Engineering Electromagnetics

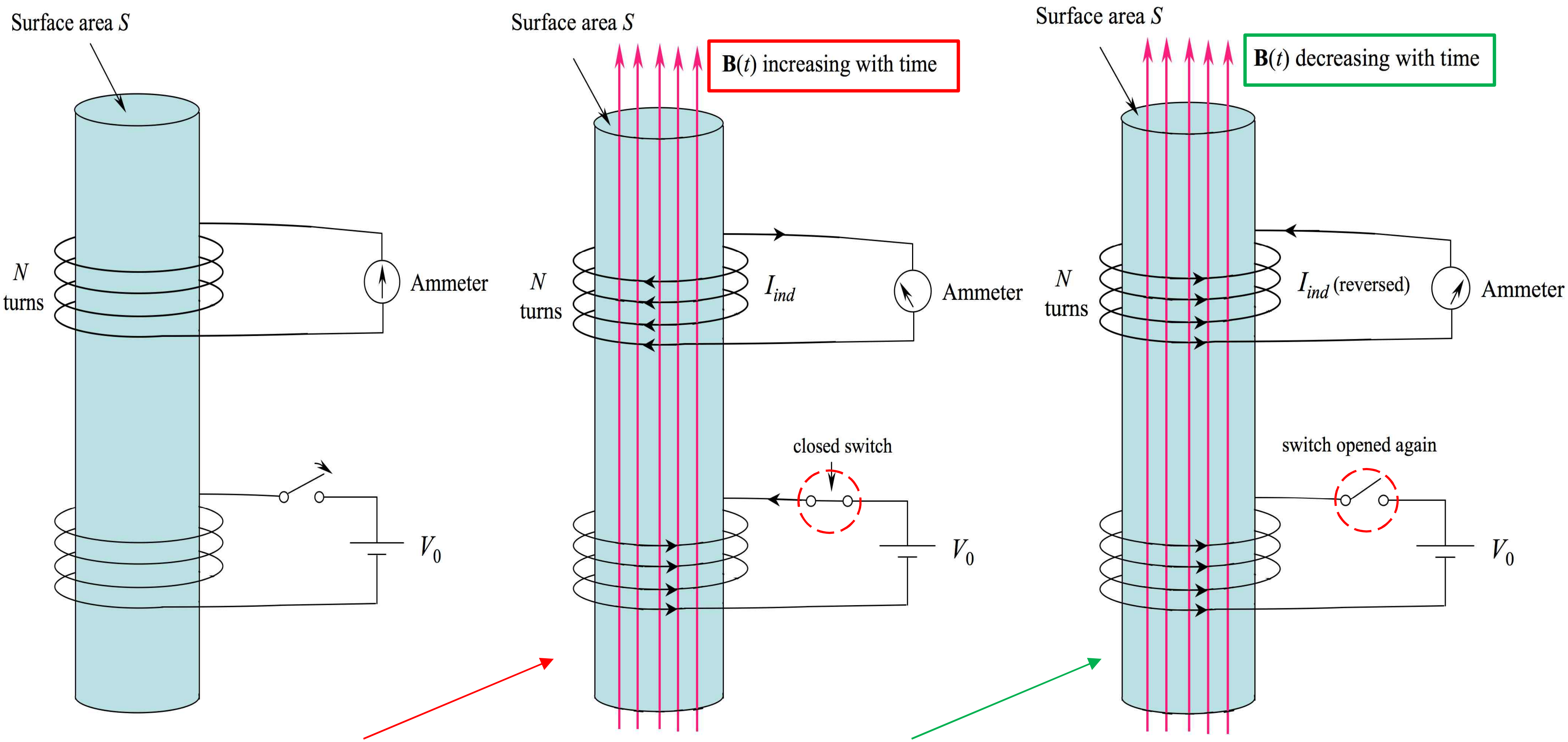
W.H. Hayt Jr. and J. A. Buck

Chapter 9:

Time-Varying Fields

and Maxwell's Equations

9.1 Faraday's Law
Faraday's Experiment



Magnetic flux density **B** generated lower circuit links the upper windings.

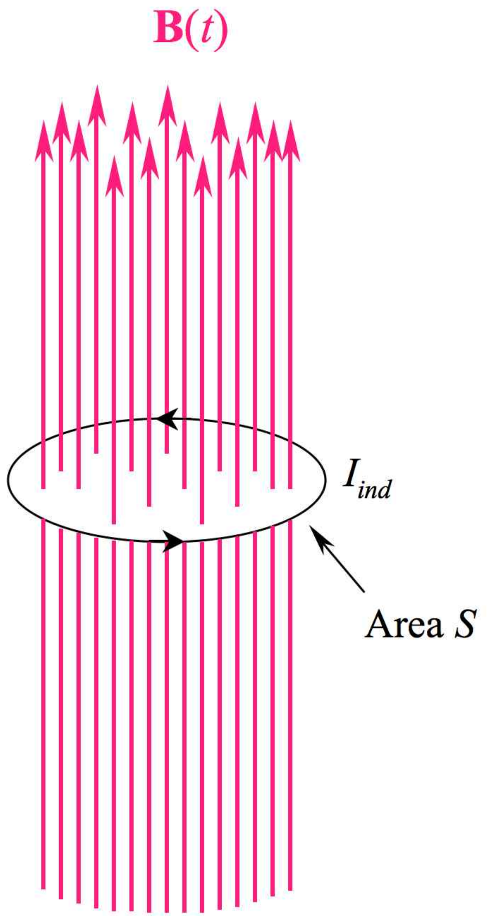
➔ I_{ind} ➔ Existed *as long as* **B** *increases with time*

➔ Proportional to time rate of change of *magnetic flux*

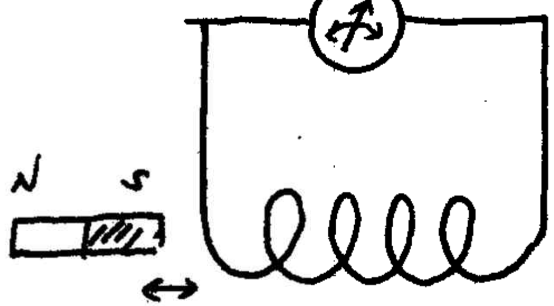
Opening the switch results in current that initially decreases with time in the lower windings.

