Engineering Electromagnetics

W.H. Hayt Jr. and J. A. Buck

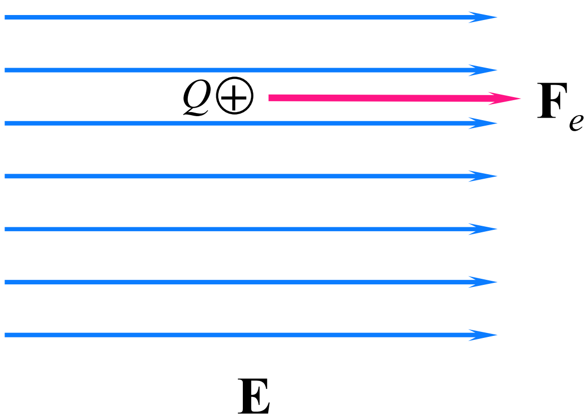
Chapter 8:

Magnetic Forces, Materials, and Inductance

8.1 Force on a Moving Charge

- Definition of electric field intensity

$$\boxed{\mathbf{F}_e = Q\mathbf{E}}$$



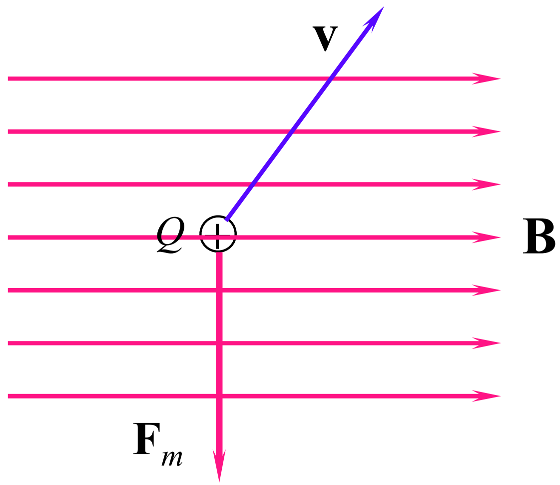
$$\vec{F}_t = \frac{Q_1 Q_t}{4\pi\varepsilon_0 R^2_{1t}}\vec{a}_{1t}$$

$$\vec{E} = \frac{\vec{F}_t}{Q_t}$$

$(\vec{F}_e // \vec{E}, \vec{F} = f(Q, \vec{E}))$

- Definition of magnetic field intensity

charge is moving at velocity **v** in magnetic flux density $\vec{B}$

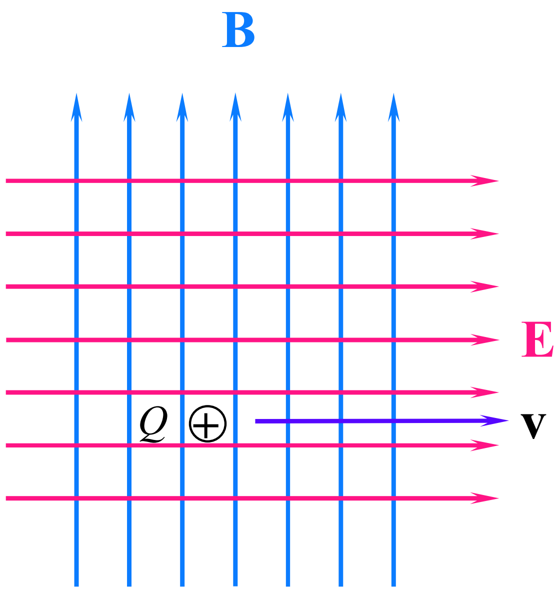


$$\boxed{\mathbf{F}_m = Q\,(\mathbf{v} \times \mathbf{B})}$$

$(\overrightarrow{F_m} \perp \vec{v},\ \overrightarrow{F_m} \perp \vec{B})$

Lorentz Force Law

Generally, with both electric and magnetic fields present, we have both forces:



The electric field will, in this case, accelerate the charge in the direction of **E**, making it cross the **B** field lines in the perpendicular sense; this gives a magnetic force component that is out of the screen

The total force on the moving charge is then the sum of the two, or

$$\boxed{\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})}$$

This is the *Lorentz Force Law* (sometimes called the "fifth Maxwell equation")

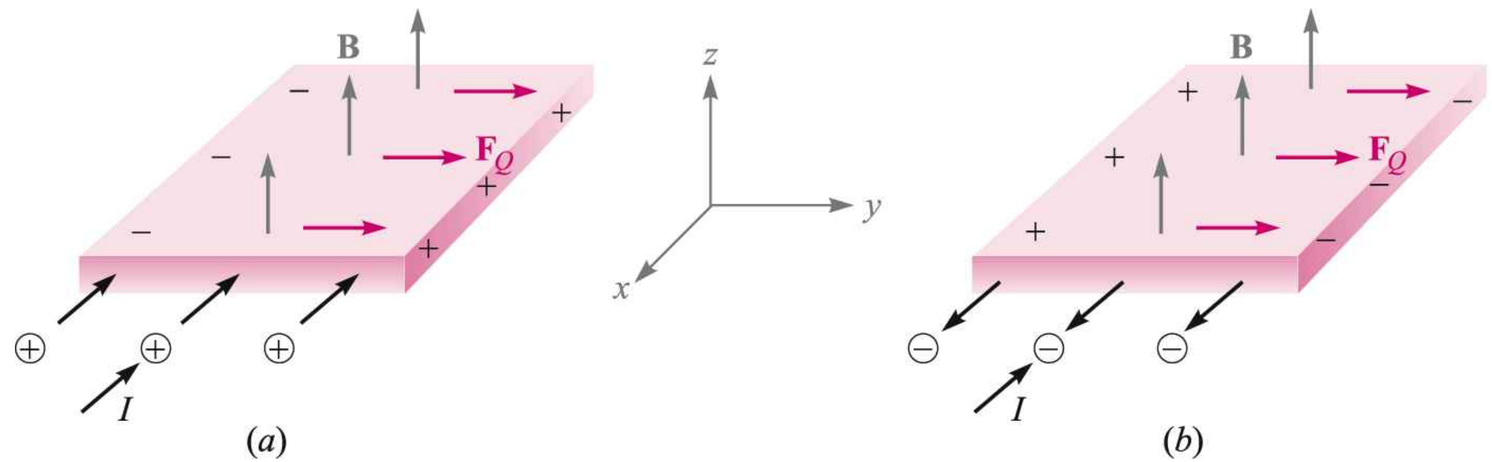
8.2 Force on a differential current element

- Definition force on a charged differential particle moving through a steady magnetic field

$$d\vec{F} = d\underline{Q}\vec{v} \times \vec{B}$$

$\rho_v dv$

Hall Effect



- Hole injection: $\vec{v} = |\vec{v}|(-\vec{a}_x)$

$$\vec{F} = Q(\vec{v} \times \vec{B}) = Q|\vec{v}||\vec{B}|(-\vec{a}_x \times \vec{a}_z)$$

$$= Q|\vec{v}||\vec{B}|\vec{a}_y$$

- On hole moving, hole receive a tension to right side. ➔ P-type semiconductor

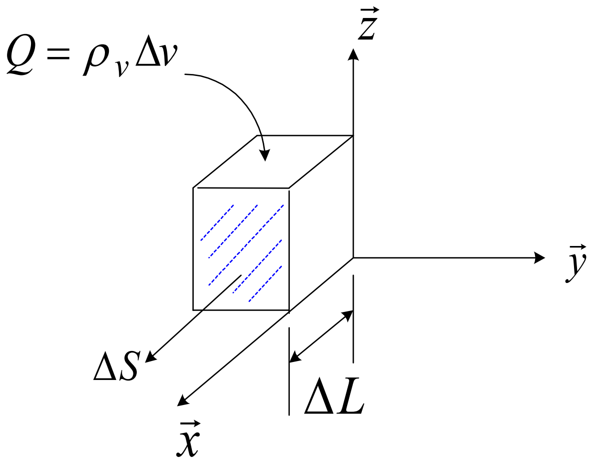
- Electron out-going: $\vec{v} = |\vec{v}|\,\vec{a}_x$

$$\vec{F} = (-Q)(\vec{v} \times \vec{B}) = (-Q)|\vec{v}||\vec{B}|(\vec{a}_x \times \vec{a}_z)$$

$$= Q|\vec{v}||\vec{B}|\vec{a}_y$$

- On electron moving, electron receive a tension to right side. ➔ N-type semiconductor

Force on a Differential Current Element

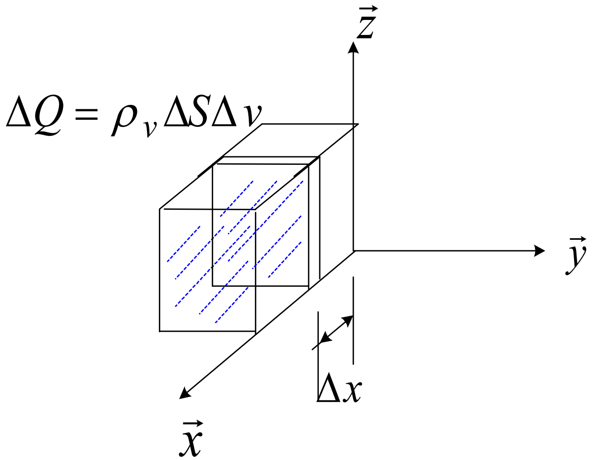


$$Q = \rho_v \Delta v = \rho_v \Delta S \Delta L$$

- In the time interval Δt, the element of charge has moved a distance Δx,

$$\Delta Q = \rho_v \Delta S \Delta x$$

- Let's assume surface ΔS moves Δx for Δt,



$$\Delta \vec{I} = \frac{\Delta Q}{\Delta t}\vec{a}_x = \rho_v \Delta S \frac{\Delta x}{\Delta t}\vec{a}_x = \rho_v \Delta S \vec{v}_x$$

$$\therefore \quad \vec{J} = \frac{\Delta \vec{I}}{\Delta S} = \rho_v \vec{v}_x$$

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}}$$

- Since $dQ = \rho_v dv$,

$$d\vec{F} = dQ(\vec{v} \times \vec{B}) = \rho_v dv(\vec{v} \times \vec{B}) = \rho_v \vec{v} \times \vec{B} dv = \vec{\mathrm{J}} \times \vec{B} dv$$

▪ For different current elements (filament current, surface current, or volume current),

$$I d\vec{\mathrm{L}} = \vec{\mathrm{K}} dS = \vec{\mathrm{J}} dv$$

$$\therefore \; d\vec{F} = (\vec{\mathrm{J}} \times \vec{B}) dv = (\vec{\mathrm{K}} \times \vec{B}) dS = I d\vec{\mathrm{L}} \times \vec{B}$$

: (modified) Lorentz force equation (3, 2, 1- dimension)

$$\vec{F} = \int_{vol} (\vec{\mathrm{J}} \times \vec{B}) dv$$ ➔ $$\vec{F} = f(I(\text{or } \vec{K}, \overleftarrow{J}), \vec{B})$$

$$= \int_{s} (\vec{\mathrm{K}} \times \vec{B}) dS$$

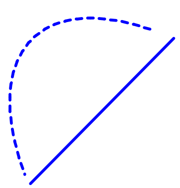
$$= \oint I d\vec{\mathrm{L}} \times \vec{B} = -I \oint \vec{B} \times d\vec{\mathrm{L}}$$

▪ For straight line conductor in a uniform magnetic field

$$\vec{F} = I\vec{\mathrm{L}} \times \vec{B}$$

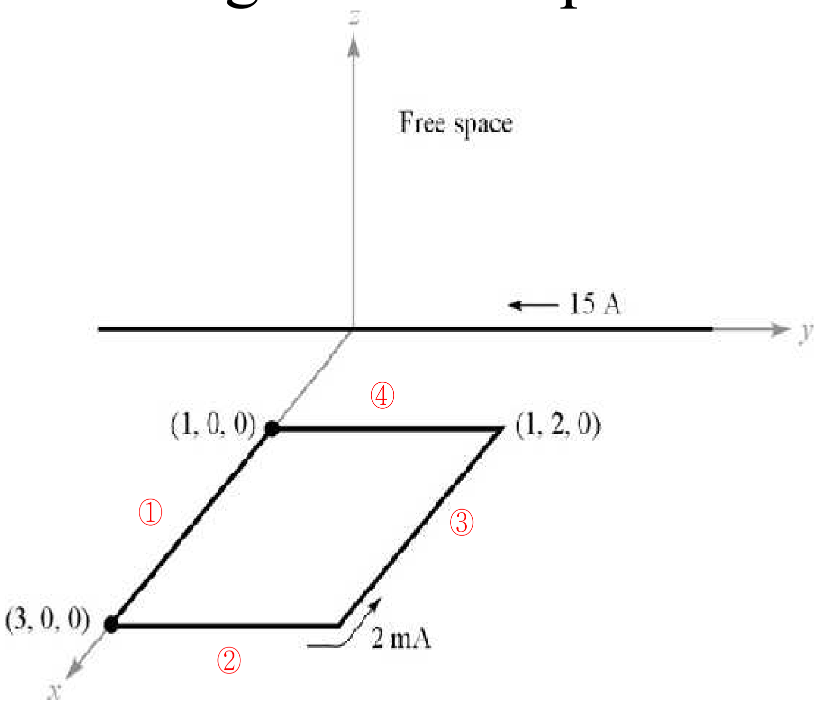
($\vec{F} = \oint I d\vec{\mathrm{L}} \times \vec{B}$ 을 이용하되 도선의 $\pm\infty$ 지점이 서로 연결되었다고 가정함)

$$F = BIL \sin\theta$$



[Ex. 8.1] Force on a Square Current Loop

- Magnetic field produced in plane of loop by straight filament wire:



$$\vec{H} = \frac{I}{2\pi x}\vec{a}_z = \frac{15}{2\pi x}\vec{a}_z \quad \text{[A/m]}$$

$$\left(\vec{H} = \oint \frac{I\,d\vec{L} \times \vec{a}_R}{4\pi R^2} \quad \left(d\vec{L} = dy(-\vec{a}_y), \quad \vec{a}_R = \vec{a}_x \right) \quad \vec{H} = \frac{I}{2\pi\rho}\vec{a}_\rho \right)$$

$$\vec{B} = \mu_0 \vec{H} = 4\pi \times 10^{-7}\,\vec{H} = \frac{3\times 10^{-6}}{x}\vec{a}_z$$

$$\vec{F} = \oint I d\vec{L} \times \vec{B} = -I\oint \vec{B} \times d\vec{L} = -(2\times 10^{-3})\int \frac{3\times 10^{-6}}{x}\vec{a}_z \times d\vec{L}$$

$$= -(2\times 10^{-3})\times(3\times 10^{-6})\left[\int_{x=1}^{3} \frac{\vec{a}_z}{x}\times dx\vec{a}_x + \int_{y=0}^{2}\frac{\vec{a}_z}{3}\times dy\vec{a}_y + \int_{x=3}^{1}\frac{\vec{a}_z}{x}\times dx\vec{a}_x + \int_{y=2}^{0}\frac{\vec{a}_z}{1}\times dy\vec{a}_y\right]$$

