \right)$$

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}$$: Poisson's equation

$$\Leftrightarrow \begin{cases} \nabla^2 A_x = -\mu_0 \vec{J}_x \\ \nabla^2 A_y = -\mu_0 \vec{J}_y \\ \nabla^2 A_z = -\mu_0 \vec{J}_z \end{cases} \quad \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\therefore \nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}$$

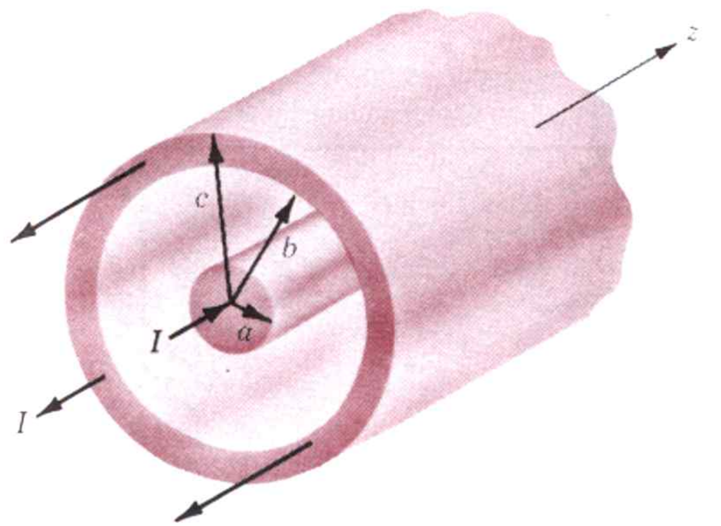
$$= \frac{1}{\mu_0}\left[\nabla\left(\nabla \cdot \vec{A}\right) - \nabla^2 \vec{A}\right] = -\frac{1}{\mu_0} \nabla^2 \vec{A} \qquad \left(\because \nabla\left(\nabla \cdot \vec{A}\right) = 0\right)$$

$$= \vec{J} \qquad (\text{proven})$$

[Ex.] Coaxial cable

At $a < \rho < b$, $\vec{J} = 0$ (in dielectiric)

$$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J} = 0 \quad \text{(Laplace eqation)}$$



In general, $\nabla^2 \vec{A} = \nabla^2 A_x \vec{a}_x + \nabla^2 A_y \vec{a}_y + \nabla^2 A_z \vec{a}_z$

$$\neq \nabla^2 A_\rho \vec{a}_\rho + \nabla^2 A_\phi \vec{a}_\phi + \nabla^2 A_z \vec{a}_z$$

(But $\vec{a}_z$ component is same in both coordiantes.)

$$\nabla^2 \vec{A}\Big|_z = \nabla^2 A_z \qquad \therefore \nabla^2 A_z = \mu_o J_z = 0 \qquad \left(\leftarrow \nabla^2 A_z = \nabla \cdot \nabla A_z\right)$$

(Since the coaxial cable is directed to z-axis and $J_z = 0$ in $a < \rho < b$, we can consider only $\vec{A} = A\vec{a}_z$ component.)

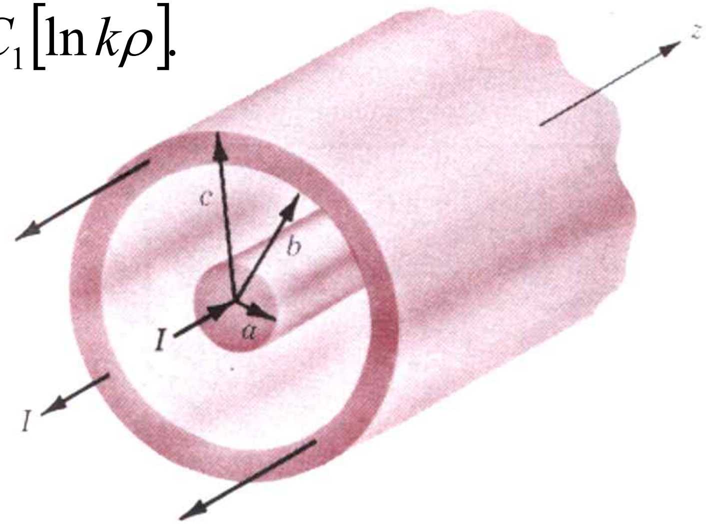
$$\frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0$$

($\because$ The current is varying for ρ, symmetrical for ϕ, and constant for z.)

$$\Rightarrow \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial A_z}{\partial \rho}\right) = 0$$

- General solution:

$$A_z = C_1 \ln \rho + C_2 = C_1\left[\ln \rho + \frac{C_2}{C_1}\right] = C_1[\ln \rho + \ln k] = C_1[\ln k\rho].$$



- Since $A_z = 0$ @ $\rho = b$,

$$kb = 1 \qquad \therefore k = \frac{1}{b}$$

$$\therefore A_z = C_1 \ln \frac{\rho}{b}$$

$$\nabla \times \vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\vec{a}_\rho + \left(\frac{1}{\rho}\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\vec{a}_\phi + \left[\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right]\vec{a}_z$$

$$= -\frac{\partial A_z}{\partial \rho}\vec{a}_\phi = -C_1 \cdot \frac{1}{\rho} \cdot \frac{1}{b}\vec{a}_\phi = -C_1 \cdot \frac{1}{\rho}\vec{a}_\phi \qquad (\because \text{비례상수이므로})$$

$$= \vec{B} = \mu_0 \vec{H}$$

$$\therefore \quad \vec{H} = -\frac{C_1}{\mu_0 \rho}\vec{a}_\phi$$

$$\oint \vec{H} \cdot d\vec{L} = \int_0^{2\pi} \left(-\frac{C_1}{\mu_0 \rho} \right) \vec{a}_\phi \cdot \rho d\phi \vec{a}_\phi = -\frac{C_1}{\mu_0} \cdot 2\pi = I$$

$$C_1 = -\frac{\mu_0 I}{2\pi}$$

$$A_z = C_1 \ln \frac{\rho}{b} = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho}$$

$$\vec{H} = -\frac{C_1}{\mu_0 \rho} \vec{a}_\phi = \frac{I}{2\pi\rho} \vec{a}_\phi$$

[Ex.] A plot of A_z versus ρ for $b = 5a$

$$A_z = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho} = \frac{\mu_0 I}{2\pi} \ln \frac{5a}{\rho}$$

$$= \frac{\mu_0 I}{2\pi} \left[\ln 5 + \ln \frac{a}{\rho} \right] = \frac{\mu_0 I}{2\pi} \left[\ln 5 - \ln \frac{\rho}{a} \right]$$