➔ I_{ind} in opposite direction from before.

➔ Existed *as long as the magnetic flux decreases with time.*



- Consider a single turn of wire, through which an externally-applied magnetic flux is present. The flux varying with time generates I_{ind}.
- A [1])time-varying magnetic field produces an electromotive force (emf) which may establish a current in a closed circuit.
- Emf is a voltage that arises also from [2])conductors moving in a magnetic field or from changing magnetic field.



- Faradays's law

$$\text{emf} = -\frac{d\Phi}{dt} \quad [\text{V}]$$

where $\frac{d\Phi}{dt}$: flux changing rate according to time

(-) : opposing flux direction sign

- $\dfrac{d\Phi}{dt} \neq 0$

  1) A time-changing flux linking a stationary closed path
  2) Relative motion between a steady flux and a closed path
  3) A combinations of 1) and 2)

- If the closed path is taken by an *N*-turns filamentary conductor,

$$\text{emf} = -N\frac{d\Phi}{dt}$$ ← Φ: flux passing through any one of *N* coincident paths.

$$\text{emf} = \oint \vec{E} \cdot d\vec{L}$$ @eletromagetic fields are time-varying

$$= -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{\mathrm{S}} \quad \leftarrow \quad \oint \vec{E} \times d\vec{L} = 0$$ @time-invariant

➔ $d\vec{S}$ 방향으로 $\vec{B}$ 의 증가는 폐경로(closed-path)에서 왼쪽 나사 방향으로 $\vec{E}_{zverage}$를 일으킴.

▪ 1st investigation: Total emf made by a changing field within a stationary path

$$\text{emf} = \oint \vec{E} \cdot d\vec{L} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \vec{B} \cdot d\vec{S} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \qquad \leftarrow \text{Stoke's theorem}$$

$$= \int_S (\nabla \times \vec{E}) \cdot d\vec{S}$$

$$\therefore \quad \nabla \times \vec{E} \cdot d\vec{S} = -\frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

$$(\nabla \times \vec{E}) = -\frac{\partial \vec{B}}{\partial t}$$

: point form of Faraday's Law

cf.) For electrostatic case,

$$\vec{B} = f(t) \;\rightarrow\; -\frac{\partial \vec{B}}{\partial t} = 0$$

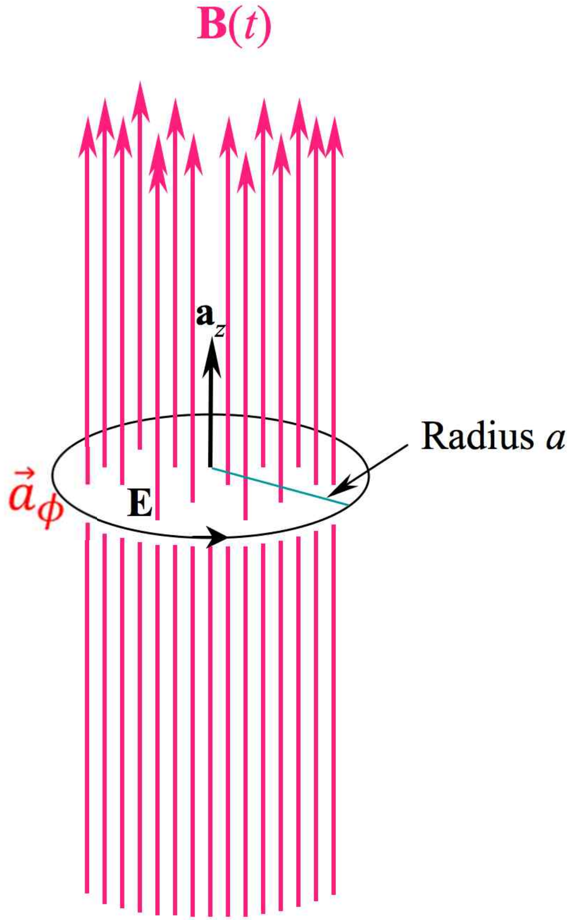
$$\rightarrow\; \nabla \times \vec{E} = 0,\; \int (\nabla \times \vec{E}) \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{L} = 0$$

➔ 정상자계에서 전하를 폐 loop 상에서 옮기는 일은 "0"

