Chapter 2
Transmission Line

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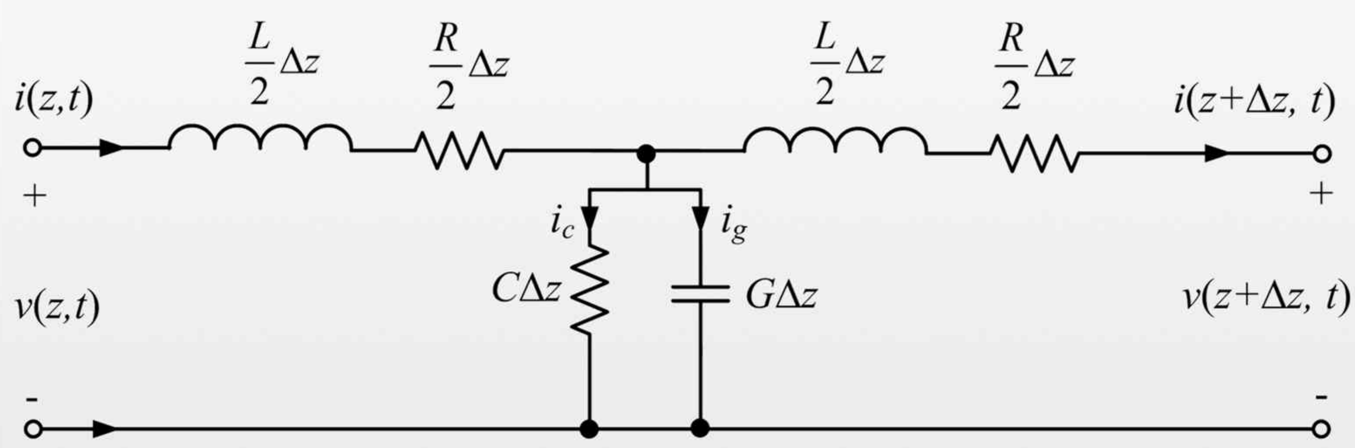


Learning Objectives

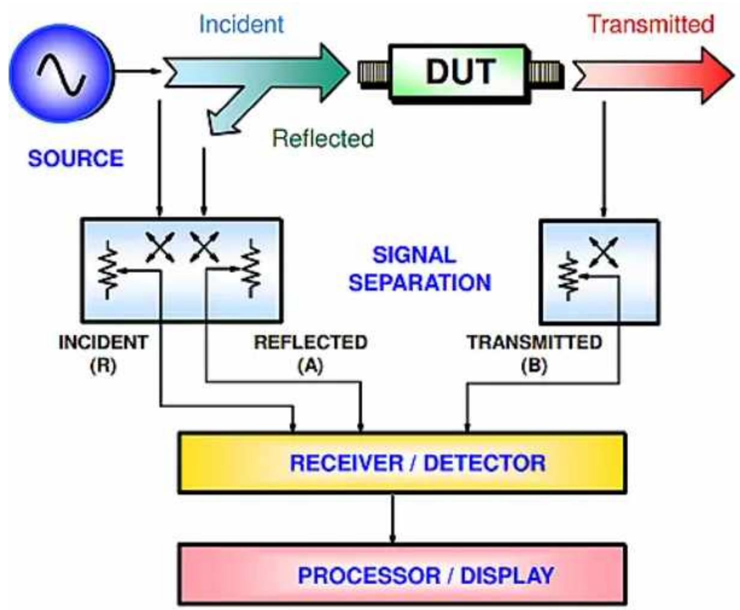
- Learn circuit parameters measurement with network analyzer
- Understanding $\lambda/4$ impedance transformer operation
- Understanding $\lambda/4$ impedance transformer with multiple reflections
- What is the condition for impedance matching?

Learning contents

- Circuit Measurements: Vector Network Analyzer (VNA)
- Quarter-wavelength ($\lambda/4$) Transformer
- Multiple Reflection Viewpoint for $\lambda/4$ Impedance Transformer
- Impedance Matchings

1 Circuit Measurements: Vector Network Analyzer (VNA)

- In stead of slotted line, modern network analyzers can measure the related parameters.
- Scalar Network analyzer: only signal magnitude measurable
- Vector Network Analyzer (VNA): Magnitude+ phase
- VNAs are extremely versatile instruments that can characterize *S*-parameters, match complex impedances, (V)SWR, group delay, insertion phase.
- Figure* shows general block diagram of VNA
- Signal can be sent through the device under test (DUT) from input port to output port.
- Network analyzer's receivers measure the incident, reflected, and transmitted signal to calculate *S*-parameters.