① ② ③ ④

[Ex. 8.1] Force on a Square Current Loop (continued)

$$= -6\times10^{-9}[\ln x|_1^3\,\vec{a}_y + \frac{1}{3}y|_0^2\,(-\vec{a}_x) + \ln x|_3^1\,\vec{a}_y + y|_2^0\,(-\vec{a}_x)]$$

$$= -6\times10^{-9}[\ln 3\,\vec{a}_y - \frac{2}{3}\vec{a}_x + \ln(\frac{1}{3})\vec{a}_y + 2\vec{a}_x]$$

$$= -8\times10^{-9}\vec{a}_x \ \ [\mathrm{N}]$$

8.3 Force Between Differential Current Elements

- Expression of the force on one current element directly in terms of second current element without finding the magnetic field. (직접관계식)

  → $\vec{B}$ 발생 → 1st current element 와 작용하여 $\vec{F}$ 발생 (간접 관계식)

- Magnetic field at point 2 due to current element at point 1

$$d\vec{H}_2 = \frac{I_1 d\vec{\mathrm{L}}_1 \times \vec{a}_{R12}}{4\pi R_{12}^2}$$

▪ Differential force on a differential current element

$$d\vec{F} = Id\vec{L} \times \underline{\vec{B}}$$

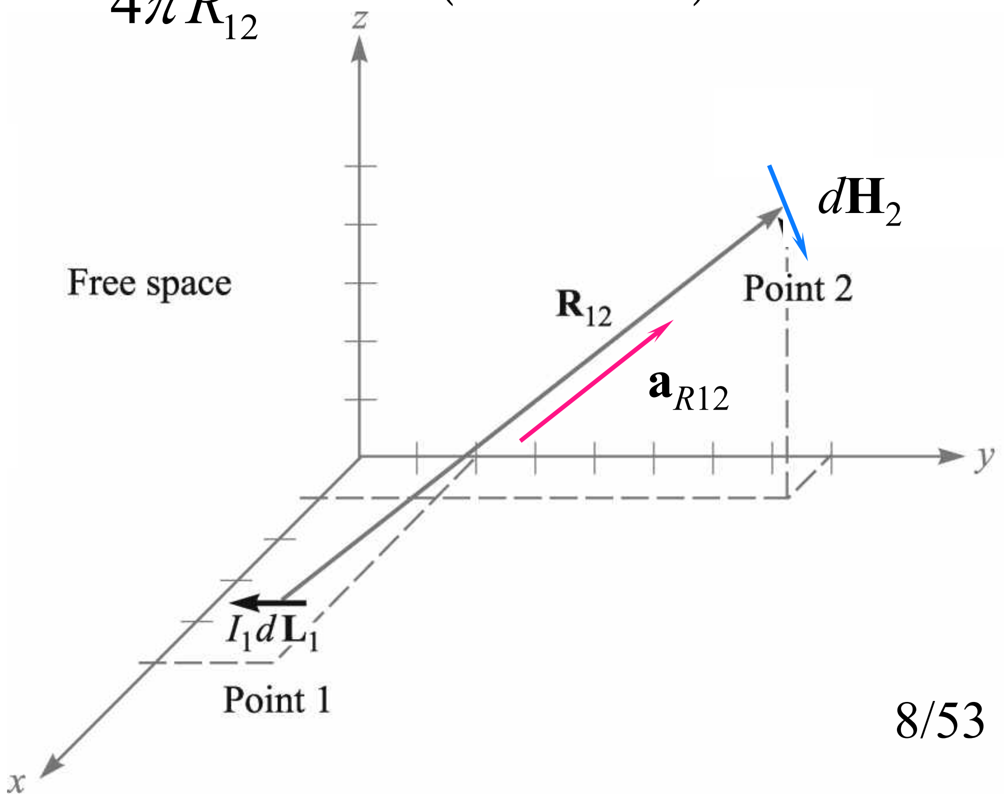
특정 point 1만이 아닌 point 1을 포함한 전체에 의한 magnetic flux density

Whereas $d\vec{B}_2$: differential flux density at point 2 caused by current element 1

$$d(dF_2) = I_2 d\vec{L}_2 \times d\vec{B}_2 = I_2 d\vec{L}_2 \times \mu_o \frac{I_1 d\vec{L}_1 \times \vec{a}_{R12}}{4\pi R_{12}^2} = \mu_o \frac{I_1 I_2}{4\pi R_{12}^2} d\vec{L}_2 \times \left(d\vec{L}_1 \times \vec{a}_{R12} \right)$$

$$= \frac{\mu_o}{4\pi R_{12}^3} I_2 d\vec{L}_2 \times \left(I_1 d\vec{L}_1 \times \vec{R}_{12} \right) = \mu_o \frac{I_1 I_2}{4\pi R_{12}^3} d\vec{L}_2 \times \left(d\vec{L}_1 \times \vec{R}_{12} \right)$$

($\because$ $d\vec{F}_2 = I_2 d\vec{L}_2 \times \vec{B}_2$ 라고 생각할 수도 있으나 point 2에서의 $\vec{B}$ 를 일으키는 전류가 한 점 (point 1) 이므로)

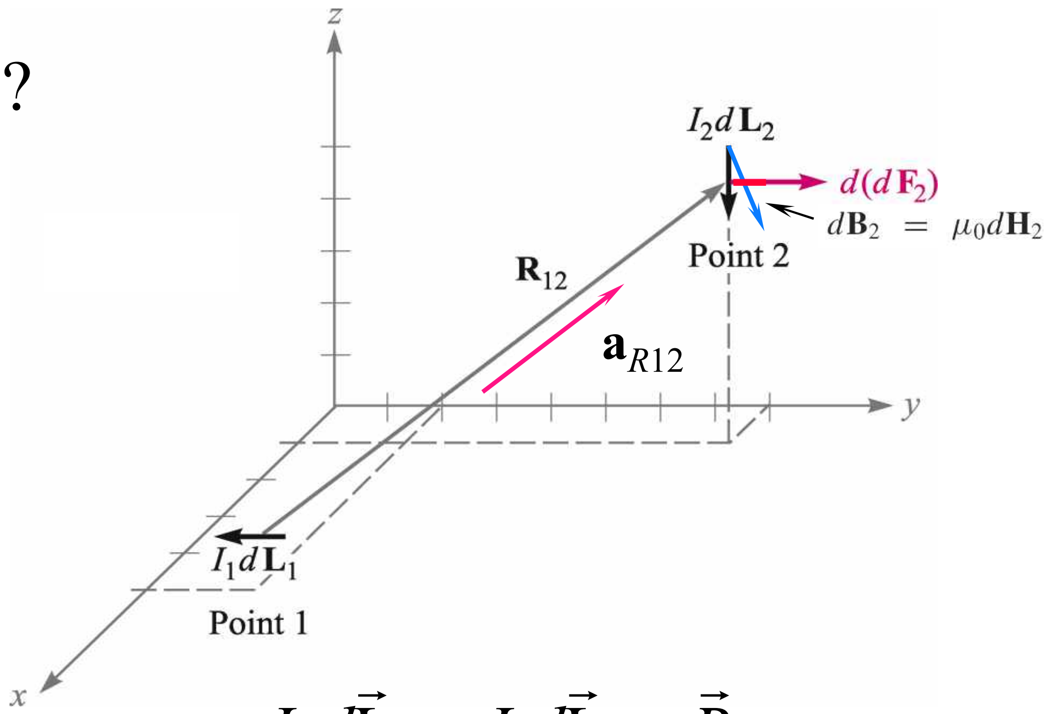


[Ex. 8.2] Differential force at P_2?

$$I_1 d\vec{L}_1 = -3\vec{a}_y \quad @\, P_1(5, 2, 1)$$

$$I_2 d\vec{L}_2 = -4\vec{a}_z \quad @\, P_2(1, 8, 5)$$

$$\vec{R}_{12} = -4\vec{a}_x + 6\vec{a}_y + 4\vec{a}_z$$



$$d(dF_2) = \mu_o \frac{I_1 I_2}{4\pi R_{12}^3} d\vec{L}_2 \times \left(d\vec{L}_1 \times \vec{R}_{12}\right) = \frac{\mu_o}{4\pi} \frac{I_2 d\vec{L}_2 \times I_1 d\vec{L}_1 \times \vec{R}_{12}}{R_{12}^3}$$

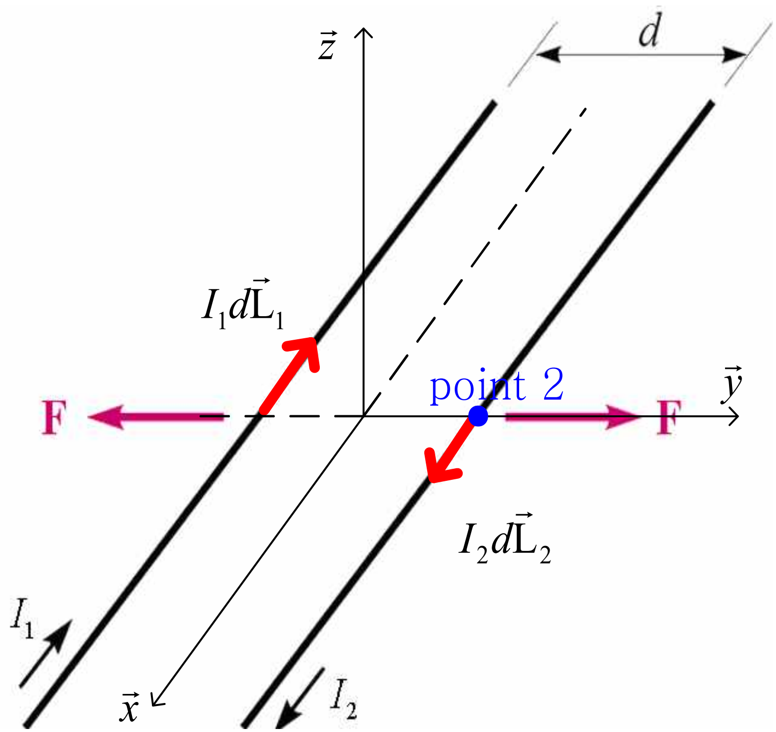
$$= \frac{4\pi \times 10^{-7}}{4\pi} \frac{(-4\vec{a}_z) \times (-3\vec{a}_y) \times (-4\vec{a}_x + 6\vec{a}_y + 4\vec{a}_z)}{(16+36+16)^{1.5}}$$

$$= 10^{-7} \times \frac{(-4\vec{a}_z) \times (-12\vec{a}_z - 12\vec{a}_x)}{560.7} = 8.56 \times 10^{-9} \vec{a}_y \quad [\mathrm{N}]$$

- Total force between two filamentary circuits:

$$\vec{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\vec{L}_2 \times \oint \frac{d\vec{L}_1 \times \vec{a}_{R12}}{R_{12}^2} \right] = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{\vec{a}_{R12} \times d\vec{L}_1}{R_{12}^2} \right] \times d\vec{L}_2$$

- Force of repulsion between two infinitely long, straight, parallel, filamentary conductors with separation d, and carrying equal but opposite current I:



$$d(d\vec{F}_2) = \underbrace{\mu_0 \frac{I_1 I_2}{4\pi R_{12}^2}}_{k} d\vec{L}_2 \times (d\vec{L}_1 \times \vec{a}_{R12})$$

$$= k(\vec{a}_x) \times \left[(-\vec{a}_x) \times (\vec{a}_y)\right]$$

$$= k(\vec{a}_x) \times (-\vec{a}_z) = k\vec{a}_y$$

$$\vec{H}_2 = \frac{I}{2\pi d}(-\vec{a}_z)$$ (I_1 에 의하여 point 2 에서 느끼는 magnetic field intensity)

$$\vec{F}_2 = \oint I_2 d\vec{L} \times \vec{B}_2 = \underline{\oint I_2 d\vec{L}} \times \left(-\frac{\mu_0 I}{2\pi d}\vec{a}_z\right)$$

$$= \underline{I\vec{a}_x} \times \left(-\frac{\mu_0 I}{2\pi d}\vec{a}_z\right) = \frac{\mu_0 I^2}{2\pi d}\vec{a}_y$$

(Straight conductor line 이므로
$\vec{F} = \oint I d\vec{L} \times \vec{B} = -I \int \vec{B} \times d\vec{L}$
➔ $\vec{F} = I\vec{L} \times \vec{B}$)

8.4 Force and torque on a closed circuit

- Force exerted on current system:

$$\vec{F} = -I\oint \vec{B} \times d\vec{L}$$

For an uniform magnetic flux density,

$$\vec{F} = -I\vec{B} \times \oint d\vec{L} \quad \longleftarrow \quad \oint d\vec{L} = 0$$

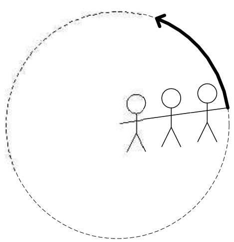
$$= 0 \qquad (Id\vec{L} = \vec{K}dS = \vec{J}dv \text{ are all applicable.})$$

$\therefore$ The force on a closed filamentary circuit in a uniform magnetic field is zero.
But the torque is? (Not '0')

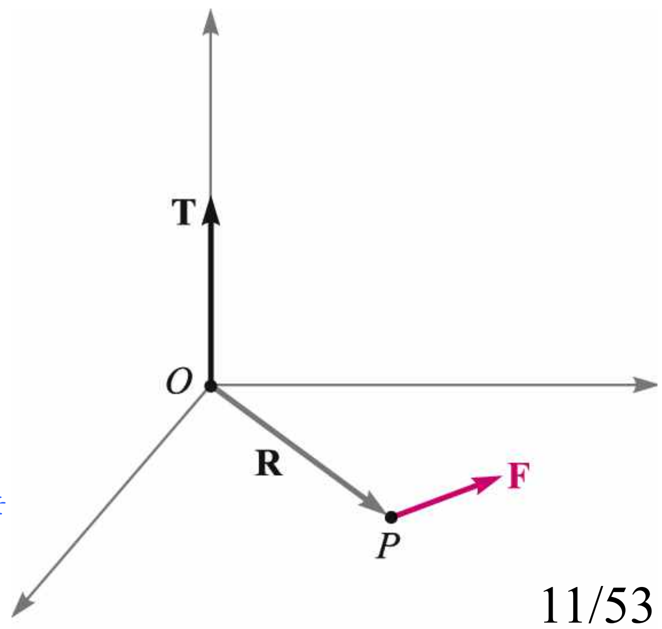
- Torque (or Moment, 회전력)

$$\vec{T} = \vec{R} \times \vec{F}$$

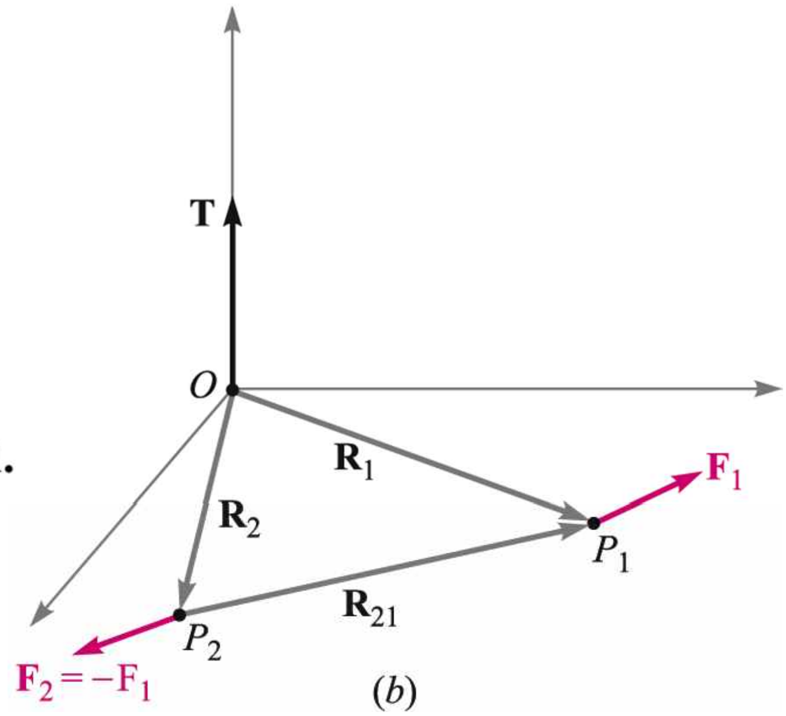
(원점으로부터 점 P에 이르는 rigid lever arm에 힘 $\vec{F}$ 가 주어졌을 때의 회전력)



동그라미 그릴 때 중심에서는
자리를 지키려는 힘이 작용.