[Ex.] $\vec{B} = B_0 e^{kt} \vec{a}_z$ for $\rho < b$ (cylindrical region)

where B_0: constant

Choose the circular path $\rho = a$ ($a < b$) in $z = 0$ plane



Method i) $\text{emf} = \oint \vec{E} \cdot d\vec{L} = \pi(2a)E_\phi$

$$= -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{B} \cdot d\vec{S}$$

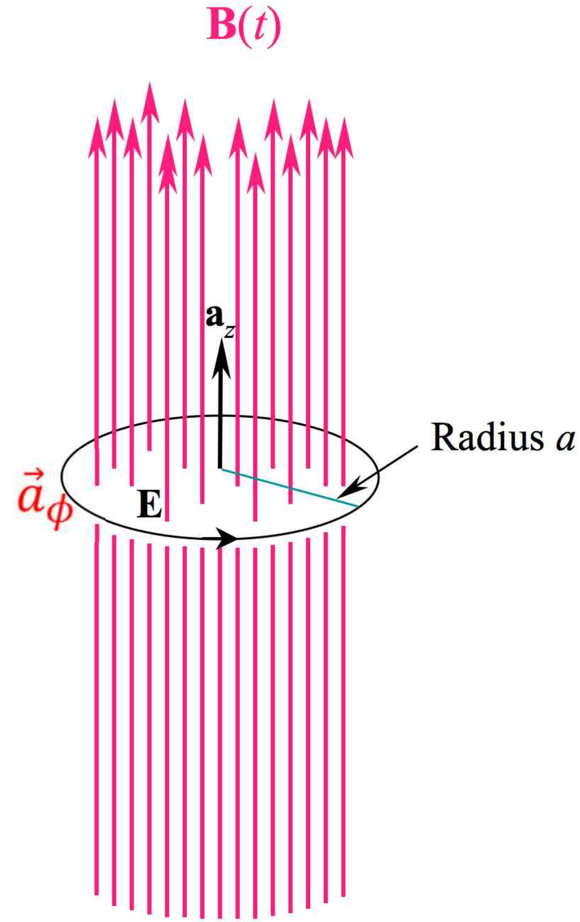
$$= -\frac{d}{dt}\left(B_0 e^{kt} \cdot \pi a^2\right)$$

$$= -kB_0 e^{kt} \cdot \pi a^2$$

$$E_\phi = \frac{-kB_0 e^{kt} \cdot \pi a^2}{2a\pi} = -\frac{1}{2}kB_0 e^{kt} a$$

$$\mathbf{E} = -\tfrac{1}{2}kB_0 e^{kt} \rho \mathbf{a}_\phi \quad \Leftarrow (a \rightarrow \rho)$$

Method ii) $(\nabla\times\vec{E})_z = -\dfrac{\partial B_z}{\partial t} = -kB_0e^{kt}$



$$= \frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial \rho}$$

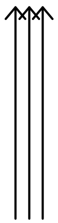
method I 과 같은 이유

$$(\text{참고}: \nabla\times\vec{E} = (\ )\vec{a}_\rho + (\ )\vec{a}_\phi + \left(\frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial E_\rho}{\partial \phi}\right)\vec{a}_z)$$

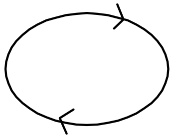
$$\rho E_\phi = -\frac{1}{2}kB_0e^{kt}\rho^2$$

$$\vec{E} = -\frac{1}{2}kB_0e^{kt}\rho\vec{a}_\phi$$

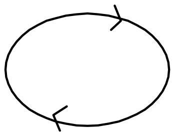
최초의 $\vec{B}$ 의 시간적 증가를 상쇄시킴.



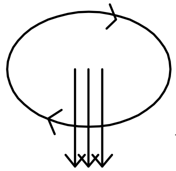
$\vec{B} = B_0e^{kt}\vec{a}_z$ 로 시간적 증가 발생



$\vec{E} = -E_0e^{kt}\vec{a}_z$ 로 시간적 증가 발생



$\dfrac{\partial\vec{E}}{\partial t}$ 이므로 $\vec{E}$ 도 시간적 증가발생



$\vec{H} = -H_0e^{kt}\vec{a}_z$ 이 시간적 증가 발생

➔ Lenz's Law

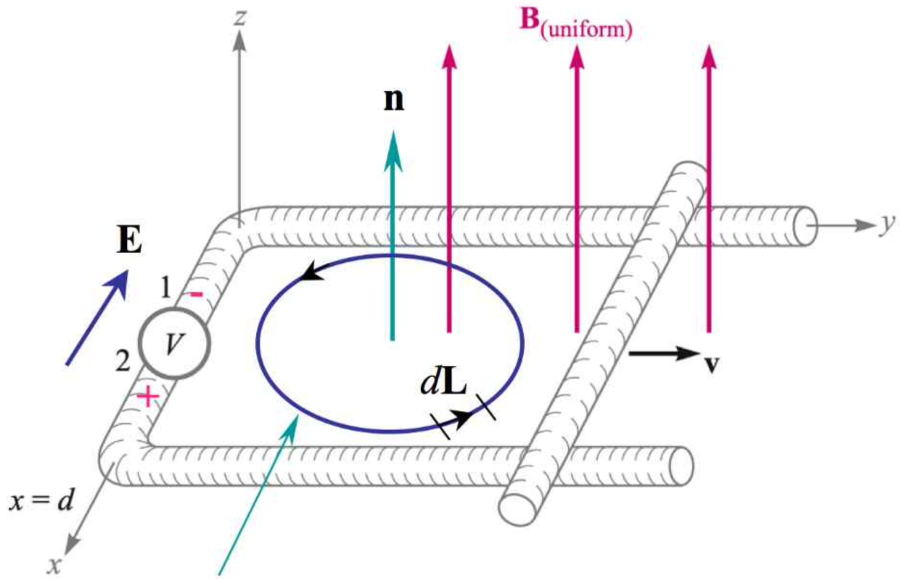
▪ 2^{nd} investigation: Time-constant flux and a moving closed path

- Sliding bar, moving at constant velocity v
- Flux passing through perfect-conducting sliding bar surface with closed path

