Chapter 2
Transmission Line

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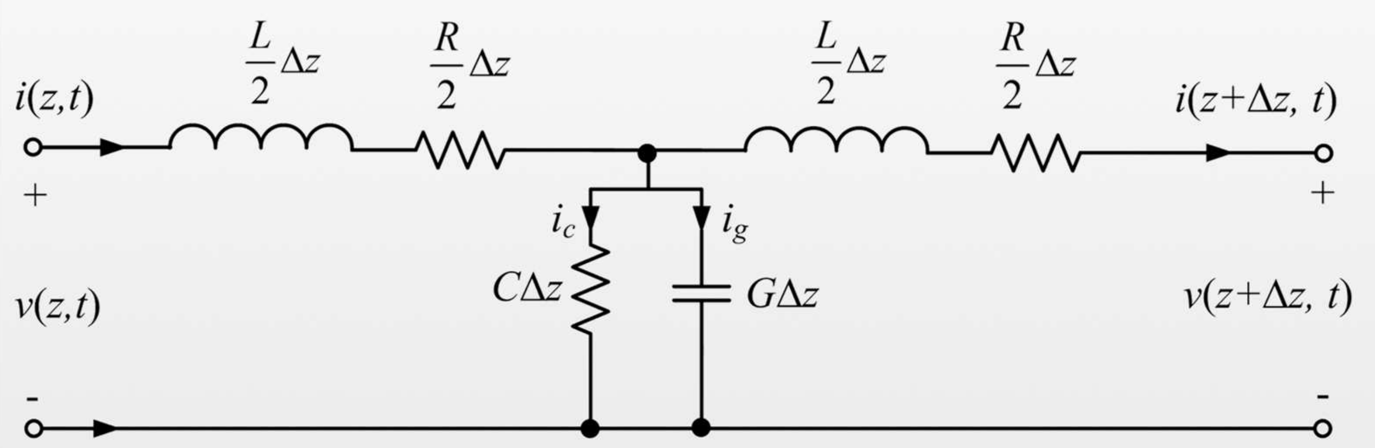


Learning Objectives

- Learn limitation of low frequency circuit elements in microwave frequency.
- Learn general transmission line characteristics.
- Learn operation of terminated lossless transmission line

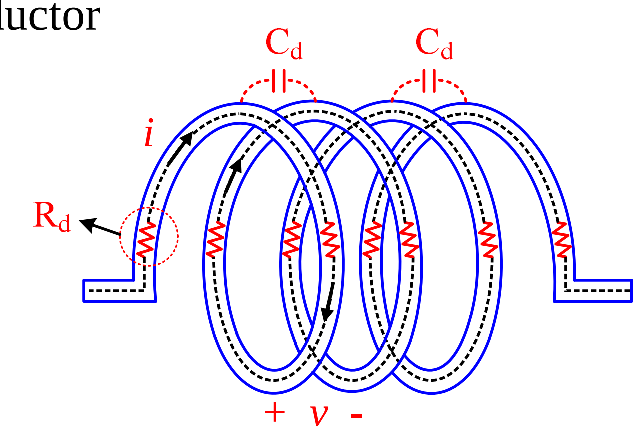
Learning contents

- Frequency dependance of microwave components
- Wave equations of transmission Line
- Terminated lossless transmission lines

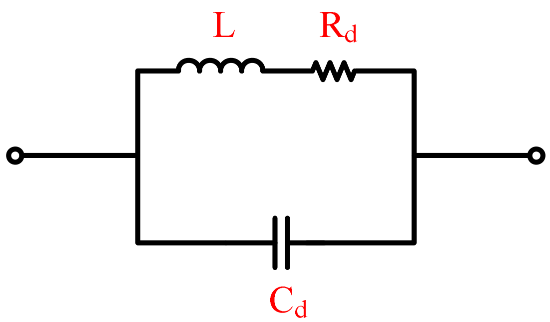
1 Frequency Dependance of Microwave Components

- General lumped elements (*R*, *L*, *C*, etc.) frequency dependent characteristics