- $\vec{F}_1$ @ $P_1 \rightarrow$ lever arm $\vec{R}_1$
  $\vec{F}_2$ @ $P_2 \rightarrow$ lever arm $\vec{R}_2$
  Common origin: O



Object does not undergo any translation.

$$\mathbf{T} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2$$

where $\mathbf{F}_1 + \mathbf{F}_2 = 0$

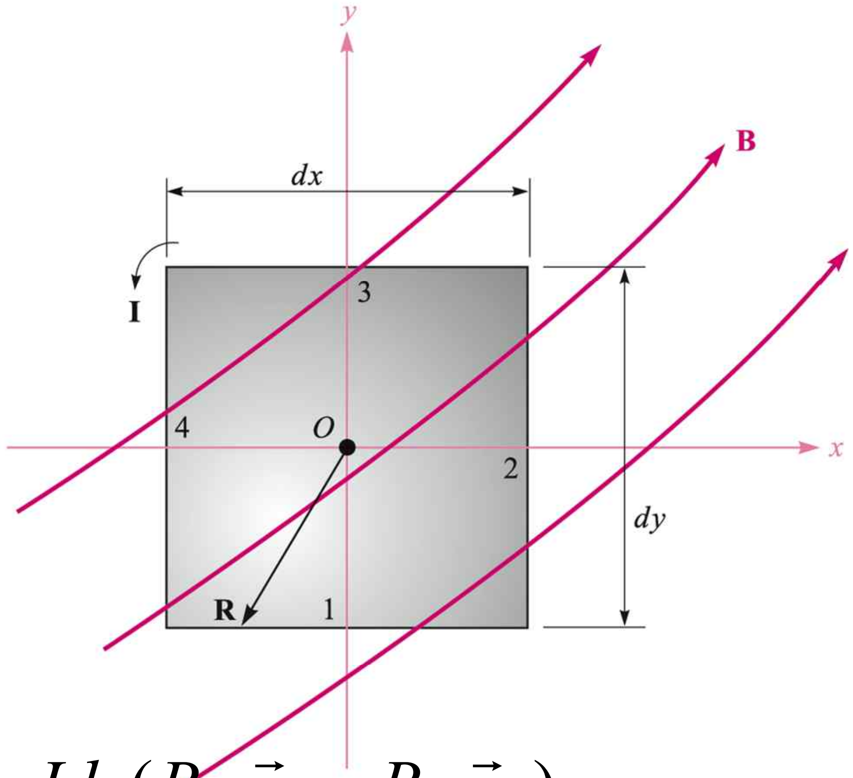
$$\mathbf{T} = (\mathbf{R}_1 - \mathbf{R}_2) \times \mathbf{F}_1 = \mathbf{R}_{21} \times \mathbf{F}_1$$

➔ *Independent of the choice of origin* for $\vec{R}_1$ and $\vec{R}_2$.
($\vec{R}_{21}$ is a relative distance and not relative on an origin.)

➔ The *torque is independent of the choice of origin* if total force is zero.

Torque on a Differential Current Loop

- Consider the torque on a differential current loop in a magnetic field $\vec{B}$.
- Magnetic flux density at center: $\vec{B}_0$
- Since the loop is of differential size, the value of $\vec{B}$ at any points of the loop can be assumed as $\vec{B}_0$.
- Vector force on side 1:



$$d\vec{F}_1 = Idx\vec{a}_x \times \vec{B}_0 \longleftarrow \boxed{\vec{F} = I\vec{L} \times \vec{B}}$$

$$= Idx\vec{a}_x \times (B_{0x}\vec{a}_x + B_{0y}\vec{a}_y + B_{0z}\vec{a}_z) = Idx(B_{0y}\vec{a}_z - B_{0z}\vec{a}_y)$$

Since $\vec{R}_1 = -\frac{1}{2}dy\vec{a}_y$,

$$d\vec{T}_1 = \vec{R}_1 \times d\vec{F}_1 = (-\frac{1}{2}dy\vec{a}_y) \times Idx(B_{0y}\vec{a}_z - B_{0z}\vec{a}_y)$$

$$= -\frac{1}{2}dxdyI B_{0y}\vec{a}_x$$

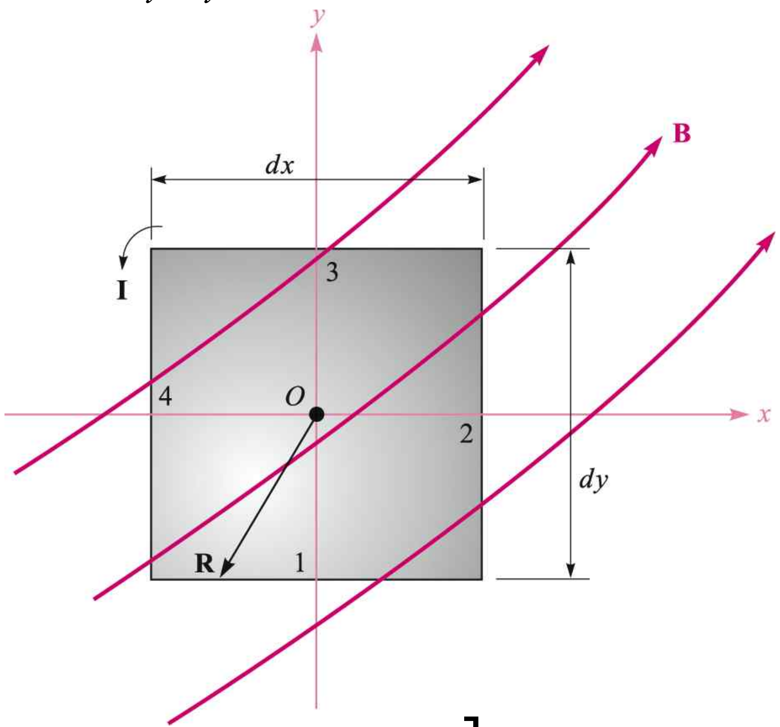
- By the same procedure on side 3,

$$d\vec{T}_3 = \vec{R}_3 \times d\vec{F}_3 = (\frac{1}{2}dy\vec{a}_y) \times \left[(-Idx\vec{a}_x) \times (B_{0x}\vec{a}_x + B_{0y}\vec{a}_y + B_{0z}\vec{a}_z)\right]$$

$$= (\frac{1}{2}dy\vec{a}_y) \times \left[-Idx(B_{0y}\vec{a}_z - B_{0z}\vec{a}_y)\right]$$

$$= -\frac{1}{2}dxdyIB_{0y}\vec{a}_x = d\vec{T}_1$$

$$\therefore\ d\vec{T}_1 + d\vec{T}_3 = -dxdyIB_{0y}\vec{a}_x$$



- By the same procedure on side 2,

$$d\vec{T}_2 = \vec{R}_2 \times d\vec{F}_2 = (\frac{1}{2}dx\vec{a}_x) \times \left[(Idy\vec{a}_y) \times (B_{0x}\vec{a}_x + B_{0y}\vec{a}_y + B_{0z}\vec{a}_z)\right]$$

$$= (\frac{1}{2}dx\vec{a}_x) \times \left[Idy(-B_{0x}\vec{a}_z + B_{0z}\vec{a}_x)\right] = \frac{1}{2}dxdyIB_{0x}\vec{a}_y = d\vec{T}_4$$

$$\therefore\ d\vec{T}_2 + d\vec{T}_4 = dxdyIB_{0x}\vec{a}_y$$

- Total torque:

$$
\begin{aligned}
d\vec{T} &= d\vec{T}_1 + d\vec{T}_2 + d\vec{T}_3 + d\vec{T}_4 \\
&= I\,dxdy(-B_{0y}\vec{a}_x + B_{0x}\vec{a}_y) = I\,dxdy\vec{a}_z \times (B_{0x}\vec{a}_x + B_{0y}\vec{a}_y + B_{0z}\vec{a}_z) \\
&= I\,dxdy\left(\vec{a}_z \times \vec{B}_o\right) \\
&= I\left[(dxdy\vec{a}_z) \times \vec{B}_0\right] = I\left[(dx\vec{a}_x \times dy\vec{a}_y) \times \vec{B}_0\right] \\
&= I(d\vec{S} \times \vec{B}_0) = Id\vec{S} \times \vec{B}_{(0)}
\end{aligned}
$$

where $d\vec{S}$: vector area of differential current loop

- Differential magnetic dipole moment: $d\vec{m}$ [A· m²]

$$d\vec{m} = I\,d\vec{S}$$

$$\therefore \quad d\vec{T} = d\vec{m} \times \vec{B}$$

- Similar procedure in $\vec{E}$ -field,

$$d\vec{T} = d\vec{p} \times \vec{E}$$: Torque produced on differential electric dipole by an electric field

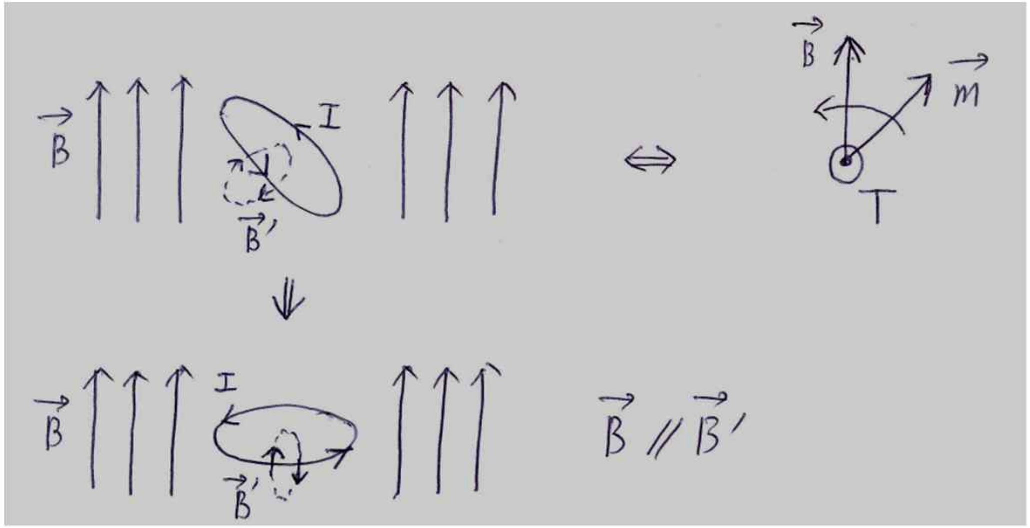
- Torque on a planar loop of any size or shape in an uniform magnetic field:

magnetic dipole moment

$$\vec{T} = I\vec{S} \times \vec{B} = \vec{m} \times \vec{B}$$

(loop 전류에 의해 생기는 자계방향을 외부의 자계방향과 일치시키도록 (전류 loop를 회전시키려는) 회전력이 가해짐.

→ **The easiest way to determine the direction of the torque**

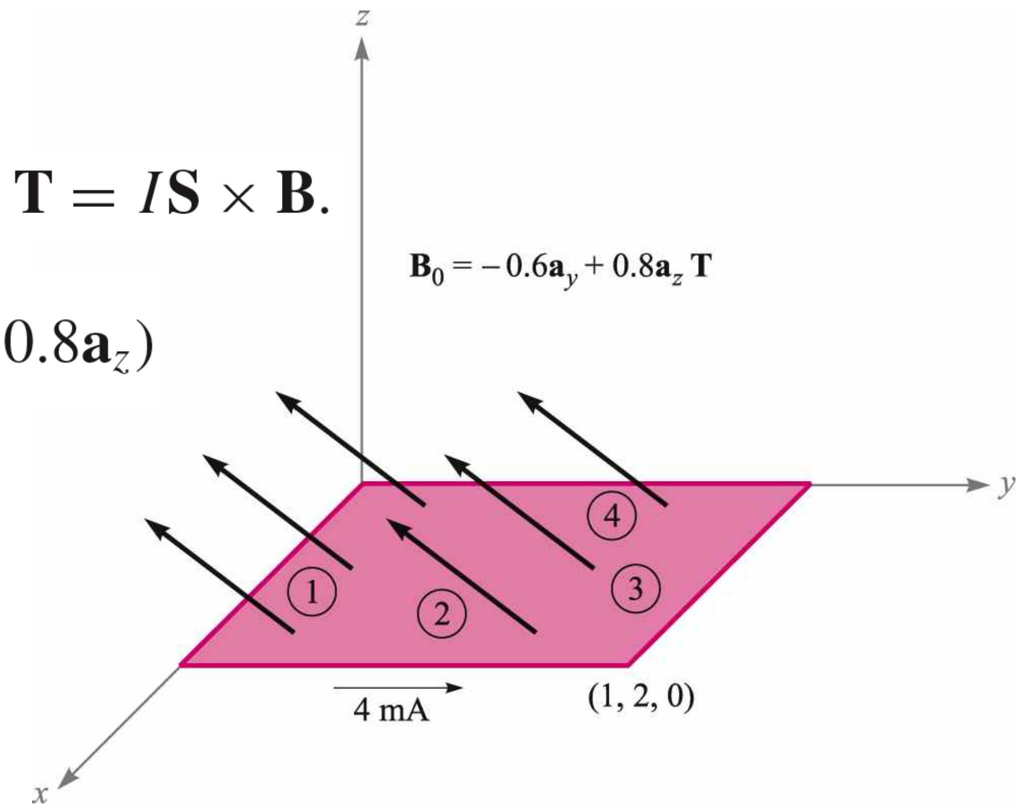


[Ex. 8.3] Calculate the torque by using $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$.

$$\mathbf{T} = 4 \times 10^{-3}[(1)(2)\mathbf{a}_z] \times (-0.6\mathbf{a}_y + 0.8\mathbf{a}_z)$$

$$= 4.8\mathbf{a}_x \text{ mN} \cdot \text{m}$$

(Thus, the loop tends to rotate about an axis parallel to the positive x-axis. The small magnetic field produced by the 4 mA loop current tends to line up with $\vec{B}$.)