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S \mathbf{B} \cdot \mathbf{n}\, da$$

$$= Byd$$

$$\text{emf} = -\frac{d\Phi}{dt} = -B\frac{dy}{dt}d = -Bvd$$



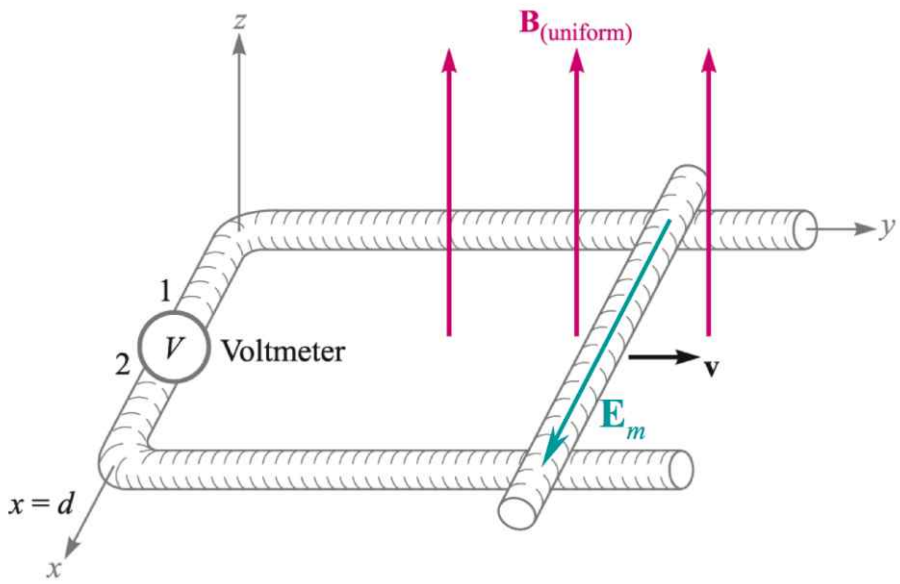
- $\vec{E}_t = 0$ @conducting bar
  $\rightarrow \int \vec{E} \cdot d\vec{L} = 0$ @except volmeter region
- Due to '-*Bvd*', terminals 2 and 1 are positive and negative, respectively.

Motional EMF

- Force on a charge Q moving with a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$:

$$\mathbf{F} = Q\mathbf{v} \times \mathbf{B}$$

$$\frac{\mathbf{F}}{Q} = \mathbf{v} \times \mathbf{B}$$



- Motional (electric) field intensity:

$$\boxed{\mathbf{E}_m = \mathbf{v} \times \mathbf{B}}$$

- The *motional emf* is produced by the moving conductor on uniform magnetic filed.

$$\text{emf} = \oint \vec{E}_m \cdot d\vec{L} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{L}$$

$$= \int_d^0 vB\,dx = -Bvd \quad \leftarrow \vec{v} \times \vec{B} = |v|\vec{a}_y \times |B|\vec{a}_z = |v||B|\vec{a}_x$$

: Same result with the previous method

Two Contributions to emf

- If **B** is also changing with time beside of moving conducting bar,

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = \underbrace{-\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}}_{\text{Due to time changing rate } \mathbf{B}} + \underbrace{\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L}}_{\text{Due to motional circuit}}$$

$$= -\frac{d\Phi}{dt}$$

9.2 Displacement Current

- Faraday's experimental law:

$$\nabla \times \mathbf{E} = -\underbrace{\frac{\partial \mathbf{B}}{\partial t}}_{\text{Source}} \quad \leftarrow \int (\nabla \times \vec{E}) \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{L} = -\frac{d\Phi}{dt} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

- Point form of Ampere's circuital law at steady magnetic fields:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

$= 0$ ➔ only true @ $\frac{\partial \rho_v}{\partial t} = 0$

(Curl의 divergence는 '0')

$$I = \oint_S \vec{J} \cdot d\vec{S} = -\frac{dQ_i}{dt}$$

$$= \int (\nabla \cdot \vec{J}) dv = -\frac{d}{dt} \int \rho_v dv$$

$$\therefore \quad \nabla \cdot \vec{J} = -\frac{\partial \rho_v}{\partial t}$$

: an unrealistic limitation

($\frac{\partial \rho_v}{\partial t} \neq 0$ 이려면, 즉 전류가 발생하려면 시간에 따른 전하 변화량이 있어야.
➔ 연속 방정식(continuity eq.)에 의해 폐곡면 외부로 전하가 유출되면,
언젠가 내부 전하는 고갈됨. ➔ $\mathbf{J} = 0$)

So what is wrong?

- For time-varying field,

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{G}$$

Error correction term

$$0 = -\frac{\partial \rho_v}{\partial t} + \nabla \cdot \vec{G} \qquad \therefore \quad \nabla \cdot \vec{G} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = \nabla \cdot \frac{\partial \vec{D}}{\partial t}$$

$$\mathbf{G} = \frac{\partial \mathbf{D}}{\partial t}$$

- Ampere's circuital law in point form for time-varying field,

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$= \vec{J} + \vec{J}_d$$

Dimension of current density [A/m^2]

: **displacement current density**

- Currents

$\vec{J} = \sigma\vec{E}$: conduction current density (motion of charge in region of zero net charge density)

$\vec{J} = \rho_v\vec{v}$: convection current density (motion of volume charge density)

$$\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$$: in nonconduction medium (no volume charge density. time varying current)

$$\overset{\text{analogy}}{\leftrightarrow} \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

(Note the symmetry when comparing to Faraday's law)

- Total displacement current crossing any given surface:

$$I_d = \int_S \vec{J}_d \cdot d\vec{S} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