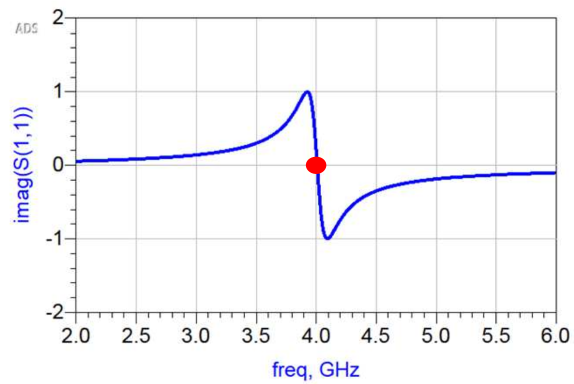
- Inductor



Coil inductor



Equivalent circuit



Electrical characteristic

$$\phi = Li, \qquad v = \frac{d\phi}{dt} = L\frac{di}{dt}$$

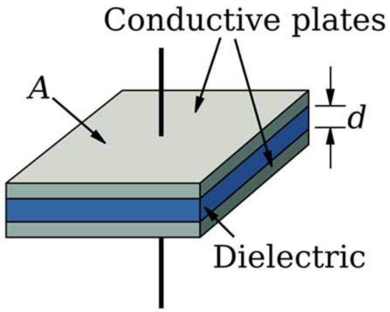
$$R = \frac{L}{\sigma S}, \qquad Q = Cv$$

- **Self resonance frequency (SRF)**: a resonant frequency where the reactive component is zero. (Im(*Z*) = 0)

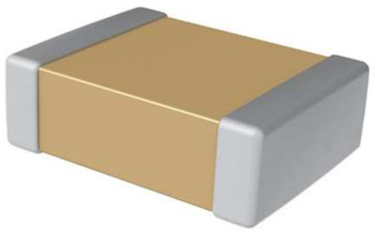
➔ It is not easy to fabricated inductor on microwave frequency range.