[Ex. 8.4] Decomposite solution

- On side ①, $\vec{F}_1 = I\vec{L}_1 \times \vec{B}_0 = (4\times10^{-3})\vec{a}_x \times (-0.6\vec{a}_y + 0.8\vec{a}_z)$

$$= (-2.4\vec{a}_z - 3.2\vec{a}_y)\times10^{-3} \quad [\text{N}]$$

- On side ③, $\vec{F}_3 = I\vec{L}_3 \times \vec{B}_0 = 4\times10^{-3}(-\vec{a}_x)\times(-0.6\vec{a}_y + 0.8\vec{a}_z)$

$$= \left(2.4\vec{a}_z + 3.2\vec{a}_y\right)\times10^{-3} \ [\text{N}]$$

- On side ②, $\vec{F}_2 = I\vec{L}_2 \times \vec{B}_0 = (4\times10^{-3})(2\vec{a}_y)\times(-0.6\vec{a}_y + 0.8\vec{a}_z)$

$$= 6.4\vec{a}_x \times 10^{-3} \text{ [N]}$$

- On side ④, $\vec{F}_4 = I\vec{L}_4 \times \vec{B}_0 = -6.4\vec{a}_x \times 10^{-3}$ [N]

- Since these forces are distributed uniformly along each of the sides, each forces were applied at the center (1/2, 1).

$$\vec{T} = \vec{T}_1 + \vec{T}_2 + \vec{T}_3 + \vec{T}_4 = \vec{R}_1\times\vec{F}_1 + \vec{R}_2\times\vec{F}_2 + \vec{R}_3\times\vec{F}_3 + \vec{R}_4\times\vec{F}_4$$

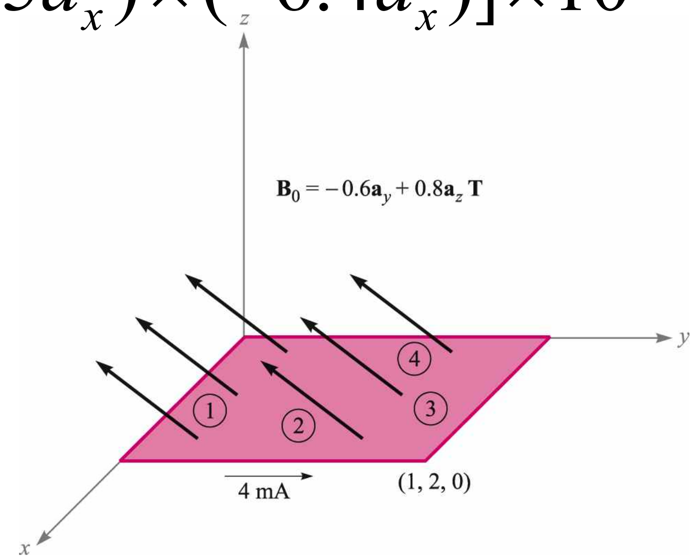
$$= [(-1\vec{a}_y)\times(-3.2\vec{a}_y - 2.4\vec{a}_z) + (0.5\vec{a}_x)\times(6.4\vec{a}_x)$$

$$+(1\vec{a}_y)\times(3.2\vec{a}_y + 2.4\vec{a}_z) + (-0.5\vec{a}_x)\times(-6.4\vec{a}_x)]\times10^{-3}$$

$$= (2.4\vec{a}_x + 2.4\vec{a}_x)\times10^{-3}$$

$$= 4.8\times10^{-3}\,\vec{a}_x \quad \text{[N/m]}$$

➔ Same result with Ex. 8.3



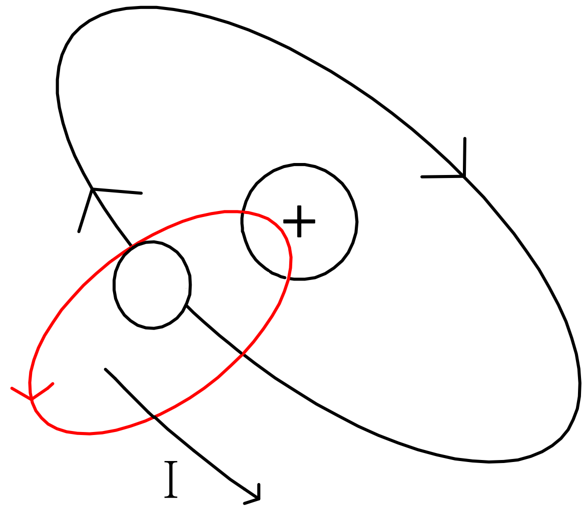
8.5 The Nature of Magnetic Materials

1) The 1st moment in magnetic fields

- An electron in an orbit is analogous to a small current loop.

  $\rightarrow$ $\vec{T}$ occurs due to $\vec{T} = I\vec{S} \times \vec{B}$.

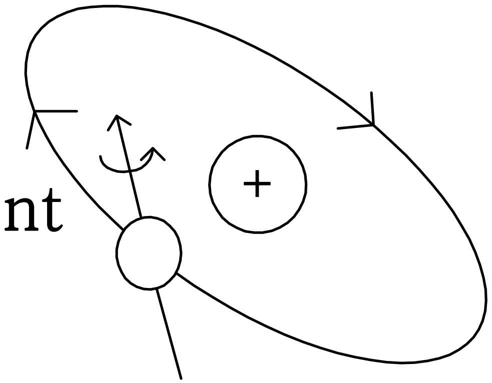
  $\rightarrow$ The torque tend to align the magnetic field produced by the orbiting election with the external magnetic fields.



2) The 2nd moment is attributed to electron spin

$$d\vec{m} = \pm 9 \times 10^{-24}\, A \cdot m^2$$: spin magnetic moment

alignment aiding or opposing an external magnetic field.



3) The 3rd moment is caused by nuclear spin. ➔ Negligible effect

Magnetic Material Summary

Classification	Magnetic Moments	B Values	Comments
Diamagnetic	$\mathbf{m}_{\text{orb}} + \mathbf{m}_{\text{spin}} = 0$	$B_{\text{int}} < B_{\text{appl}}$	$B_{\text{int}} \doteq B_{\text{appl}}$
Paramagnetic	$\mathbf{m}_{\text{orb}} + \mathbf{m}_{\text{spin}} = \text{small}$	$B_{\text{int}} > B_{\text{appl}}$	$B_{\text{int}} \doteq B_{\text{appl}}$
Ferromagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} \gg B_{\text{appl}}$	Domains
Antiferromagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} \doteq B_{\text{appl}}$	Adjacent moments oppose
Ferrimagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} > B_{\text{appl}}$	Unequal adjacent moments oppose; low σ
Superparamagnetic	$\lvert\mathbf{m}_{\text{spin}}\rvert \gg \lvert\mathbf{m}_{\text{orb}}\rvert$	$B_{\text{int}} > B_{\text{appl}}$	Nonmagnetic matrix; recording tapes

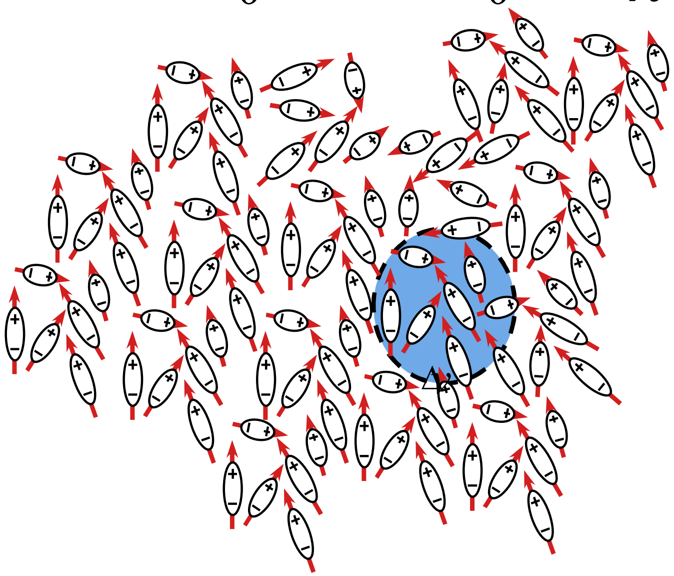
8.6 Magnetization and Permeability

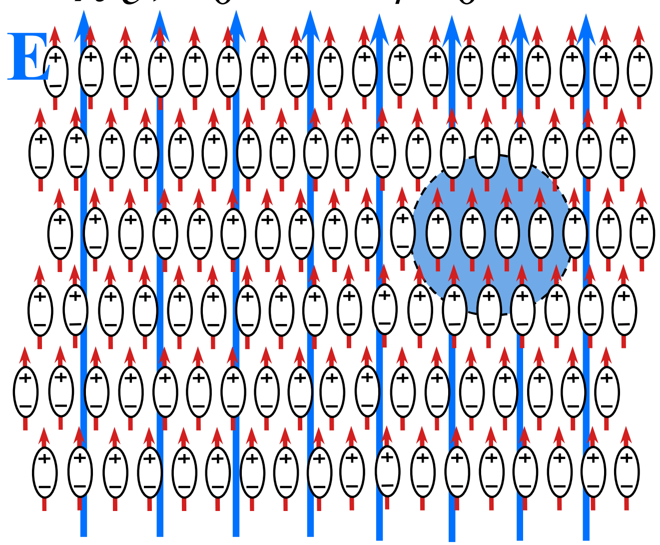
- The magnetic dipoles act as a distributed source for the magnetic field.

  cf.) Electric dipole: $\vec{P} = Q\vec{d} = \chi_e \varepsilon_0 \vec{E}$

  Electric flux density in dielectric materials

$$\vec{D} = \vec{D}_0 + \vec{P} = \varepsilon_0 \vec{E} + \chi_e \varepsilon_0 \vec{E} = (1+\chi_e)\varepsilon_0 \vec{E} = \varepsilon_r \varepsilon_0 \vec{E}$$

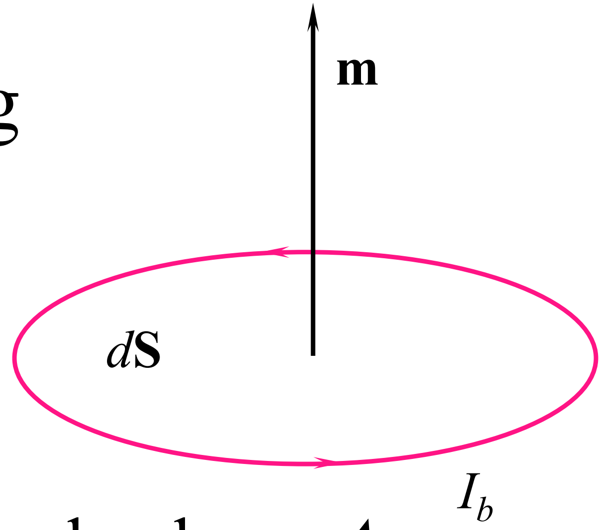




- Movement of bound charges (orbital electrons, electron spin, nuclear spin): bound current or Amperian current

  → A new field cause by the bound charged currents has the same dimension of $\vec{H}$ and is called the magnetization $\vec{M}$.

Magnetic Dipole Ensembles

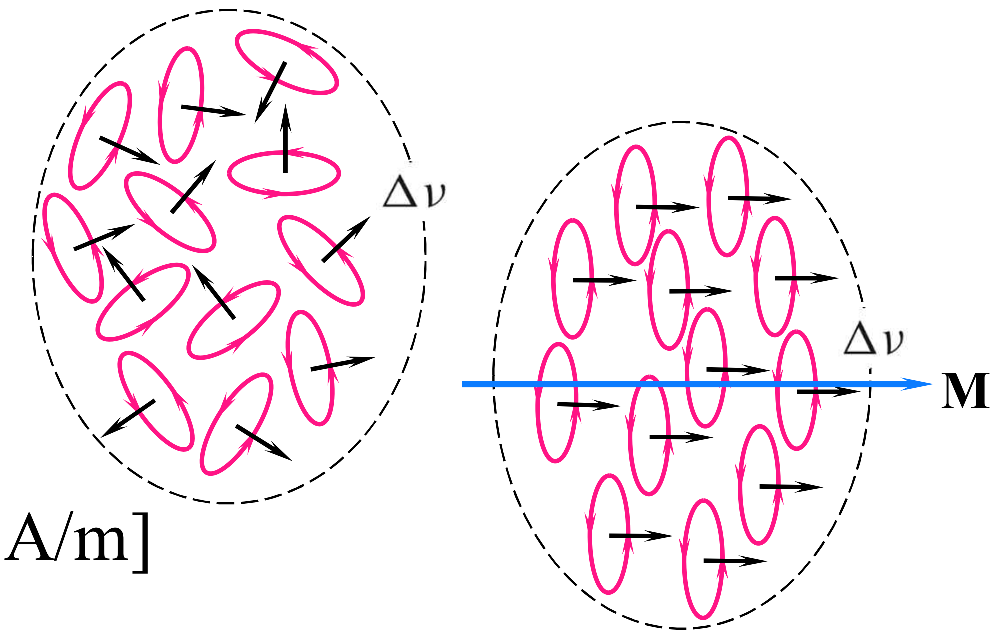


- Consider a bound current, I_b, surrounding a differential area, dS.
- Magnetic dipole moment:

$$\mathbf{m} = I_b d\mathbf{S}$$

- Consider n magnetic dipoles per unit volume and volume Δv

$$\mathbf{m}_{\text{total}} = \sum_{i=1}^{n\Delta v} \mathbf{m}_i$$



- *Magnetization* $\vec{M}$: magnetic dipole moment per unit volume as the volume shrinks to a point

$$\vec{M} = \lim_{\Delta v \to 0} \frac{\vec{m}_{total}}{\Delta v} = \lim_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \vec{m}_i \quad \text{[A/m]}$$

- If all have the same orientation, the magnetization is simplified as

$$\mathbf{M} = n\mathbf{m} = nI_b d\mathbf{S}$$

Bound Current Formulation

- Identical magnetic moments ($\vec{m}$, n per unit volume) makes an angle θ with the element of path $d\mathbf{L}$.
- $\vec{m}$ consists of a bound current I_b circulation about an area $d\vec{S}$.
- Differential *bound current*, crossing the surface along length dL:

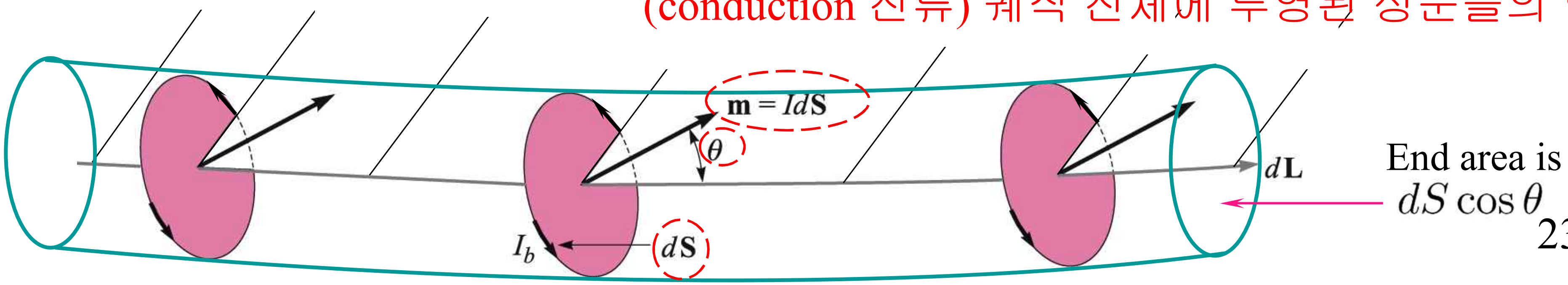
$$dI_B = \underbrace{nI_b}_{\text{Dipole current per unit volume}} \underbrace{d\mathbf{S} \cdot d\mathbf{L}}_{\text{Differential volume, } dv} = \mathbf{M} \cdot d\mathbf{L}$$