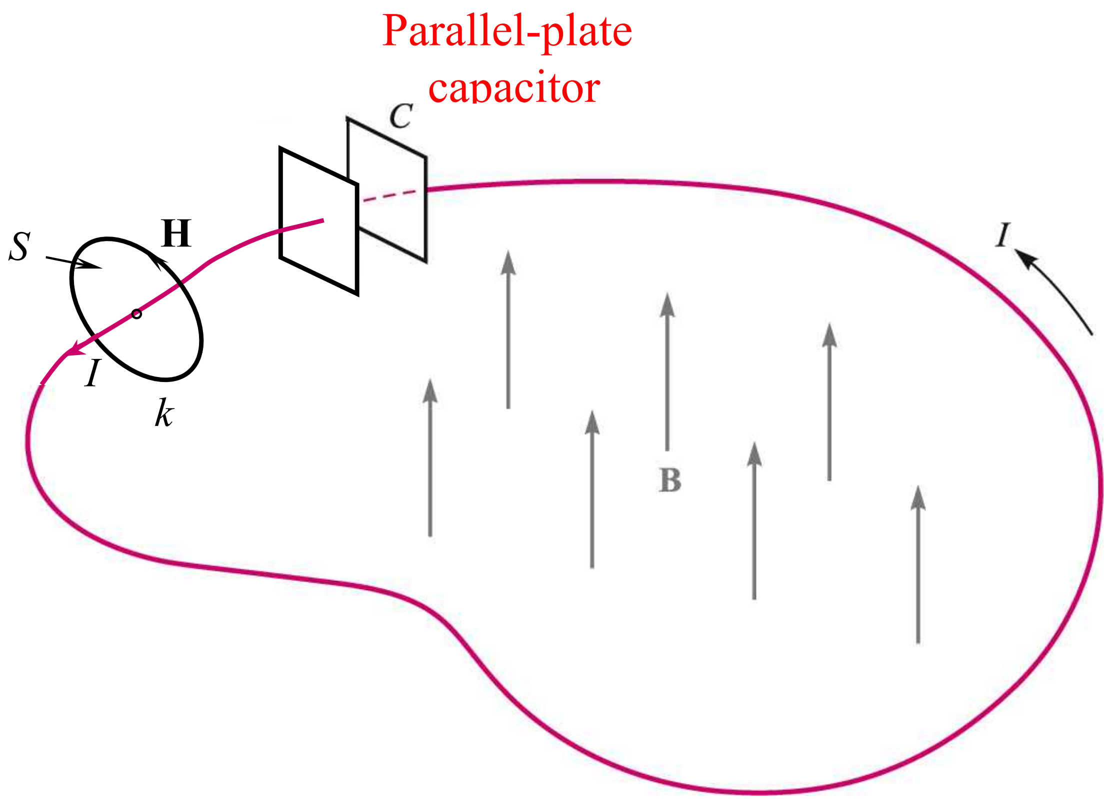
- Time-varying version of Ampere's circuital law:

$$\int_S (\nabla \times \vec{H}) \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

$$\oint \vec{H} \cdot d\vec{L} = I + I_d = I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S}$$

Demonstration of Displacement Current

- The magnetic field is presumed time-varying within the loop, thus it generates emf, which in turn provides the current.



$$\text{emf} = V_0 \cos \omega t \quad [\text{V}]$$

$$I = \frac{dQ}{dt} = \frac{d}{dt}(CV)$$

$$= -\omega C V_0 \sin \omega t$$

$$= -\omega \frac{\varepsilon S}{d} V_0 \sin \omega t$$

- Apply Ampere's circuital law about the small closed circular path *k*.

$$\oint_k \vec{H} \cdot d\vec{\mathrm{L}} = I_k$$ ← conduction current

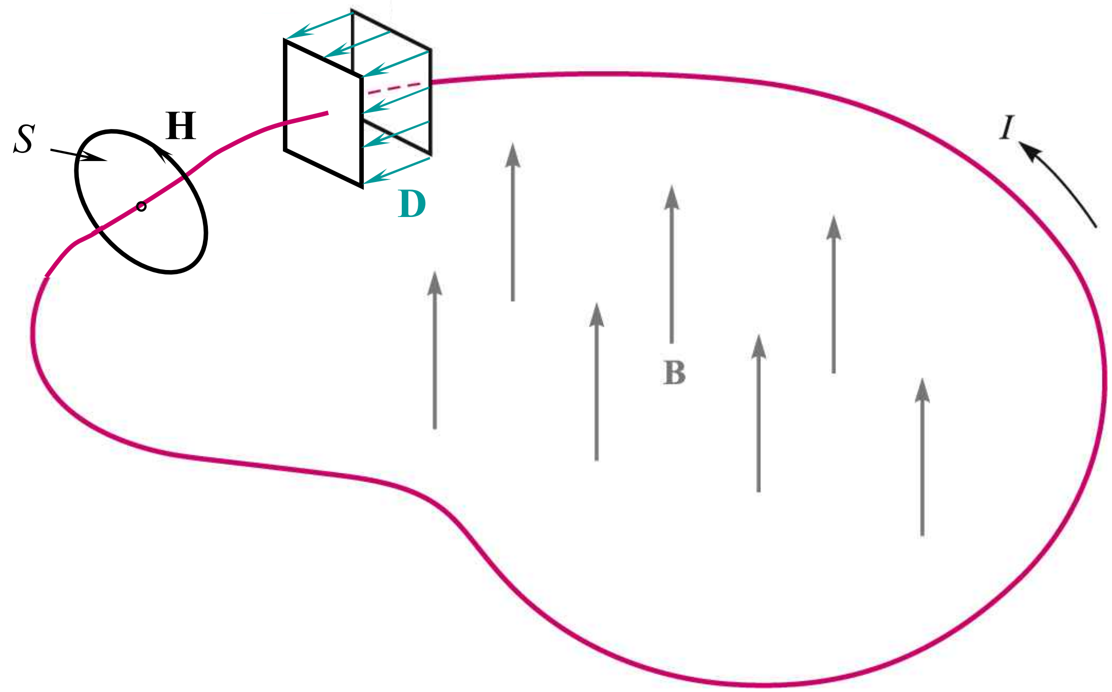
▪ At parallel-plate capacitor, the conduction current is zero due to no connection conductor between two plates in capacitor.

$$D = \varepsilon E = \varepsilon \frac{V_0 \cos\omega t}{d}$$

$$I_d = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{\mathrm{S}}$$

$$= (-\varepsilon\omega \frac{V_0}{d} \sin\omega t) \cdot S$$

$$= -\omega \frac{\varepsilon S}{d} V_0 \sin\omega t = I_k$$



➔ Displacement current is associated with time-varying electric fields and therefore exists in all imperfect conductor carrying a time-varying conduction current.