1 Frequency Dependance of Microwave Components

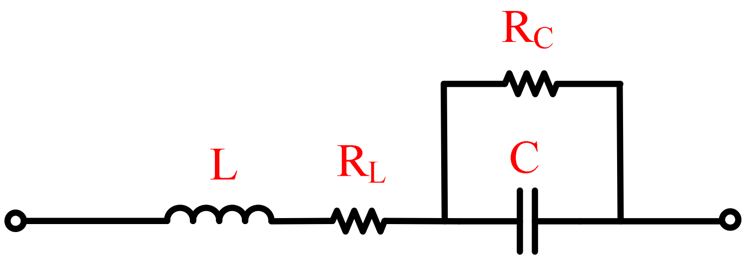
- Capacitor



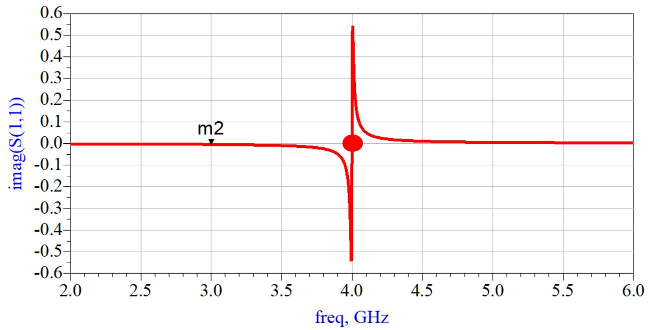
Typical capacitor coil inductor



Chip capacitor



Equivalent circuit

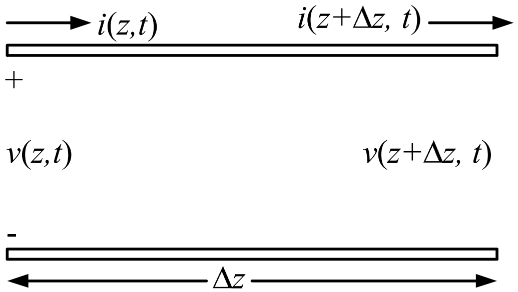


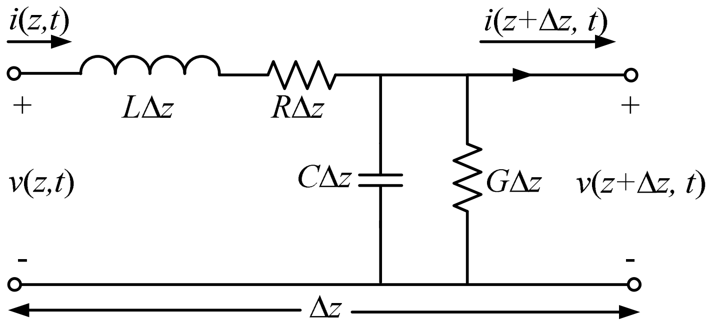
➔ Even though relatively easy to fabricated than inductor, it is also not easy to fabricated capacitor in microwave frequency.

☻ ***How to realize R, L, and C on the microwave/millimeter frequency ranges?***

2 Wave Equations of Transmission Line

- **Two-wire line** as typical transmission line and **its equivalent circuits** for short piece length (Δz)



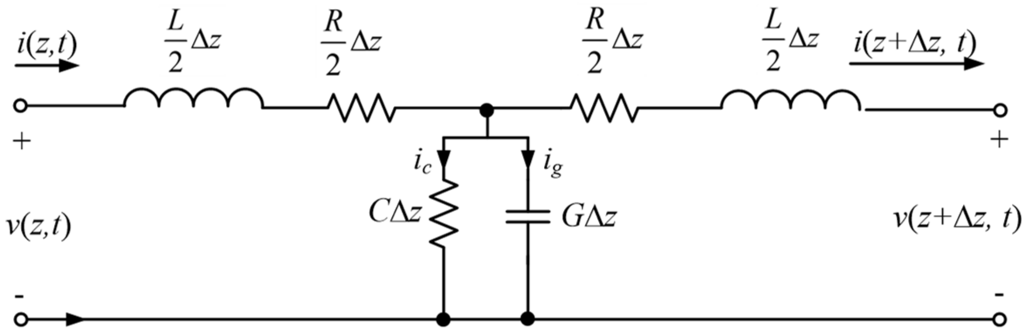


- Low pass characteristics

- $\phi = Li, \quad v = \dfrac{d\phi}{dt} = L\dfrac{di}{dt}$

$$Q = Cv, \quad i = \frac{dQ}{dt} = C\frac{dv}{dt}$$

$$R = \frac{L}{\sigma S}, \ C = \frac{\varepsilon S}{d} \ \leftarrow \varepsilon = \varepsilon' - j\varepsilon''$$

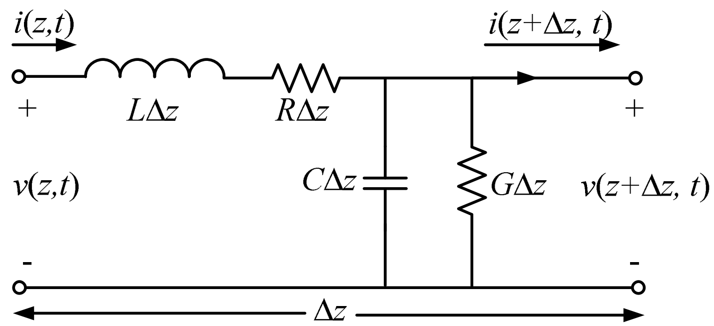


- *L*: series inductance per unit length [H/m] *R*: series resistance per unit length [Ω/m]
  C: shunt capacitance per unit length [F/m] *G*: shunt conductance per unit length [S/m or Ʊ/m]

2 Wave Equations of Transmission Line

- Kirchhoff's voltage law

$$v(z,t) - R\Delta z i(z,t) - L\Delta z \frac{\partial i(z,t)}{\partial t} - v(z+\Delta z,t) = 0 \qquad (1)$$