(Magnetic moment를 일으키는 전체 전류 (nI_b) 중에 $d\mathbf{L}$ 방향으로 투영된 n개의 bounded current를 고려)

$$I_B = \oint \mathbf{M} \cdot d\mathbf{L} \qquad (21)$$

: within and entire closed contour

(bounded 전류가 실제로 움직이는 (conduction 전류) 궤적 전체에 투영된 성분들의 합)



General Relation Between **B** and **H** in specific media except free space

- The *total current* (I_T)in a general medium consist of the sum of bound and free currents
- Defining **B** as the fundamental magnetic field quantity, Ampere's Circuital Law for the total current becomes:

$$\oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} = I_T = I_{\mathbf{B}} + I$$

- *Total free current* (I):

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = I_T - I_{\mathbf{B}} = \oint \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \cdot d\mathbf{L}$$

Thus: $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$ …and finally: $\boxed{\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})}$

▪ Current and Related Current Densities with Stoke's theorem

$$I_T = I + I_B$$

$$I = \oint \vec{H} \cdot d\vec{L} = \int (\nabla \times \vec{H}) \cdot d\vec{S} = \int \vec{J} \cdot d\vec{S}$$

Bound Current: $$I_B = \int_s \mathbf{J}_B \cdot d\mathbf{S} \qquad (21)$$

$$= \oint \vec{M} \cdot d\vec{L} = \int (\nabla \times \vec{M}) \cdot d\vec{S} \quad \rightarrow \nabla \times \vec{M} = \vec{J}_B$$

Total Current: $$I_T = \int_s \mathbf{J}_T \cdot d\mathbf{S} \qquad (22)$$

$$= \oint \frac{\vec{B}}{\mu_0} \cdot d\vec{L} = \int (\nabla \times \frac{\vec{B}}{\mu_0}) \cdot d\vec{S} \quad \rightarrow \nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}_T$$

Conduction Current: $$I = \int_s \mathbf{J} \cdot d\mathbf{S} \qquad (26)$$

$$= \oint \vec{H} \cdot d\vec{L} = \int (\nabla \times \vec{H}) \cdot d\vec{S} \quad \rightarrow \nabla \times \vec{H} = \vec{J}$$

- Relation between $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$

$$\mathbf{M} = \chi_m \mathbf{H}$$ for linear isotropic media

magnetic susceptibility function
(frequency domain)

$$\vec{B} = \mu_0(\vec{H} + \chi_m \vec{H}) = \mu_0(1 + \chi_m)\vec{H} = \mu_0 \mu_r \vec{H}$$

where χ_m: magnetic susceptibility
$\mu_r = 1 + \chi_m$: relative permeability

- *Permeability*: $\underline{\mu = \mu_0 \mu_r}$

- Consequently: $\boxed{\mathbf{B} = \mu \mathbf{H}}$

[Ex. 8.5] Ferrite material ($\mu_r = 50$). $|B| = 0.05$ [T]

$$\chi_m = \mu_r - 1 = 49$$

(isotropic material 이므로 vector로 표기하지 않음 $\vec{H} // \vec{B} // \vec{M}$)

$$H = \frac{B}{\mu_0 \mu_r} = \frac{0.05}{4\pi \times 10^{-7} \times 50} = 796 \quad [\text{A/m}]$$

$$M = \chi_m H = 49H = 39000 \quad [\text{A/m}]$$

($\gg \vec{H}$)

➔ Magnetic flux density depends on the motion of the bound charges mainly

- Permeability of an anisotropic magnetic material

$$\begin{aligned} B_x &= \mu_{xx} H_x + \mu_{xy} H_y + \mu_{xz} H_z \\ B_y &= \mu_{yx} H_x + \mu_{yy} H_y + \mu_{yz} H_z \\ B_z &= \mu_{zx} H_x + \mu_{zy} H_y + \mu_{zz} H_z \end{aligned} \quad \Longrightarrow [B] = [\underline{\mu}][H]$$

Permeability tensor

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) : \text{valid} \quad (\vec{H} \nparallel \vec{B},\ \vec{H} \nparallel \vec{M},\ \vec{B} \nparallel \vec{M})$$

8.7 Magnetic (Field) Boundary Conditions

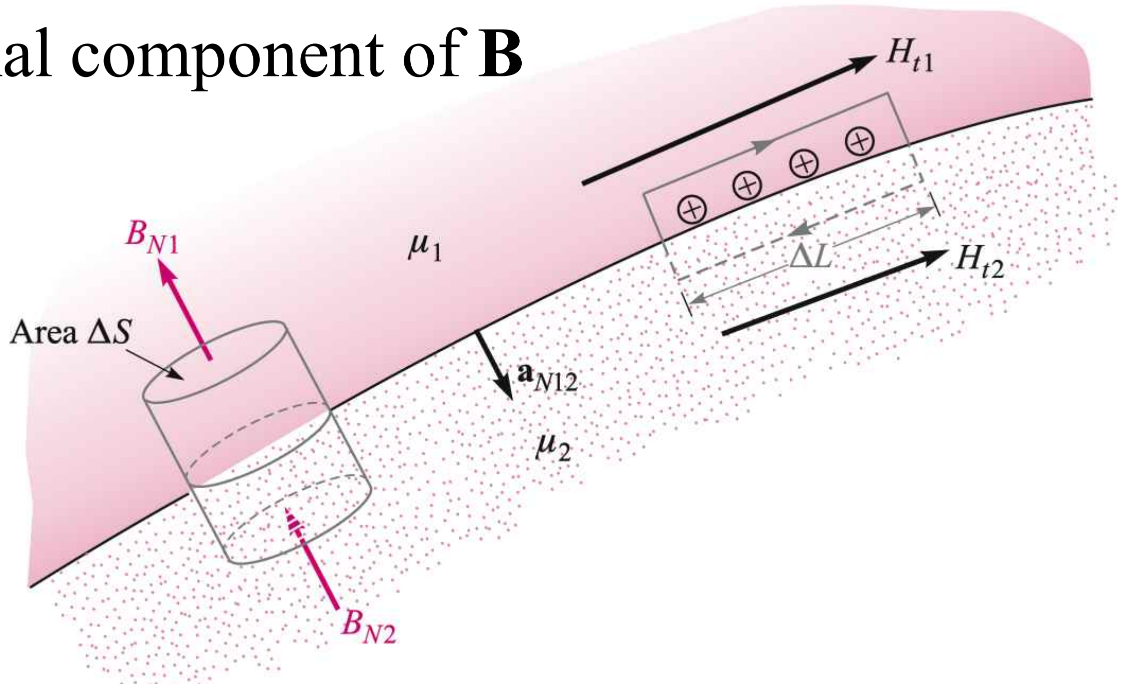
- Two isotropic homogeneous linear material with permeabilities μ_1 and μ_2.

- Boundary condition for normal component of **B** (with Gauss' Law)

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

$$B_{N1}\Delta S - B_{N2}\Delta S = 0$$

$$@\ h \rightarrow 0$$



$$\boxed{B_{N2} = B_{N1}}$$ ➔ The normal component of **B** is continuous across a boundary.

$$\Longrightarrow H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \quad \longleftarrow \quad M = \chi_m H$$

$$\frac{M_{N2}}{\chi_{m2}} = \frac{\mu_1}{\mu_2}\frac{M_{N1}}{\chi_{m1}} \quad \Rightarrow \quad M_{N2} = \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1}$$

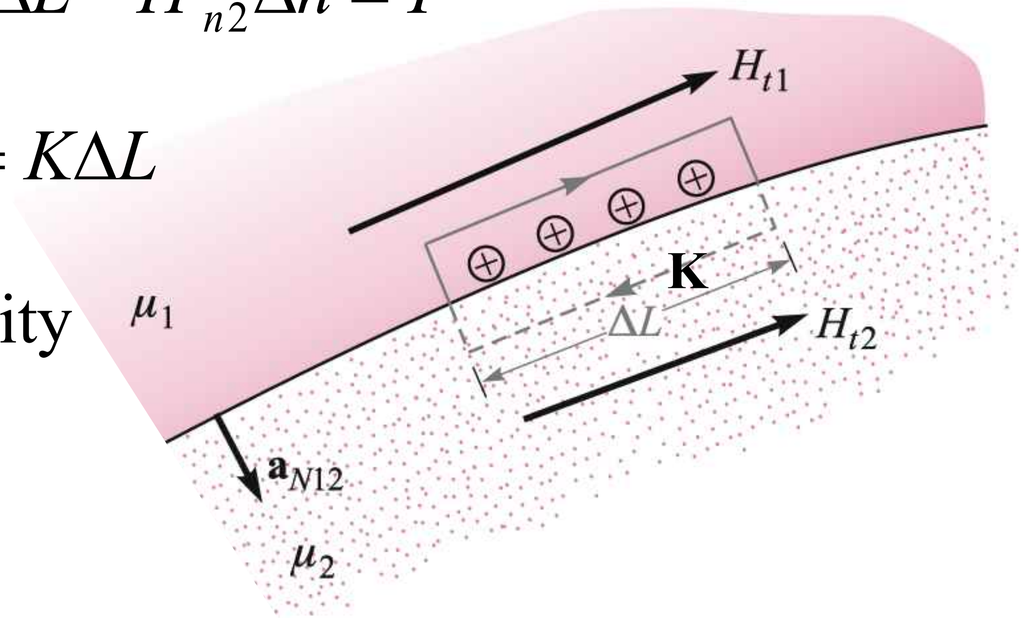
▪ Boundary condition for tangential component of **H** with Ampere's circuital law

$$\oint \vec{H} \cdot d\vec{\mathrm{L}} = H_{t1}\Delta L + H_{n1}\Delta h - H_{t2}\Delta L - H_{n2}\Delta h = I$$

$$\lim_{\Delta h \to 0} \oint \vec{H} \cdot d\vec{\mathrm{L}} = H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

where **K**: surface current density

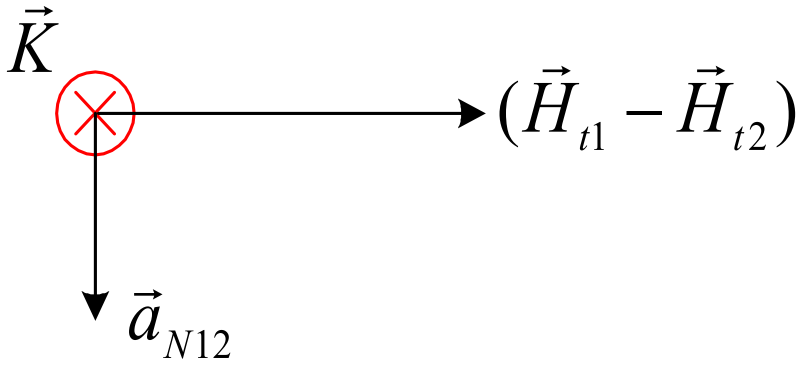
$$\therefore \quad \boxed{H_{t1} - H_{t2} = K}$$



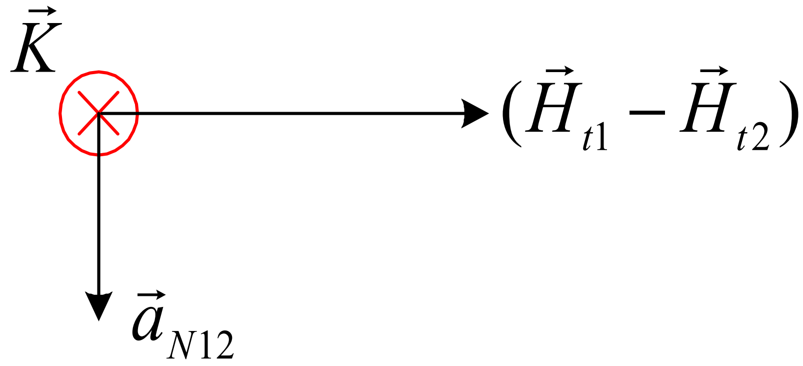
▪ More generally:

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{N12} = \mathbf{K}$$

or $$\mathbf{H}_{t1} - \mathbf{H}_{t2} = \mathbf{a}_{N12} \times \mathbf{K}$$



- An equivalent formulation



$$\vec{H}_{t1} - \vec{H}_{t2} = \vec{a}_{N12} \times \vec{K}$$

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \qquad \leftarrow \qquad B = \mu_0 \mu_r H = \mu_0 \mu_r \frac{M}{\chi_m} = \mu \frac{M}{\chi_m}$$

$$\frac{\cancel{\mu_1} \dfrac{M_{t1}}{\chi_{m1}}}{\cancel{\mu_1}} - \frac{\cancel{\mu_2} \dfrac{M_{t2}}{\chi_{m2}}}{\cancel{\mu_2}} = K$$

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K$$

[Ex. 8.6] $\mu_1 = 4$ [μH/m] @ $z > 0$ (region 1)

$\mu_2 = 7$ [μH/m] @ $z < 0$ (region 2)

$\vec{K} = 80\vec{a}_x$ [A/m] @ $z = 0$, $\vec{B}_1 = 2\vec{a}_x - 3\vec{a}_y + \vec{a}_z$ [mT] $\vec{B}_2 = ?$

$$\Rightarrow \mathbf{B}_{N1} = (\mathbf{B}_1 \cdot \mathbf{a}_{N12})\mathbf{a}_{N12} = [(2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z) \cdot (-\mathbf{a}_z)](-\mathbf{a}_z) = \mathbf{a}_z \text{ mT}$$

$$\therefore \underline{\mathbf{B}_{N2} = \mathbf{B}_{N1} = \mathbf{a}_z \text{ mT}}$$

$$\mathbf{B}_{t1} = \mathbf{B}_1 - \mathbf{B}_{N1} = 2\mathbf{a}_x - 3\mathbf{a}_y \text{ mT}$$

$$\mathbf{H}_{t1} = \frac{\mathbf{B}_{t1}}{\mu_1} = \frac{(2\mathbf{a}_x - 3\mathbf{a}_y)10^{-3}}{4 \times 10^{-6}} = 500\mathbf{a}_x - 750\mathbf{a}_y \text{ A/m}$$

$$\mathbf{H}_{t2} = \mathbf{H}_{t1} - \mathbf{a}_{N12} \times \mathbf{K} = 500\mathbf{a}_x - 750\mathbf{a}_y - (-\mathbf{a}_z) \times 80\mathbf{a}_x$$

$$= 500\mathbf{a}_x - 750\mathbf{a}_y + 80\mathbf{a}_y = 500\mathbf{a}_x - 670\mathbf{a}_y \text{ A/m}$$

$$\mathbf{B}_{t2} = \mu_2\mathbf{H}_{t2} = 7 \times 10^{-6}(500\mathbf{a}_x - 670\mathbf{a}_y) = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y \text{ mT}$$

$$\mathbf{B}_2 = \mathbf{B}_{N2} + \mathbf{B}_{t2} = \underline{3.5\mathbf{a}_x - 4.69\mathbf{a}_y + \mathbf{a}_z \text{ mT}}$$