9.3 Maxwell's Equations in Point Form

- Maxwell's equations for time-varying fields

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ Faraday's Law of Induction

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$ Ampere's Circuital Law

$$\nabla \cdot \vec{D} = \rho_v$$ Gauss' Law for the electric field

$$\nabla \cdot \vec{B} = 0$$ Gauss's Law for the magnetic field

- Auxiliary equations

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P}$$ → polarization

$$\vec{B} = \mu\vec{H} = \mu_0(\vec{H} + \vec{M})$$ → magnetization

For linear materials, $\vec{P} = \chi_e\varepsilon_0\vec{E}$ and $\vec{M} = \chi_m\vec{H}$

- Conduction current density: $\vec{J} = \sigma\vec{E}$

- Convection current density: $\vec{J} = \rho_v\vec{v}$

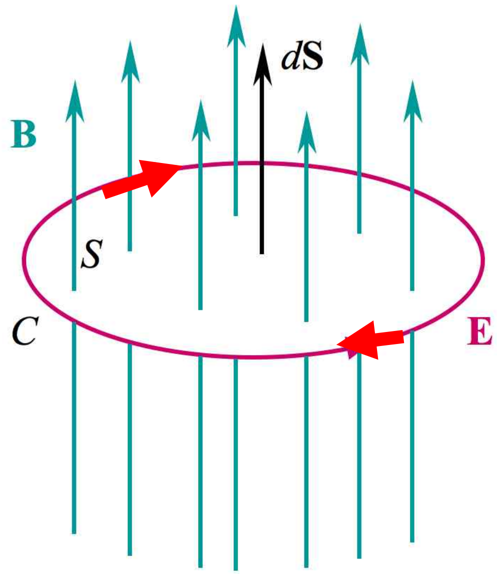
- Lorentz force equation in point form: $\vec{f} = \rho_v(\vec{E} + \vec{v}\times\vec{B})$

9.3 Maxwell's Equations in Integral Form

- Faraday's law using Stoke's theorem

$$\int (\nabla \times \vec{E}) \cdot d\vec{S} = \oint \vec{E} \cdot d\vec{L} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S}$$

$$= -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = -\frac{d\Phi_m}{dt}$$

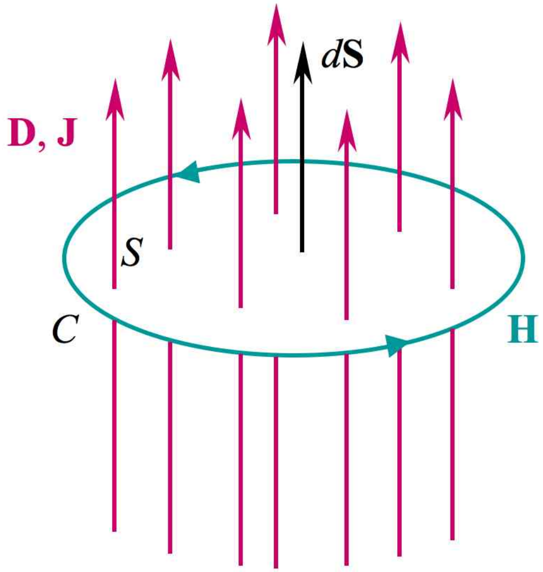


- Ampere's circuital law

$$\int (\nabla \times \vec{H}) \cdot d\vec{S} = \oint \vec{H} \cdot d\vec{L}$$

$$= \underbrace{\int_S \mathbf{J} \cdot d\mathbf{S}}_{I} + \underbrace{\int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}}_{I_d}$$

$$= I + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} = I + I_d$$



- Gauss's law

$$\int (\nabla \cdot \vec{D}) dv = \oint_S \vec{D} \cdot d\vec{S} = \int_{vol} \rho_v dv = Q$$

$$\int (\nabla \cdot \vec{B}) dv = \oint_S \vec{B} \cdot d\vec{S} = 0$$

- Boundary conditions

$$\begin{cases} E_{t1} = E_{t2} \\ H_{t1} = H_{t2} \end{cases} \qquad \begin{cases} D_{N1} - D_{N2} = \rho_S \\ B_{N1} = B_{N2} \end{cases}$$

- For a perfect conductor($\sigma = \infty$, $\vec{J}$ = finite)

$\vec{J} = \sigma \vec{E}$: $\vec{J}$ → finite, σ → infinite, $\vec{E}$ → $\therefore 0$

$$\vec{E} = 0$$

$\Rightarrow \quad \vec{H} = 0 \quad$ ($\because$ For satisfying $\nabla \times \vec{H} = \frac{\partial \vec{D}}{\partial t}$)

$\Rightarrow \quad \vec{J} = 0 \quad$ ($\because \ \vec{H} = 0$)

$\therefore$ Current must be carried on the conductor surface as a surface current $\vec{K}$. ($\because$ volume current density $\vec{J} = 0$)