Kirchhoff's current law

$$i(z,t) - C\Delta z \frac{\partial v(z+\Delta z,t)}{\partial t} - G\Delta z v(z+\Delta z,t) - i(z+\Delta z,t) = 0 \qquad (2)$$

- Dividing (2.1a) and (2.1b) by Δz and taking the limit as $\Delta z \to 0$:

$$\lim_{\Delta z \to 0} \frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = \frac{\partial v(z,t)}{\partial z} = -L\frac{\partial i(z,t)}{\partial t} - Ri(z,t)$$

$$\lim_{\Delta z \to 0} \frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = \frac{\partial i(z,t)}{\partial z} = -C\frac{\partial v(z,t)}{\partial t} - Gv(z,t)$$

- By assuming sinusoidal steady state condition, time domain forms can be changed cosine-based **phasor forms**.

$$\frac{dV(z)}{dz} = -(R + j\omega L)I(z) \qquad (3) \qquad \frac{dI(z)}{dz} = -(G + j\omega C)V(z) \qquad (4)$$

cf.) $v(z,t) \leftrightarrow V(z)$

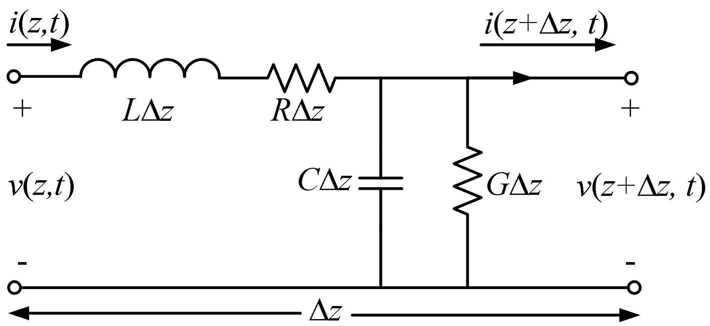
$i(z,t) \leftrightarrow I(z)$

2 Wave Equations of Transmission Line

- By differentiating (3) with (4)

$$\frac{d^2V(z)}{dz^2} = -(R + j\omega L)\frac{dI}{dz} = (R + j\omega L)(G + j\omega C)V(z) = \gamma^2 V(z)$$

$$\frac{d^2V(z)}{dz^2} - \gamma^2 V(z) = 0 \qquad (5)$$



By same manner

$$\frac{d^2I(z)}{dz^2} - \gamma^2 I(z) = 0 \qquad (6)$$

where $\gamma = \alpha + j\beta = \sqrt{(R + j\omega L)(G + j\omega C)} = f(\omega)$: complex propagation constant

α: attenuation constant, β: phase constant

- Traveling wave equations on space domain:

$$V(z) = V_o^+ e^{-\gamma z} + V_o^- e^{\gamma z} \qquad (7) \qquad I(z) = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z} \qquad (8)$$

where $e^{-\gamma z}$: wave propagation for +z direction (or forward direction)

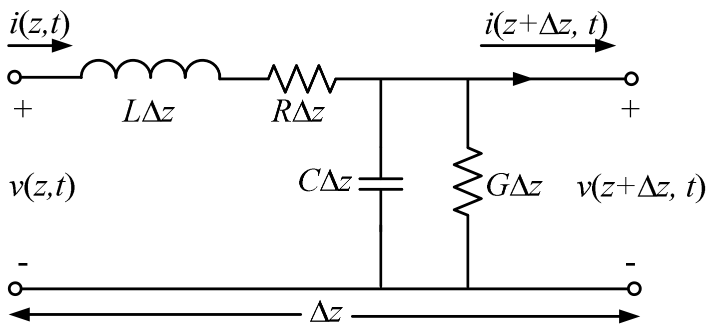
$e^{\gamma z}$: wave propagation for -z direction (or backward direction)