8.8 The Magnetic Circuit

- Relationship between the electrostatic potential and electric field intensity:

$$\mathbf{E} = -\nabla V$$

- Its analogous relation to the magnetic field intensity.

$$\mathbf{H} = -\nabla V_m \qquad @\vec{J} = 0$$

where V_m: scalar magnetic potential or magnetomotive force (mmf)

- Electric potential difference between A and B

$$V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{L}$$ (이동 경로에 무관)

By the corresponding relation between the mmf and $\vec{H}$,

$$V_{mAB} = \int_A^B \mathbf{H} \cdot d\mathbf{L}$$ (특정 이동 경로를 지정하면 specific value 발생)

- Ohm's law for electric field: $\mathbf{J} = \sigma \mathbf{E}$

- The magnetic flux density will be the analog of the current density.

$$\mathbf{B} = \mu \mathbf{H}$$

- Total (electric) current: $I = \int_S \mathbf{J} \cdot d\mathbf{S}$

  ↔ Total magnetic flux flowing through the cross section of a magnetic circuit

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

- Resistance: $V = IR$

  ↔ Reluctance: $V_m = \Phi \underline{\underline{\Re}}$

[A · turn/Wb]

$\Re = \dfrac{V_m}{\Phi}$: a ratio of magnetomotive force (mmf) to total flux

Reluctance as an Analogy to Resistance

- $R = \frac{d}{\sigma S}$ $\leftrightarrow$ $\boxed{\Re = \frac{d}{\mu S}}$

where d: isotropic homogeneous material length

S: uniform cross section

- Source voltage in electric circuit: $\oint \mathbf{E} \cdot d\mathbf{L} = 0$

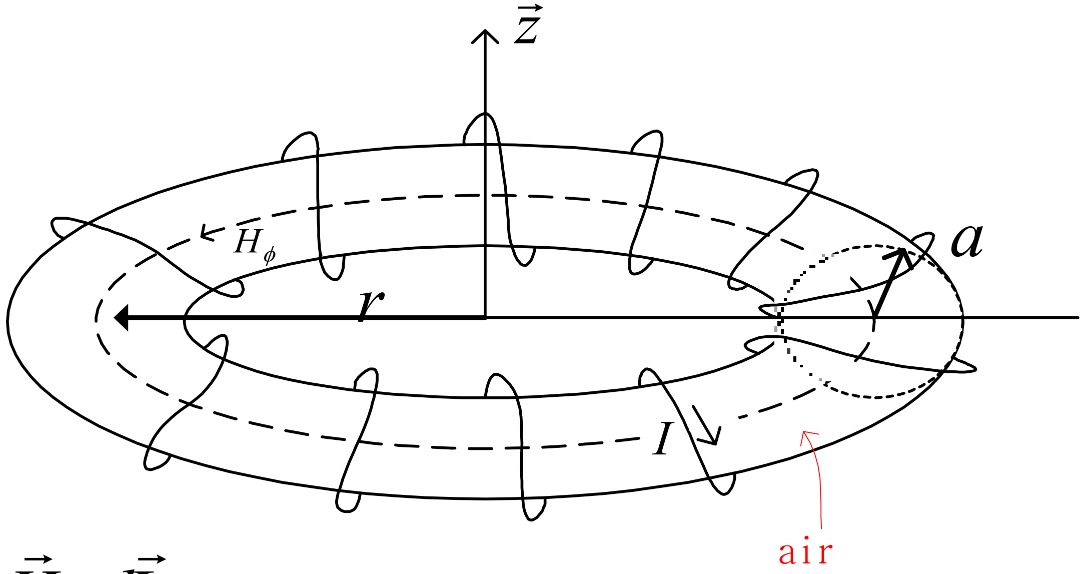
➔ KVL: The rise in potential (electromotive force, emf) through the source is exactly equal to the fall in potential through the load.

$\leftrightarrow$ $\oint \mathbf{H} \cdot d\mathbf{L} = I_{\text{total}}$

$(= NI)$ (in case of a current I flown through an N-turn coil)

$V_{mAB} = \int_A^B \mathbf{H} \cdot d\mathbf{L}$: This quantity is the **mmf** around a closed path, which we use as V_m in our magnetic circuit equation.

[Ex.] $N = 500$ turns, $S = 6\ \text{cm}^2$, $r = 15$ cm, $I = 4$ A, Toroid



$$V_{mAB} = \int_A^B \vec{H} \cdot d\vec{\mathrm{L}}$$

$$V_m = \oint \vec{H} \cdot d\vec{\mathrm{L}} = N\, I = 500 \times 4 = 2000 \quad [\mathrm{A \cdot t}] \quad \leftarrow \ \text{mmf}$$

$$\Re = \frac{d}{\mu S} = \frac{2\pi \times 0.15}{4\pi \times 10^{-7} \times 6 \times 10^{-4}} = 1.25 \times 10^{9} \quad [\mathrm{A \cdot t/Wb}] \quad \leftarrow \text{reluctance}$$

$$\Phi = \frac{V_m}{\Re} = \frac{2000}{1.25 \times 10^{9}} = 1.6 \times 10^{-6} \quad [\mathrm{Wb}] \quad \leftarrow \text{total flux}$$

$$B = \frac{\Phi}{S} = \frac{1.6 \times 10^{-6}}{6 \times 10^{-4}} = 2.67 \times 10^{-3} \quad [\mathrm{T}] \quad \leftarrow \ \text{magnetic flux density}$$

[Ex.] (Continued)

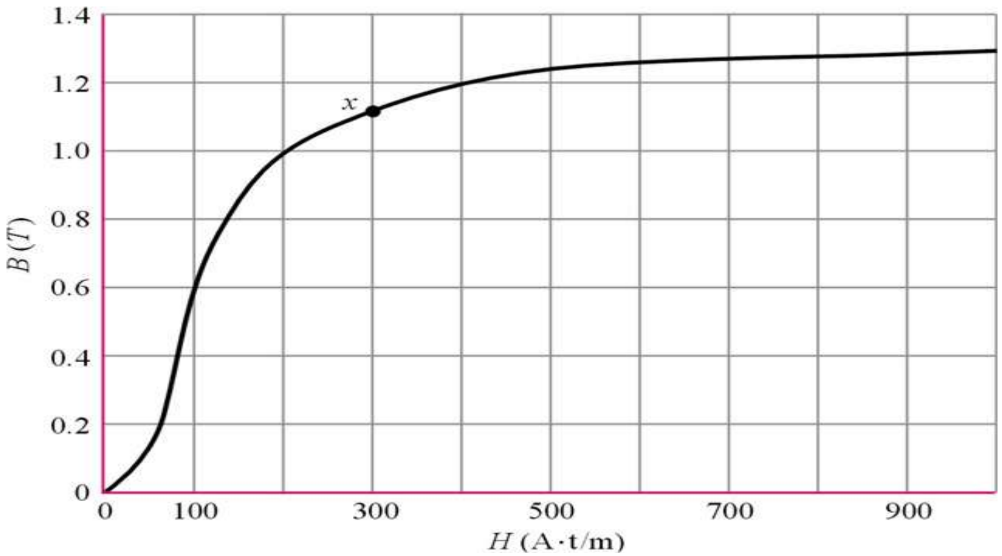
$$H = \frac{B}{\mu} = \frac{2.67\times10^{-3}}{4\pi\times10^{-7}} = 2125 \quad [\mathrm{A\cdot t/m}] \leftarrow \text{magnetic field intensity in free space}$$

By Ampere's circuital law,

$$H_\phi 2\pi r = NI$$

$$H_\phi = \frac{NI}{2\pi r} = \frac{500\times4}{2\pi\times0.15} = 2122 \quad [\mathrm{A\cdot t/m}]$$: similar results

- Ferromagnetic materials (general material): $|\vec{m}_{spin}| \gg |\vec{m}_{orb}|$



In ferromagnetic materials, **B** increases with increasing **H**, but in a nonlinear manner as shown in the typical curve

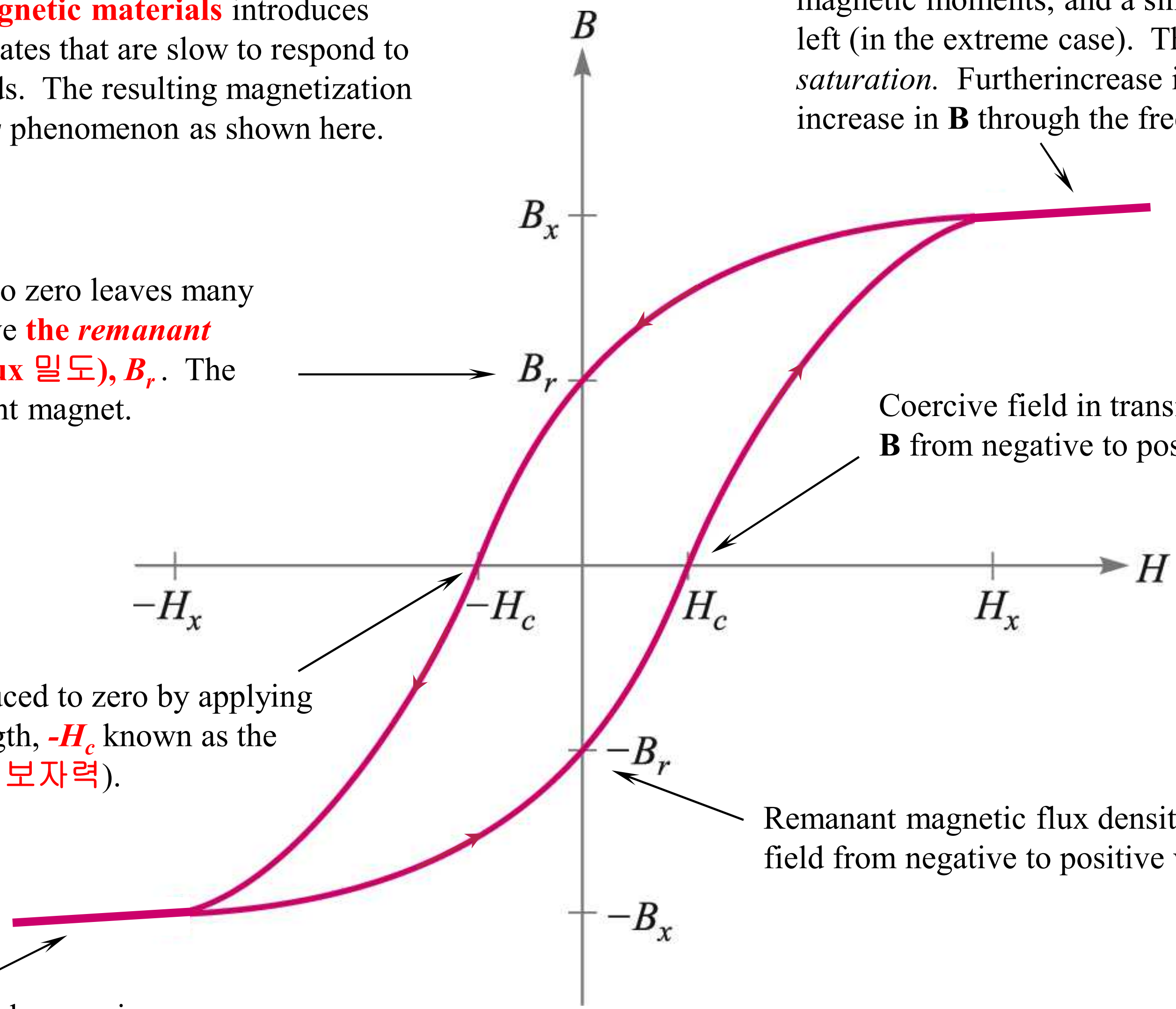
FIGURE 9.11 Magnetization curve of a sample of silicon sheet steel.

A Further Complication: Hysteresis

Domain wall shifting in **ferromagnetic materials** introduces semi-permanent magnetization states that are slow to respond to changes in applied magnetic fields. The resulting magnetization curve demonstrates the *hysteresis* phenomenon as shown here.

Increasing **H** to high positive values lines up all magnetic moments, and a single domain is left (in the extreme case). The core is thus in *saturation*. Furtherincrease in **H** leads to an increase in **B** through the free space permeability

Decreasing the applied **H** field to zero leaves many dipoles still aligned, and we have **the *remanant* magnetic flux density (잔여 flux 밀도), B_r**. The material has become a permanent magnet.



Coercive field in transistioning **B** from negative to positive values

The remanant flux density is reduced to zero by applying an opposing magentic field strength, ***-H_c*** known as the *coercive* field (or **coercive force, 보자력**).

Remanant magnetic flux density, for *increasing* **H** field from negative to positive values

Increasing **H** to high negative values again leads to saturation

[Ex. 8.7] Toroid, d(air gap) = 2 mm, N = 500 turns, S = 6 cm^2, r = 15cm, I = ? (To establish a magnetic flux density of 1 [T])

$$\Rightarrow \Re_{air} = \frac{d_{air}}{\mu S} = \frac{2\times 10^{-3}}{4\pi\times 10^{-7}\times 6\times 10^{-4}} = 2.653\times 10^{6}\ \ [\mathrm{A\cdot t/Wb}] \ \leftarrow \text{Reluctance}$$

$$\Phi = BS = 1\times(6\times 10^{-4}) = 6\times 10^{-4} \qquad [\mathrm{Wb}]$$

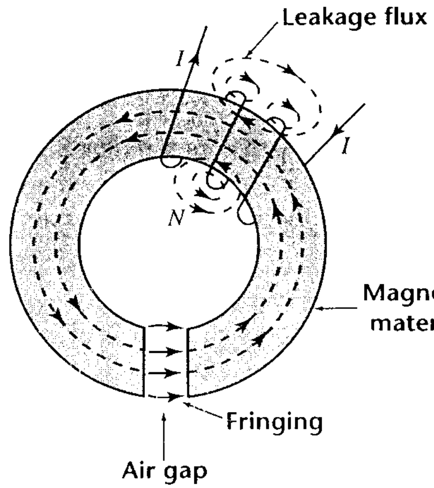
$$V_{m.air} = \Phi\,\Re_{air} = (6\times 10^{-4})\times(2.653\times 10^{6}) = 1592 \quad [\mathrm{A\cdot t}]$$

From Fig. 8.11, H = 200 [A·t/m] at B = 1 [T]

$$H_{steel} = 200 \quad [\mathrm{A\cdot t/m}]$$

$$V_{m.steel} = \oint \vec{H}\cdot d\mathrm{L} \cong H\cdot(\pi\times 2r) = 200\times(\pi\times 2\times 0.15)$$

$$= 188 \quad [\mathrm{A\cdot t}]$$



$$\therefore V_m = V_{m.air} + V_{m.steel} = 1592+188=1780 \quad [\mathrm{A\cdot t}]$$

$$= NI$$

$$I = V_m / N = 1780/500 = 3.56\ [\mathrm{A}]$$

[Ex. 8.8] Given a coil current 4 A in previous example, B=?