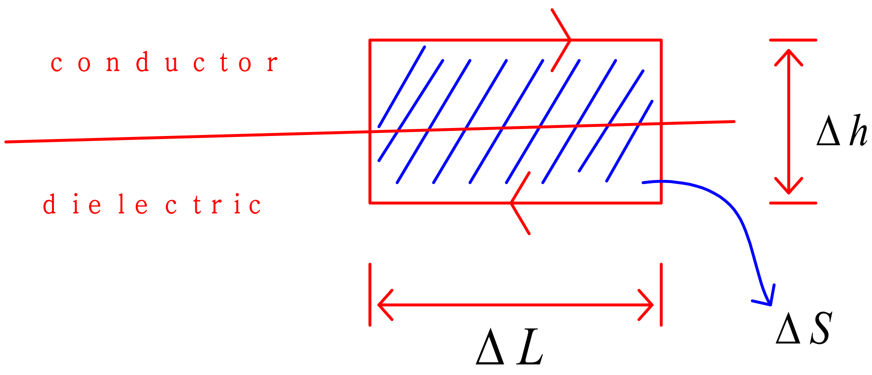
▪ If region 1 is a dielectric material and region 2 is a perfect conductor,

$$E_{t1} = 0$$

$$H_{t1} = K \qquad (\vec{H}_{t1} = \vec{K} \times \vec{a}_N) \qquad \leftarrow H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

$$D_{N1} = \rho_S$$

$$B_{N1} = 0$$



$$\oint \vec{H}_c \cdot d\vec{L} = \int (\nabla \times \vec{H}) \cdot dS = \int (\vec{J} + \frac{\partial \vec{D}}{\partial t}) \cdot d\vec{S}$$

$$\lim_{h \to 0} \oint \vec{H} \cdot d\vec{L} = H_{1t}\Delta L \quad (\because H_{2t} = 0 \text{ at 도체})$$

$$= \lim_{h \to 0} \int_S j\omega \vec{D} \cdot d\vec{S} + \lim_{h \to 0} \int \vec{J} \cdot d\vec{S}$$

$$= K\Delta L$$

9.5 The Retarded Potentials (or Time-varying Potential)

▪ Scalar electric potential

$$V = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon R} \quad \text{(static)}$$

$$V_{AB} = -\int_B^A \vec{E}\cdot d\vec{L} = -\int_B^A \frac{Q\,dr}{4\pi\varepsilon_0 r^2} = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right) = V_A - V_B$$

$$V_A = \frac{Q}{4\pi\varepsilon_0 r_A} = \frac{1}{4\pi\varepsilon r_A}\int \rho_v dv$$

▪ Vector magnetic potential

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J}\, dv}{4\pi R} \quad \text{(dc)}$$

▪ Differential equations:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \text{(static)}$$

(: Poisson's eq. $\nabla\cdot\vec{D} = \rho_v$, $\nabla\cdot\vec{D} = \nabla\cdot(\varepsilon\vec{E}) = -\nabla\cdot(\varepsilon\nabla V) = \rho_v$

$$\nabla\cdot\nabla V = \nabla^2 V = -\frac{\rho_v}{\varepsilon}$$

↕

$$\nabla^2 \mathbf{A} = -\mu\mathbf{J} \quad \text{(dc)}$$

$\vec{J} \Leftrightarrow \rho,\ \mu_0 \Leftrightarrow \frac{1}{\varepsilon_0},\quad A \Leftrightarrow V:\quad A_x = \int_{vol} \frac{\mu_0 J_x dv}{4\pi R} \leftarrow V = \int_{vol} \frac{\rho_v dv}{4\pi\varepsilon_0 R}$

- Having (or knowing) V and $\vec{A}$, the fundamental fields are obtained by

Ⅰ. $\vec{E} = -\nabla V$ (static)

Ⅱ. $\vec{B} = \nabla \times \vec{A}$ (d.c)

Static 또는 DC의 경우에는 $\vec{E}$, $\vec{B}$를 구할 수 있으나,
Time-varying인 경우에는 적절치 않음.
Time-varying 조건을 만족할 경우에는 Eq. 유도필요.

- Time-varying case ($\vec{B} = \nabla \times \vec{A}$ 는 d.c 뿐만 아니라 time-varying case 에도 만족)

II. $\nabla \cdot \vec{B} = 0$

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

(∵ The divergence of the curl is zero. ➔ Proven!)

I. $\nabla \times \vec{E} = -\nabla \times \nabla V$

$$\nabla \times \nabla V = 0$$

(Appendix Eq.(A.19))

$\therefore\ \nabla \times \vec{E} = 0$ ➔ By Faraday's law, $\nabla \times \vec{E} \neq 0$ in general.

Hence, let $\vec{E} = -\nabla V + \vec{N}$

And curling

$$\nabla \times \vec{E} = \nabla \times (-\nabla V + \vec{N}) = -\nabla \times \nabla V + \nabla \times \vec{N} = 0 + \nabla \times \vec{N}$$

$$= -\frac{\partial \vec{B}}{\partial t} \quad \text{(By Faraday's law)}$$

$$\nabla \times \vec{N} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla \times \vec{A}) = -\nabla \times \frac{\partial \vec{A}}{\partial t}$$

$$\therefore \quad \vec{N} = -\frac{\partial \vec{A}}{\partial t}$$

So

$$\begin{cases} \vec{E} = -\nabla V + \vec{N} = -\nabla V - \dfrac{\partial \vec{A}}{\partial t} & \text{— ①} \\ \vec{B} = \nabla \times \vec{A} & \text{— ②} \end{cases}$$