2 Wave Equations of Transmission Line

- From (3) and (4),

$$I(z) = -\frac{1}{R+j\omega L}\frac{dV(z)}{dz} = \frac{\gamma}{R+j\omega L}[V_o^+ e^{-\gamma z} - V_o^- e^{\gamma z}] = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z}$$

$$I_o^+ = \frac{\gamma}{R+j\omega L}V_o^+ \qquad I_o^- = -\frac{\gamma}{R+j\omega L}V_o^-$$



- Characteristic impedance: a ratio of voltage wave to current wave

$$Z_0 = \frac{V_o^+}{I_o^+} = -\frac{V_o^-}{I_o^-} = \frac{R+j\omega L}{\gamma} = \sqrt{\frac{R+j\omega L}{G+j\omega C}} \quad (9) \qquad \leftarrow \quad \gamma = \sqrt{(R+j\omega L)(G+j\omega C)}$$

- Voltage-current combined current waveform

$$I(z) = \frac{V_o^+}{Z_0}e^{-\gamma z} - \frac{V_o^-}{Z_0}e^{\gamma z} = I_o^+ e^{-\gamma z} + I_o^- e^{\gamma z} \quad (10)$$

- Sinusoidal voltage wave can be recovered by using $v(z,t) = \mathrm{Re}[V(z)e^{j\omega t}]$ and $\gamma = \alpha + j\beta$.

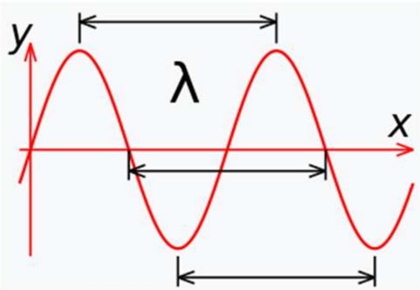
$$v(z,t) = |V_o^+|\cos(\omega t - \beta z + \phi^+)e^{-\alpha z} + |V_o^-|\cos(\omega t + \beta z + \phi^-)e^{\alpha z}$$

where $\phi^{\pm}$: phase angle of voltage wave $V_0^{\pm}$ ex.) $V_0^+ = |V_0^+|e^{j\phi^+}$

2 Wave Equations of Transmission Line

- Wavelength (: distance between consecutive corresponding points of same phase (2π)) on transmission line:

$$\lambda = \frac{2\pi}{\beta} \qquad [\frac{\text{rad}}{\text{rad/m}}] = [\text{m}] \qquad (11)$$



- Phase velocity (rate at the wave propagates in any medium) on transmission line:

$$v_p = \frac{\lambda}{t} = \frac{2\pi f}{\beta} = \frac{\omega}{\beta} = f\lambda \quad [\text{m/sec}] \qquad (12)$$

- For lossless transmission line (@ $R = G = 0$),

$$\gamma = \sqrt{(R + j\omega L)(G + j\omega C)}\Big|_{R=G=0} = j\omega\sqrt{LC} \;\; (= \alpha + j\beta) \qquad \Leftrightarrow \qquad \alpha = 0 \;\; \& \;\; \beta = \omega\sqrt{LC}$$

$$Z_0 = \sqrt{\frac{R + j\omega L}{G + j\omega C}}\Bigg|_{R=G=0} = \sqrt{\frac{L}{C}} \qquad (13)$$

Wave equations: $V(z) = V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}$ (14) $\qquad I(z) = I_o^+ e^{-j\beta z} + I_o^- e^{j\beta z} = \frac{V_o^+}{Z_0} e^{-j\beta z} - \frac{V_o^-}{Z_0} e^{j\beta z}$ (15)