➔ $V_m = NI = 500 \times 4 = 2000$ [A·t]

In the previous example, $V_{m,steel} = 188 = \Phi \cdot \Re_{steel} = 6 \times 10^{-4} \Re_{steel}$

$$\Re_{steel} = 0.313 \times 10^6$$

$$\Re_{total} = \Re_{steel} + \Re_{air} = 2.96 \times 10^6 \quad [\mathrm{A \cdot t}]$$

$$V_m = \Phi \cdot \Re_{total} = 2000$$

$$\Phi = 2000 / \Re_{total} = 6.76 \times 10^{-4} \ [\mathrm{Wb}] = B \cdot S$$

$$B = 6.76 \times 10^{-4} / S = 1.13 [\mathrm{T}]$$

8.9 Potential Energy and Force on Magnetic Materials

- Work necessary to bring the prerequisite point charge from infinity to their final resting places:

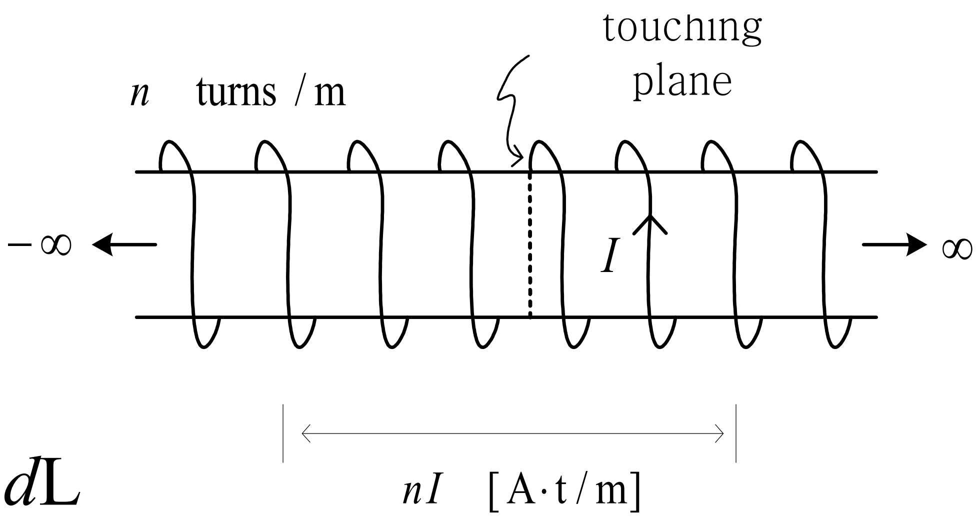
$$W_E = \int_v \frac{1}{2} \mathbf{D} \cdot \mathbf{E} \, dv \quad \text{J}$$

- Total energy stored in a steady magnetic field in which $\vec{B}$ is linear to $\vec{H}$ (in linear media):

$$W_H = \frac{1}{2}\int_{vol} \vec{B} \cdot \vec{H} \, dv = \frac{1}{2}\int_{vol} \mu H^2 \, dv = \underline{\frac{1}{2}\int_{vol} \frac{B^2}{\mu} dv}$$

It is convenient to think that of this energy as being distributed throughout the volume with an energy of $\frac{1}{2}\vec{B} \cdot \vec{H}$ [J/m³].

[Ex.]



- Apply a mechanical force to separate these two sections of the core while keeping the flux density constant: F over distance dL

- The magnetic flux density can be obtained from magnetization curve for silicon steel.: B_{ST}

- Work moving one core appears as stored energy in the air gap

$$dW_H = F\,dL = \frac{1}{2}\frac{B_{st}^2}{\mu_0}dv = \frac{1}{2}\frac{B_{st}^2}{\mu_0}S\,dL$$ where S: core-sectional area

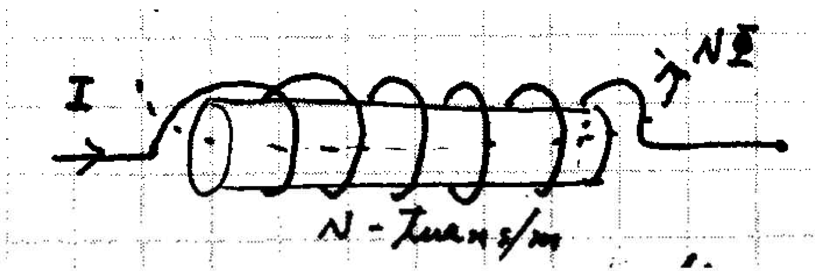
$$F = \frac{B_{st}^2 S}{2\mu_0}$$

[Ex.] If $B = 1.4$ [T], $F = \dfrac{1.4^2 S}{2\times 4\pi \times 10^{-7}} = 7.8\times 10^5 S$ [N]

8.10 Inductance and Mutual Inductance

- Resistance $\left(R = \frac{V}{I}\right)$: The ratio of potential difference between two equipotential surfaces of a conducting material to the total current crossing either equipotential surface. ($R = \frac{d}{\sigma S}$: geometric function)
- Capacitance $\left(C = \frac{Q}{V}\right)$: The ratio of total charge on either of two equipotential conducting surface to potential difference between surfaces. ($C = \varepsilon \frac{S}{d}$)
- Consider a toroid of N-turns in which a current I produces a total flux Φ. ➔ (Total) Flux linkage (λ)

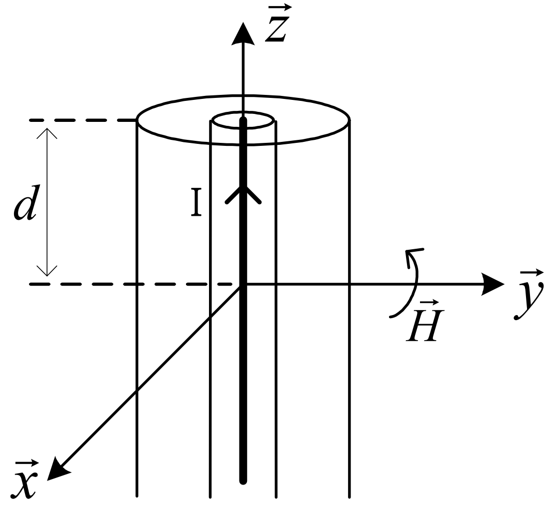
$$\lambda = N\Phi$$



- (Self-) Inductance (L):

$$L = \frac{\lambda}{I} = \frac{N\Phi}{I}$$ for linear magnetic material only.

[Ex.] $H_\phi = \dfrac{I}{2\pi\rho} \quad (a < \rho < b)$ in coaxial cable



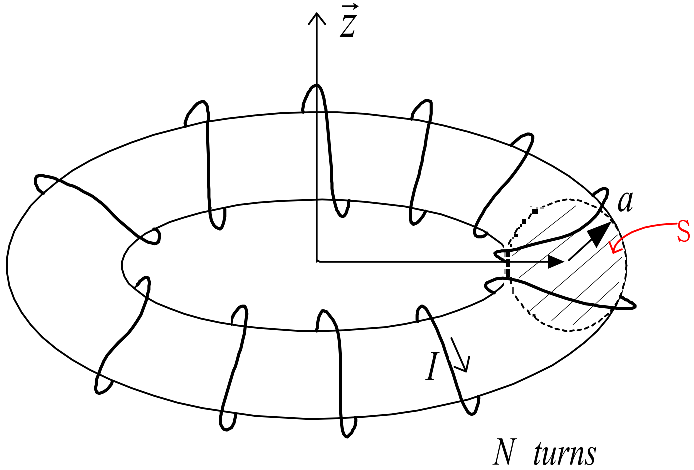
$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi\rho}\vec{a}_\phi$$

$$\Phi = \int_S \vec{B}\cdot d\vec{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\vec{a}_\varphi \cdot d\rho dz \vec{a}_\varphi = \frac{\mu_0 I d}{2\pi}\ln\frac{b}{a}$$

$$L = \frac{\lambda}{I} = \frac{\Phi}{I} = \frac{\mu_0 d}{2\pi}\ln\frac{b}{a} \quad \text{[H]}$$ ← coaxial cable이므로 $N=1$로 가정

$$= \frac{\mu_0}{2\pi}\ln\frac{b}{a} \quad \text{[H/m]}$$ ← inductance per unit length

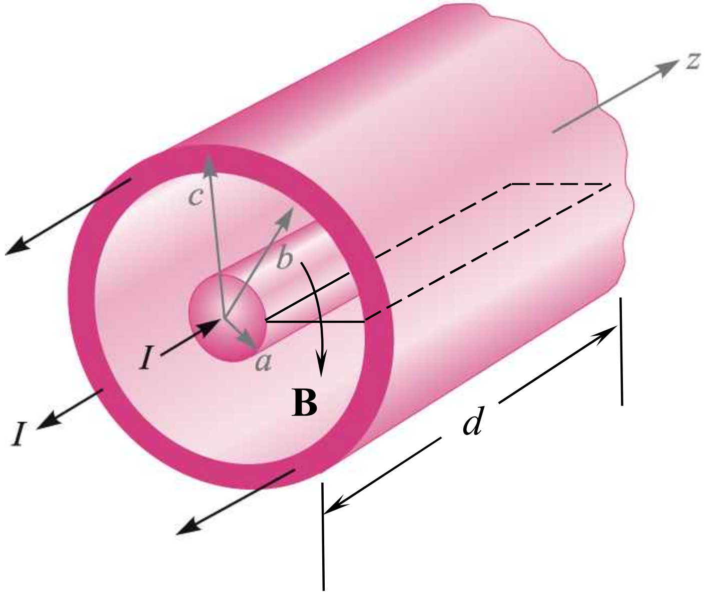
[Ex.] Toroid coil of N-turns, current: I, mean toroid radius: ρ_0



$$B_\phi = \frac{\mu_0 N I}{2\pi\rho_0} \qquad \Phi = \frac{\mu_0 N I S}{2\pi\rho_0} \quad \leftarrow \Phi = \int \vec{B}\cdot d\vec{S}$$

$$L = \frac{N\Phi}{I} = \frac{\mu_0 N^2 S}{2\pi\rho_0}$$

[Ex.] $H_\varphi = \dfrac{I}{2\pi\rho}$ @$a < \rho < b$ in coaxial cable



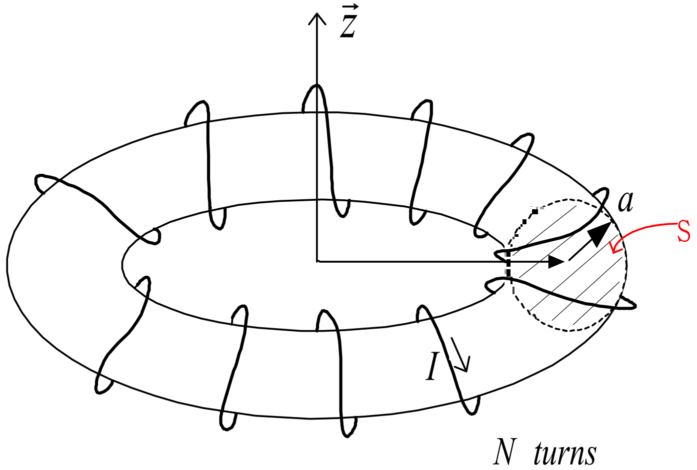
$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi\rho}\vec{a}_\phi$$

$$\Phi = \int_S \vec{B}\cdot d\vec{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho}\vec{a}_\varphi \cdot d\rho dz \vec{a}_\varphi = \frac{\mu_0 I d}{2\pi}\ln\frac{b}{a}$$

$$L = \frac{\lambda}{I} = \frac{\Phi}{I} = \frac{\mu_0 d}{2\pi}\ln\frac{b}{a} \quad [\mathrm{H}]$$ ← coaxial cable이므로 $N = 1$로 가정

$$= \frac{\mu_0}{2\pi}\ln\frac{b}{a} \quad [\mathrm{H/m}]$$ ← inductance per unit length

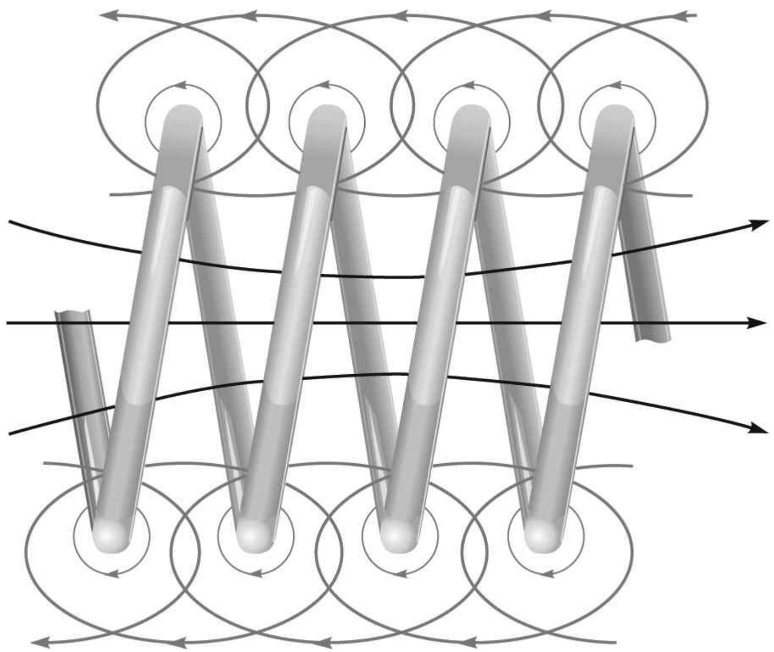
[Ex.] Toroid coil of N-turns, current: I, mean toroid radius: ρ_0



$$B_\phi = \frac{\mu_0 N I}{2\pi\rho_0} \qquad \Phi = \frac{\mu_0 N I S}{2\pi\rho_0} \qquad \leftarrow \Phi = \int \vec{B}\cdot d\vec{S}$$

$$L = \frac{N\Phi}{I} = \frac{\mu_0 N^2 S}{2\pi\rho_0}$$

Departure from the Ideal



- In reality, flux density generated by each turn may not link the entire coil. Such *fringing fields* may link only one or two turns.