▪ Substituting eqs. ① and ② into two of Maxwell equations in below

$$\begin{cases} \nabla \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \\ \nabla \cdot \vec{D} = \rho_v \end{cases}$$

$$\begin{cases} \dfrac{1}{\mu} \nabla \times \vec{B} = \dfrac{1}{\mu} \nabla \times \nabla \times \vec{A} = \vec{J} + \varepsilon \dfrac{\partial \vec{E}}{\partial t} = \vec{J} + \varepsilon(-\nabla \dfrac{\partial V}{\partial t} - \dfrac{\partial^2 \vec{A}}{\partial t^2}) \\ \varepsilon \nabla \cdot \vec{E} = \varepsilon \nabla \cdot (-\nabla V - \dfrac{\partial \vec{A}}{\partial t}) = \varepsilon(-\nabla \cdot \nabla V - \dfrac{\partial}{\partial t} \nabla \cdot \vec{A}) = \rho_v \end{cases}$$

$$\begin{cases} \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - \mu\varepsilon(\nabla \dfrac{\partial V}{\partial t} + \dfrac{\partial^2 \vec{A}}{\partial t^2}) & \text{—— ③} \\ \nabla^2 V + \dfrac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\dfrac{\rho_v}{\varepsilon} & \text{—— ④} \end{cases}$$

▪ Under static or DC condition, eqs. ③ and ④ are simplified as below.

$$\nabla \cdot \vec{A} = 0 \quad \Rightarrow \quad \begin{cases} \nabla^2 \vec{A} = -\mu \vec{J} \\ \nabla^2 V = -\dfrac{\rho_v}{\varepsilon} \end{cases}$$

$$\begin{cases} \vec{A}_2 = \displaystyle\int_{vol} \frac{\mu_0 \vec{J}_1 dv_1}{4\pi R_{12}} \\ \nabla_2 \cdot \vec{A}_2 = \dfrac{\mu_0}{4\pi} \displaystyle\int_{vol} \frac{\nabla_2 \cdot \vec{J}_1}{R_{12}} dv_1 = 0 \end{cases}$$

($\nabla_2 \cdot \vec{J}_1 \rightarrow 0$)

(비록 정전계(static)의 경우 최종 방정식으로 유도되지만 이것으로 $\vec{A}$ 와 V를 완전히 정의할 수 없음. 즉, $\vec{B} = \nabla \times \vec{A}$ 라고 하였는데, 하나의 vector는 curl로만는 정의될 수 없음.)

➔ Necessary, but not sufficient conditions!!!

- For example, choose a very simple vector potential field ($A_y = A_z = 0$).

By eq. ②, $\begin{cases} B_x = 0 \\ B_y = \dfrac{\partial A_x}{\partial z} \\ B_z = -\dfrac{\partial A_x}{\partial y} \end{cases}$

$(\vec{B} = \nabla \times \vec{A})$

➔ No information is available.
Information could be found if we knew the value of the divergence $\vec{A}$

$$\Rightarrow \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x}$$

(대신에 curl과 divergence에 대한 조건을 알면 $\vec{E}, V$ 를 항상 구할 수 있음

➔ A vector field is defined completely when both its curl and divergence are given and <u>when its value is known at any one point (including infinity)</u>.

Boundary condition

- Define below equation for seeking the simplest expression.

$$\nabla \cdot \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

($\vec{B} = \nabla \times \vec{A}$ 라는 curl 정보는 알고 있으므로 divergence $\nabla \cdot \vec{A}$ 정보가 필요.)

- From eqs. ③ and ④, ($\leftarrow \nabla \cdot \vec{A} = 0$)

$$\begin{cases} \nabla^2 \vec{A} = -\mu \vec{J} + \mu\varepsilon \dfrac{\partial^2 \vec{A}}{\partial t^2} \\ \nabla^2 V = -\dfrac{\rho_v}{\varepsilon} + \mu\varepsilon \dfrac{\partial^2 V}{\partial t^2} \end{cases}$$

⇦

$$\begin{cases} \text{Eq. ③: } \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J} - \mu\varepsilon(\nabla \frac{\partial V}{\partial t} + \frac{\partial^2 \vec{A}}{\partial t^2}) \\ \text{Eq. ④: } \nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\varepsilon} \end{cases}$$

$$\vec{B} = \nabla \times \vec{A}$$

$$\nabla \cdot \vec{A} = -\mu\varepsilon \frac{\partial V}{\partial t}$$

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

($\Rightarrow$ proven)

(Time varying 조건에서도 $\vec{B}$ 와 $\vec{E}$ 를 알 수 있는 정의. 접근 방법이 wave equation과 유사(Ch. 11))

$\mu\varepsilon$ → Propagation element (or speed)

- Propagation: Any electromagnetic wave disturbance is found to travel at a velocity $v = \frac{1}{\sqrt{\mu\varepsilon}}$ through any homogeneous medium.

- The potential at any point is due not to have the value of charge density at some distant point at the same instant, but to its value at some previous time, since the effect propagate at a finite velocity.

- Hence $V = \int_{vol} \frac{\rho_v dv}{4\pi\varepsilon R}$ $\rightarrow$ $V = \int_{vol} \frac{[\rho_v]dv}{4\pi\varepsilon R}$

$$\rho_v = e^{-r}\cos\omega t \quad \rightarrow \quad [\rho_v] = e^{-r}\cos\left[\omega\left(t - \frac{R}{v}\right)\right]$$

where $t' = t - \frac{R}{v}$: retarded time

(전자기파가 $t = \frac{R}{v}$ 시간 동안에 v라는 속도로 거리 R만큼 이동함을 나타냄.)

- Retarded vector magnetic potential:

$$\vec{A} = \int_{vol} \frac{\mu[\vec{J}]}{4\pi R} dv$$