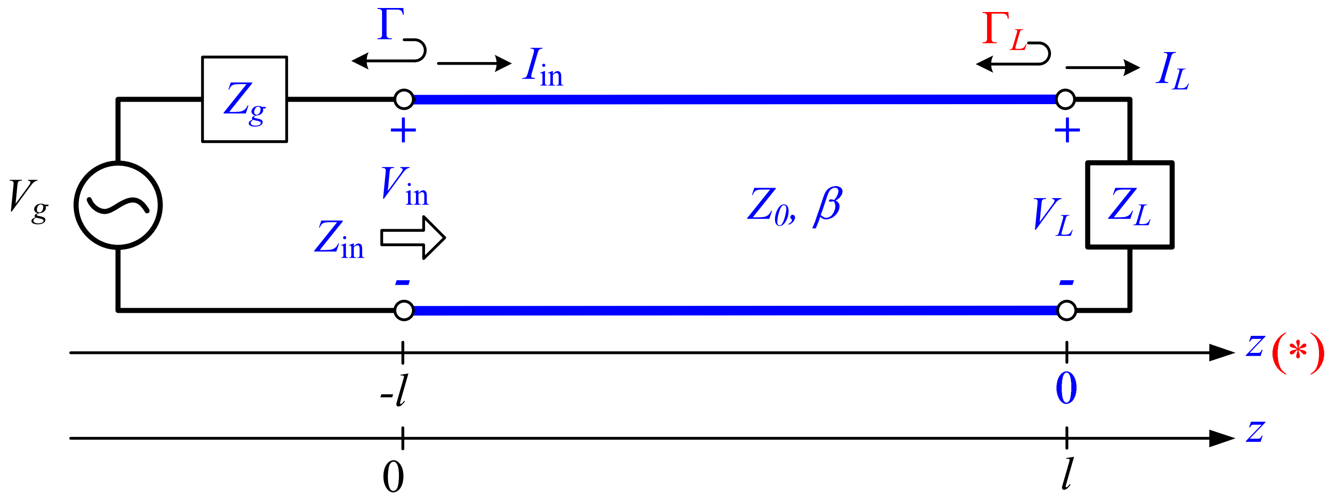
Wavelength: $\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{LC}}$ (16) $\qquad$ Phase velocity: $v_p = \frac{\omega}{\beta} = \frac{1}{\sqrt{LC}}$ (17)

3 Terminated Lossless Transmission Lines

- A lossless transmission line terminated in an arbitrary load impedance Z_L

  - Incident voltage wave: $V_0^+ e^{-j\beta z}$

  - Z_0 ($\neq Z_L$, typ.): ratio of traveling voltage wave to traveling current wave on transmission line

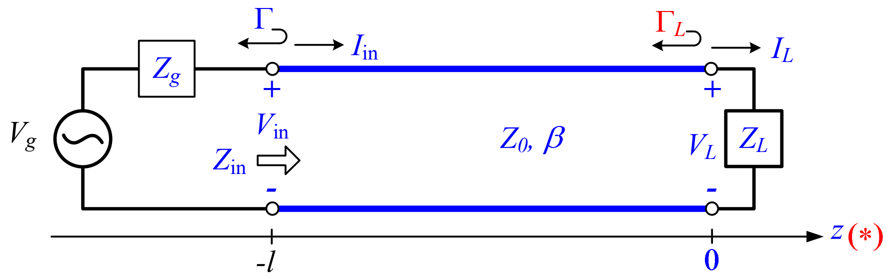


  - Total voltage and current waves consisted of incident and reflected voltage/current waves due to $Z_0 \neq Z_L$:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} \qquad (1)$$

$$I(z) = \frac{V_0^+}{Z_0} e^{-j\beta z} - \frac{V_0^-}{Z_0} e^{j\beta z} \qquad (2)$$

3 Terminated Lossless Transmission Lines



- Load impedance: $Z_L = \left.\dfrac{V(z)}{I(z)}\right|_{z=0} = \dfrac{V_0^+ + V_0^-}{V_0^+ - V_0^-} Z_0$

Reflected voltage wave: $V_o^- = \dfrac{Z_L - Z_0}{Z_L + Z_0} V_o^+$

- Voltage reflection coefficient (Γ) = Reflected voltage wave amplitude / Incident voltage wave amplitude

$$\Gamma = \frac{V_0^-}{V_0^+} = \frac{|V_0^-| e^{j\phi^-}}{|V_0^+| e^{j\phi^+}} = \frac{Z_L - Z_0}{Z_L + Z_0} = |\Gamma| e^{j\theta} \qquad (3)$$