- Total flux linkages:

$$(N\Phi)_{\text{total}} = \Phi_1 + \Phi_2 + \Phi_3 + \cdots + \Phi_N$$

$$= \sum_{i=1}^{N} \Phi_i$$

where Φ_i: flux linking the i^{th} turn

- An equivalent definition for inductance

$$L = \frac{2W_H}{I^2} \qquad (\leftarrow W_H = \frac{1}{2} L I^2)$$

where I: total current flowing in the closed path

W_H: energy in the magnetic field produced by the current

$$L = \frac{2W_H}{I^2} \qquad (\leftarrow W_H = \frac{1}{2}LI^2)$$

[Proof]

$$L = \frac{2W_H}{I^2} = \frac{2\times\frac{1}{2}\int_{vol}\vec{B}\cdot\vec{H}\,dv}{I^2} = \frac{\int_{vol}\vec{B}\cdot\vec{H}\,dv}{I^2} \qquad \leftarrow \vec{B} = \nabla\times\vec{A}$$

$$= \frac{1}{I^2}\int_{vol}\vec{H}\cdot(\nabla\times\vec{A})dv \quad \leftarrow \nabla\cdot(\vec{A}\times\vec{H}) = \vec{H}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{H})$$

$$= \frac{1}{I^2}\left[\int_{vol}\nabla\cdot(\vec{A}\times\vec{H})dv + \int_{vol}\vec{A}\cdot(\nabla\times\vec{H})dv\right]$$

$$= \frac{1}{I^2}\left[\oint_S(\vec{A}\times\vec{H})\cdot d\vec{S} + \int_{vol}\vec{A}\cdot\vec{J}\,dv\right] = \frac{1}{I^2}\int_{vol}\vec{A}\cdot\vec{J}\,dv$$

(모든 magnetic energy를 갖고 있는 체적 element들을 포함하게끔 임의의 폐곡면을 설정하면, 폐곡면 표면에는 magnetic flux density가 '0'이므로 ($\vec{B} = \nabla\times\vec{A} = \mu_0\vec{H} = 0$), 폐곡면에서 $\vec{A} = 0 = \vec{H}$)

[Proof (continued)]

Since $\vec{A} = \int_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R}$ (7.51),

$$L = \frac{1}{I^2} \int_{vol} \left(\int_{vol} \frac{\mu \vec{J} dv}{4\pi R} \right) \cdot \vec{J} dv \qquad \leftarrow \vec{J} dv = I d\vec{L}$$

$$= \frac{1}{I^2} \oint \left(\oint \frac{\mu I d\vec{L}}{4\pi R} \right) \cdot I d\vec{L} = \frac{\mu}{4\pi} \oint \left(\oint \frac{d\vec{L}}{R} \right) \cdot d\vec{L} \quad (= f\,(\text{distribution of current in space}))$$

Let hypothesize a uniform current distribution for simplicity.

$$L = \frac{1}{I^2} \int_{vol} \vec{A} \cdot \vec{J} dv = \frac{1}{I^2} \oint \vec{A} \cdot I d\vec{L} = \frac{1}{I} \oint \vec{A} \cdot d\vec{\mathrm{L}} \quad \leftarrow \text{Stoke's theorem}$$

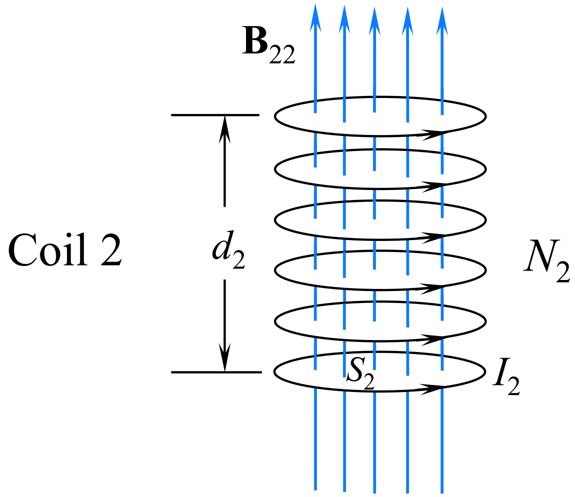
$$= \frac{1}{I} \int_S (\nabla \times \vec{A}) \cdot d\vec{\mathrm{S}} = \frac{1}{I} \int_S \vec{B} \cdot d\vec{\mathrm{S}} = \frac{\Phi}{I} \quad [\mathrm{H/turn}]$$

When the closed line integral consist of N laps about this common path.

$$L = \frac{N\Phi}{I} \quad [\mathrm{H}] \qquad (\text{Proven})$$

Two Inductors

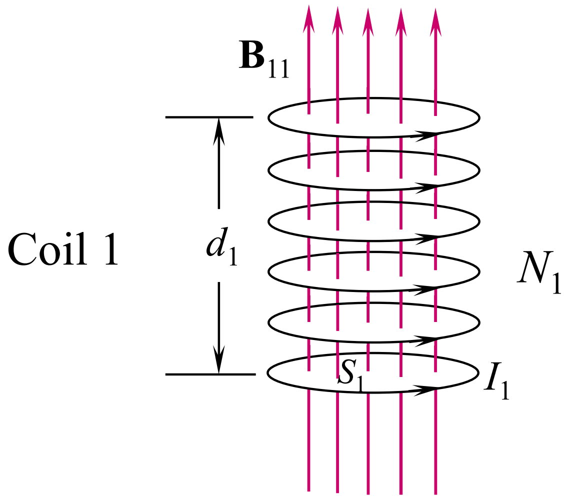
Suppose we have two solenoids, having different specifications as indicated:



The *self linkage* and *self inductance* of each coil are determined in the manner that we used before, assuming identical fluxes through each turn.

$$\lambda_{22} = N_2 \Phi_{22} = N_2 \int_{S_2} \mathbf{B}_{22} \cdot d\mathbf{S}_2$$

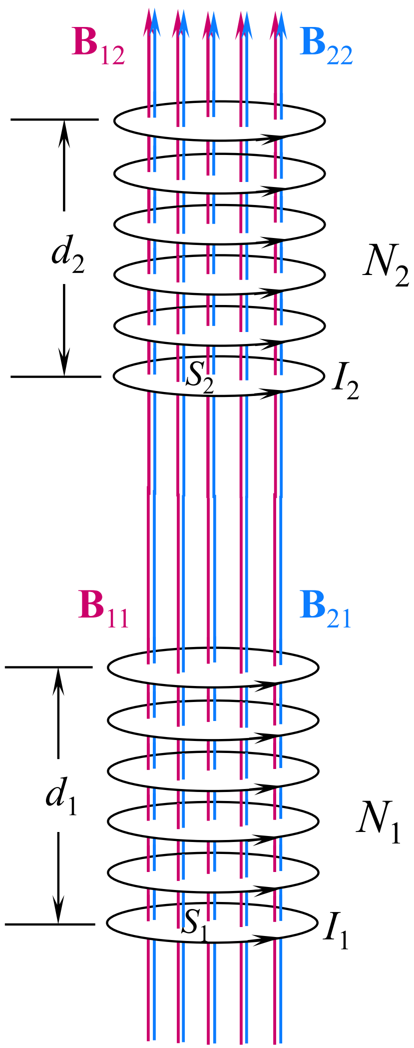
and $$L_{22} = \frac{\lambda_{22}}{I_2} = N_2^2 \frac{\mu_2 S_2}{d_2}$$



$$\lambda_{11} = N_1 \Phi_{11} = N_1 \int_{S_1} \mathbf{B}_{11} \cdot d\mathbf{S}_1$$

and $$L_{11} = \frac{\lambda_{11}}{I_1} = N_1^2 \frac{\mu_1 S_1}{d_1}$$

Interaction Between Inductors



Actually, the magnetic fields generated by each coil will link the other, as shown here. This flux overlap is the basis of *mutual inductance*.

Throughout this discussion, the field in red is that generated by Coil 1, while the blue field is generated by Coil 2

With both currents on, all the fields indicated here will be present. The fields and other quantities are kept track of by the subscripts, the meaning of which is:

$$\mathbf{B}_{ij}$$

arising from coil i

evaluated within coil j

$$i, j = 1, 2$$

Note that the diagrams shown here are oversimplified, because there will be significant spreading of the crossover fields, $\mathbf{B}_{12}$ and $\mathbf{B}_{21}$.

▪ Mutual inductance between circuits 1 and 2: M_{21}

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1}$$

where Φ_{12}: flux produced by I_1 which links the path of the filamentary current I_2

N_2: # of turns in circuit 2

$$M_{12} = \frac{1}{I_1 I_2} \int_{vol} (\vec{B}_1 \cdot \vec{H}_2) dv$$

$$= \frac{1}{I_1 I_2} \int_{vol} (\mu \vec{H}_1 \cdot \vec{H}_2) dv$$

← $L = \frac{2W_H}{I^2}$. I_1에 의해 magnetic flux $\vec{B}_1$가 발생하여 전류 I_2가 흐르는 곳에 영향을 미치는 것.

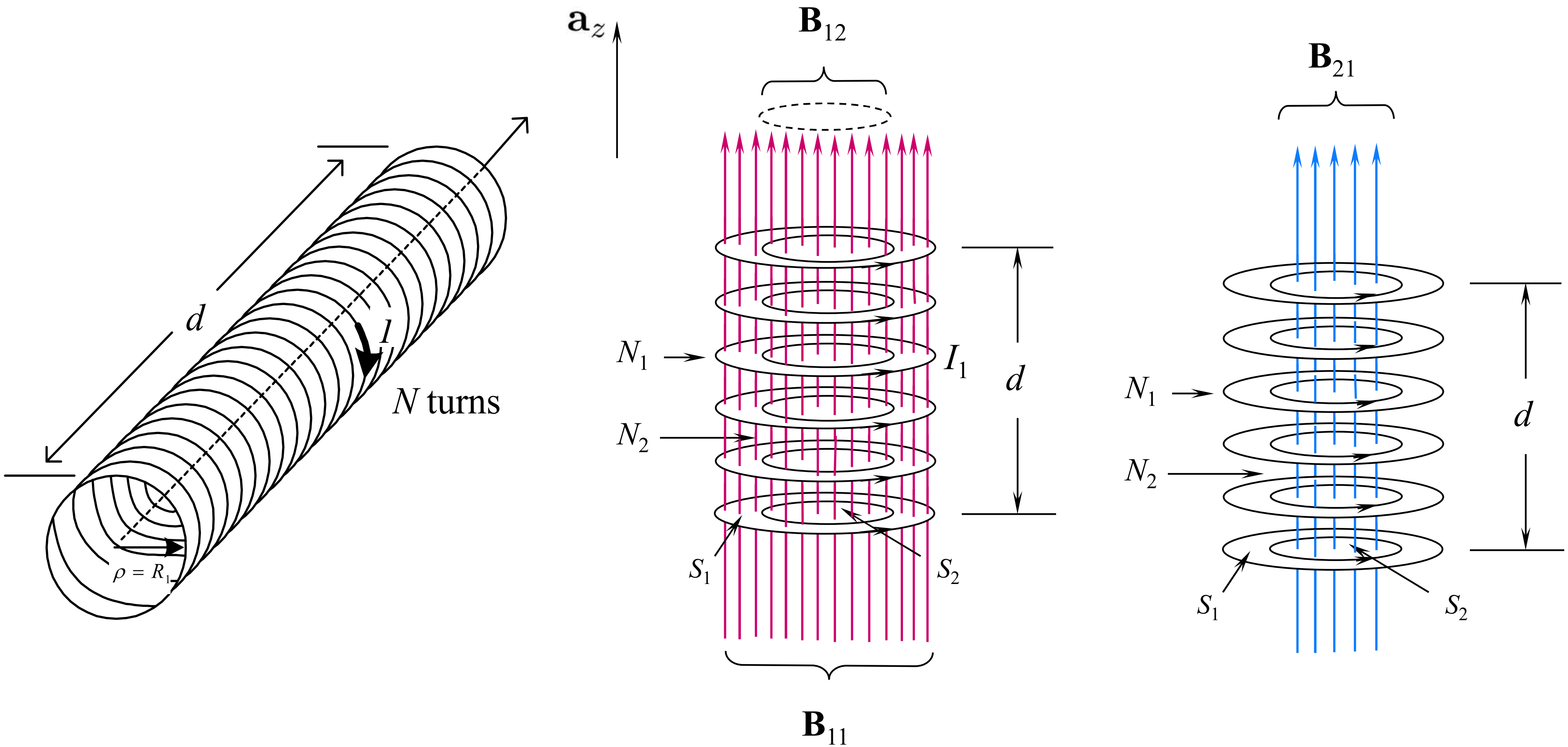
where $\vec{B}_1$: field resulting from I_1 with $I_2 = 0$

$\vec{B}_2$: field resulting from I_2 with $I_1 = 0$

$$\therefore\ M_{12} = \frac{1}{I_1 I_2} \int_{vol} (\mu \vec{H}_1 \cdot \vec{H}_2) = \frac{1}{I_2 I_1} \int_{vol} (\mu \vec{H}_2 \cdot \vec{H}_1) = \frac{1}{I_2 I_1} \int_{vol} (\vec{B}_2 \cdot \vec{H}_1)$$

$$= M_{21}$$

[Ex. 8.9] Two coaxial solenoid of radius R_1, R_2 ($R_2 > R_1$) carrying currents I_1 and I_2 with n_1, n_2 [turns/m]



$$\vec{H}_1 = \frac{NI_1}{d}\vec{a}_z = n_1 I_1 \vec{a}_z \quad (0 < \rho < R_1)$$
$$= 0 \quad (\rho > R_1)$$

$$\vec{H}_2 = n_2 I_2 \vec{a}_z \quad (0 < \rho < R_2)$$
$$= 0 \quad (\rho > R_2)$$

For uniform field,

$$\Phi_{12} = \underline{\mu_0 n_1 I_1}\,\pi R_1^{\,2} \qquad \leftarrow \Phi = \int \vec{B}\cdot d\vec{S}\,,\; M_{12} = \frac{\int_v (\vec{B}_1 \cdot \vec{H}_2)\,dv}{I_1 I_2}$$

B_1 (H_1- field 발생 가능 영역이 $0<\rho<R_1$ 이므로, coupling 영역도 $0<\rho<R_1$ 임.)

$$M_{12} = \frac{(\mu_0 n_1 I_1\; \pi R_1^{\,2})(n_2 I_2)}{I_1 I_2} = \mu_0 n_1 n_2 \pi R_1^{\,2}$$

Similarly,

$$\Phi_{21} = \underline{\mu_0 n_2 I_2}\,\pi R_1^{\,2}$$

B_1 (비록 I_2 에 의해 flux가 πR_2^2 만큼 발생해도 2차측의 면적 $\pi R_1^2 = S_1$만큼만 flux가 쇄교 가능)

$$M_{21} = \frac{(\mu_0 n_2 I_2 \pi R_1^2)(n_1 I_1)}{I_2 I_1} = \mu_0 n_1 n_2 \pi R_1^2 = M_{12}$$

If $N_1 = 50$ [turns/cm], $N_2 = 80$ [turns/cm], $R_1 = 2$[cm], and $R_2 = 3$[cm],

$$M_{12} = M_{21} = 4\pi \times 10^{-7} \times 5000 \times 8000 \times \pi \times 0.02^2 = 63.17 \times 10^{-3} \text{ [H/m]}$$

$$L_1 = \frac{n_1 \Phi_1}{I_1} = \frac{n_1 B_1 S_1}{I_1} = \frac{n_1 \mu_0 H_1 S_1}{I_1} = \frac{n_1 \mu_0 n_1 I_1 \pi R_1^2}{I_1} \qquad (\leftarrow \frac{N_1}{d} = n_1)$$

$$= n_1^2 \mu_0 \pi R_1^2 = 5000^2 \times 4\pi \times 10^{-7} \times \pi \times 0.02^2 = 39.48 \times 10^{-3} \text{ [H/m]}$$

$$L_2 = n_2^2 \mu_0 S_2$$

$$= n_2^2 \mu_0 \pi R_2^2$$

$$= 8000^2 \times 4\pi \times 10^{-7} \times \pi \times 0.03^2 = 227.4 \times 10^{-3} \text{ [H/m]}$$