: **complex** ($\because$ Z_L: complex)

➔ phase deviation between incident and reflected voltage waves

- Total voltage and current waves on transmission line:

$$V(z) = V_0^+ e^{-j\beta z} + V_0^- e^{j\beta z} = V_0^+ [e^{-j\beta z} + (V_0^- / V_0^+) e^{j\beta z}] = V_0^+ [e^{-j\beta z} + \Gamma e^{j\beta z}] \qquad (4)$$

$$I(z) = \frac{V_0^+}{Z_0} [e^{-j\beta z} - \Gamma e^{j\beta z}] \qquad (5)$$

- Standing waves: superposition of an incident and reflected waves as $V_o^+ e^{-j\beta z} + V_o^- e^{j\beta z}$ and $I_o^+ e^{-j\beta z} + I_o^- e^{j\beta z}$

- **If $\Gamma = 0$, no reflected wave**. ➔ **$Z_L = Z_0$** (**(Impedance) Matching!!**)

3 Terminated Lossless Transmission Lines

- Time-average power flowing along transmission line (@ z)

$$P_{av} = \frac{1}{2}\text{Re}[V(z)I(z)^*] = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}\text{Re}\{1-\Gamma^* e^{-2j\beta z} + \Gamma e^{2j\beta z} - |\Gamma|^2\} = \frac{1}{2}\frac{|V_0^+|^2}{Z_0}(1-|\Gamma|^2)$$

- Incident power: $|V_0^+|^2 / 2Z_0$

- Reflected power: $|V_0^+|^2 |\Gamma|^2 / 2Z_0$

- Delivered power (P_{av}) = Incident power - Reflected power

- If $\Gamma = 0$, the maximum power is delivered to the load

If $|\Gamma| = 1$, no power is delivered.

- When the load is mismatched ($|\Gamma| \neq 0$ or $Z_L \neq Z_0$), then not all of the available power from the generator is delivered to the load.

- Return loss (RL) = Reflected power / Incident power

- RL = $-20\log|\Gamma|$ [dB]

- Ex.) RL = ∞ dB @ $\Gamma = 0$

RL = 0 dB @ $|\Gamma| = 1$

3 Terminated Lossless Transmission Lines

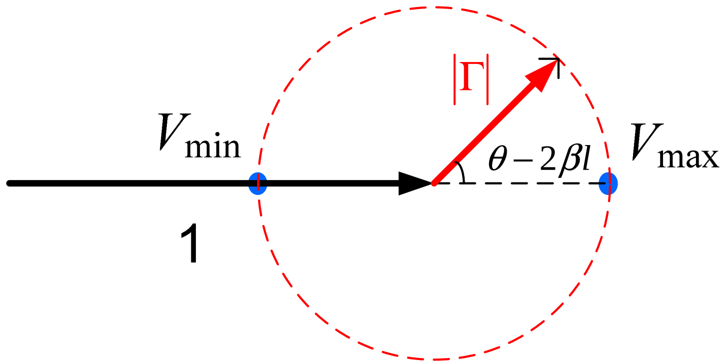
- In case of $Z_L \neq Z_0$, voltage standing wave due to reflected wave at $z = -l$

$$|V(-l)| = |V_0^+||e^{j\beta l} + \Gamma e^{-j\beta l}| = |V_0^+||e^{j\beta l}||1+\Gamma e^{-2j\beta l}| = |V_0^+||1+\Gamma e^{-2j\beta l}| = |V_0^+||1+|\Gamma| e^{j(\theta-2\beta l)}|, \qquad (6)$$

where $\Gamma = |\Gamma| e^{j\theta}$ and θ: phase of reflection coefficient

- In case of $e^{j(\theta - 2\beta l)} = 1$, $V_{max} = |V_0^+|(1+|\Gamma|)$

- In case of $e^{j(\theta - 2\beta l)} = -1$, $V_{min} = |V_0^+|(1-|\Gamma|)$



- (Voltage) Standing wave ratio (V**SWR**):

$$\text{SWR} = \frac{V_{max}}{V_{min}} = \frac{1+|\Gamma|}{1-|\Gamma|} \qquad (7)$$

➔ $1 \leq \text{SWR} \leq \infty$ $(\because 0 \leq |\Gamma| \leq 1)$: another expression of reflection coefficient

- Two successive voltage maxima (or minima) on transmission line:

$$(\theta - 2\beta l_1) - (\theta - 2\beta l_2) = 2\beta(l_2 - l_1) = 2\beta l = 2\pi \qquad \leftarrow l = l_2 - l_1$$

$$l = 2\pi / 2\beta = \pi / \beta = \pi\lambda / 2\pi = \lambda / 2$$

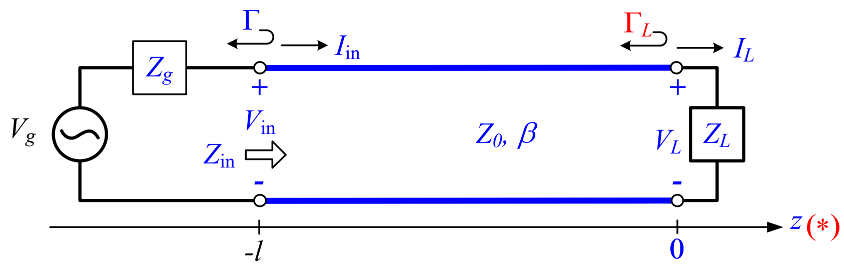
- Distance between two successive voltage maximum and minimum points: $l = \pi/2\beta = \lambda/4$

3 Terminated Lossless Transmission Lines

- Reflection coefficient at z = -l: $\Gamma(-l) = \dfrac{V_0^- e^{-j\beta l}}{V_0^+ e^{j\beta l}} = \Gamma(0) e^{-j2\beta l}$

  - $\Gamma(0)$: reflection coefficient @ z = 0

  - '**Load reflection @ z = 0**' + '**two times phase shift**'



- Input impedance seen looking toward the load at z = -l: Z_{in}

$$Z_{in} = \frac{V(-l)}{I(-l)} = \frac{V_0^+[e^{j\beta l} + \Gamma e^{-j\beta l}]}{V_0^+[e^{j\beta l} - \Gamma e^{-j\beta l}]} Z_0 = \frac{e^{j\beta l} + \Gamma e^{-j\beta l}}{e^{j\beta l} - \Gamma e^{-j\beta l}} Z_0 \qquad (8)$$

$$= \frac{e^{j\beta l} + \dfrac{Z_L - Z_0}{Z_L + Z_0} e^{-j\beta l}}{e^{j\beta l} - \dfrac{Z_L - Z_0}{Z_L + Z_0} e^{-j\beta l}} Z_0 \qquad \leftarrow \ \Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

$$= Z_0 \frac{(Z_L + Z_0) e^{j\beta l} + (Z_L - Z_0) e^{-j\beta l}}{(Z_L + Z_0) e^{j\beta l} - (Z_L - Z_0) e^{-j\beta l}} = Z_0 \frac{Z_L (e^{j\beta l} + e^{-j\beta l}) + Z_0 (e^{j\beta l} - e^{-j\beta l})}{Z_0 (e^{j\beta l} + e^{-j\beta l}) + Z_L (e^{j\beta l} - e^{-j\beta l})}$$

$$= Z_0 \frac{Z_L \cos\beta l + jZ_0 \sin\beta l}{Z_0 \cos\beta l + jZ_L \sin\beta l} = Z_0 \frac{Z_L + jZ_0 \tan\beta l}{Z_0 + jZ_L \tan\beta l} \qquad (9)$$

$$= f(Z_0, Z_L, l)$$

4 Review

- Lumped elements (*L* and *C*) frequency dependances

  - Self resonance

  - Element value variation according to frequency

- Terminated lossless transmission lines

  - Reflection coefficient (Γ)

  - (V)SWR

  - Input impedance of transmission line circuit terminated with load