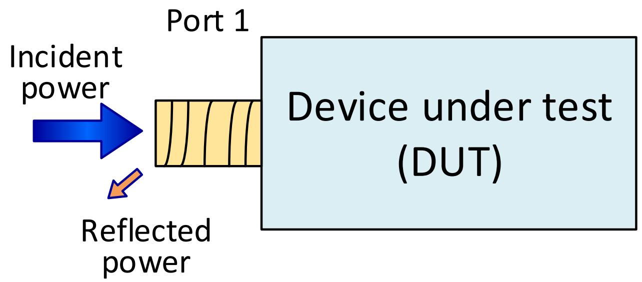


*https://www.keysight.com/us/en/solutions/measurement-fundamentals/network-analysis.html

1 Circuit Measurements: Vector Network Analyzer (VNA)

- Reflection measurement: 1-port device under test (DUT)
- Reflection coefficient (or S_{11}-parameters):

$$S_{11} = \Gamma = \frac{V_{reflected}}{V_{incident}} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0}$$



- Return loss (RL):

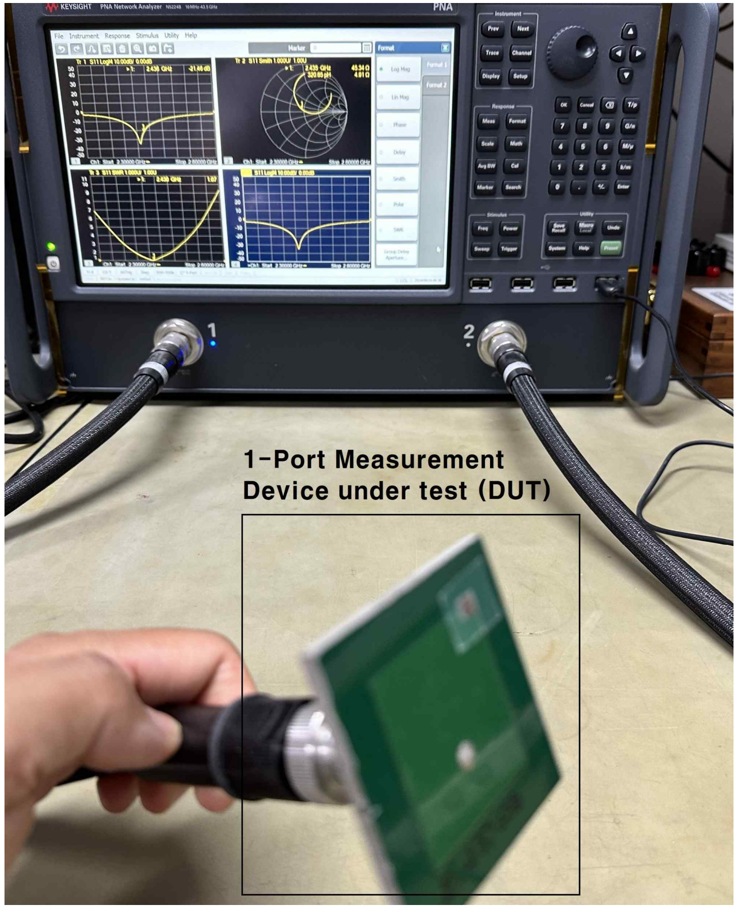
$$\text{RL} = -20\log_{10}\left(|S_{11}|\right) = -20\log_{10}\left(|\Gamma|\right)$$

- (Voltage) Standing wave ratio ((V)SWR):

$$\text{SWR} = \frac{1-|\Gamma|}{1+|\Gamma|}$$

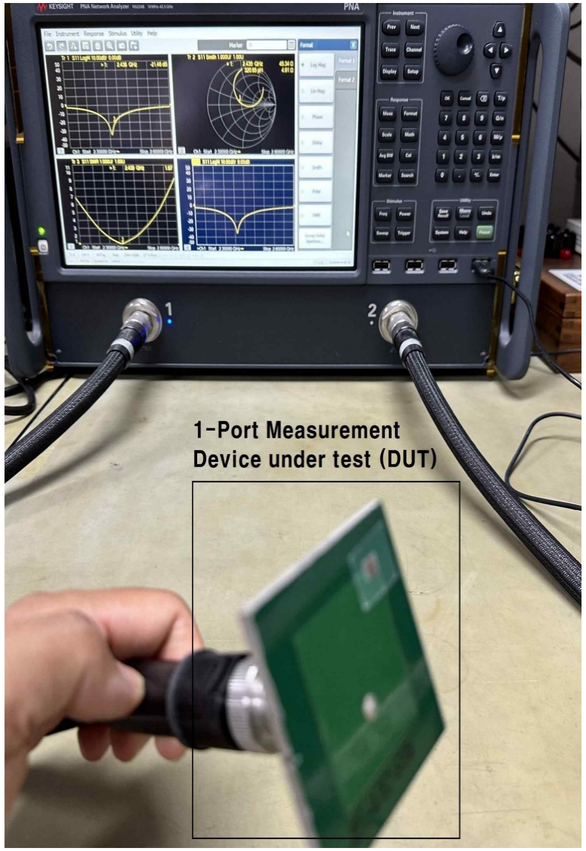
- Input impedance:

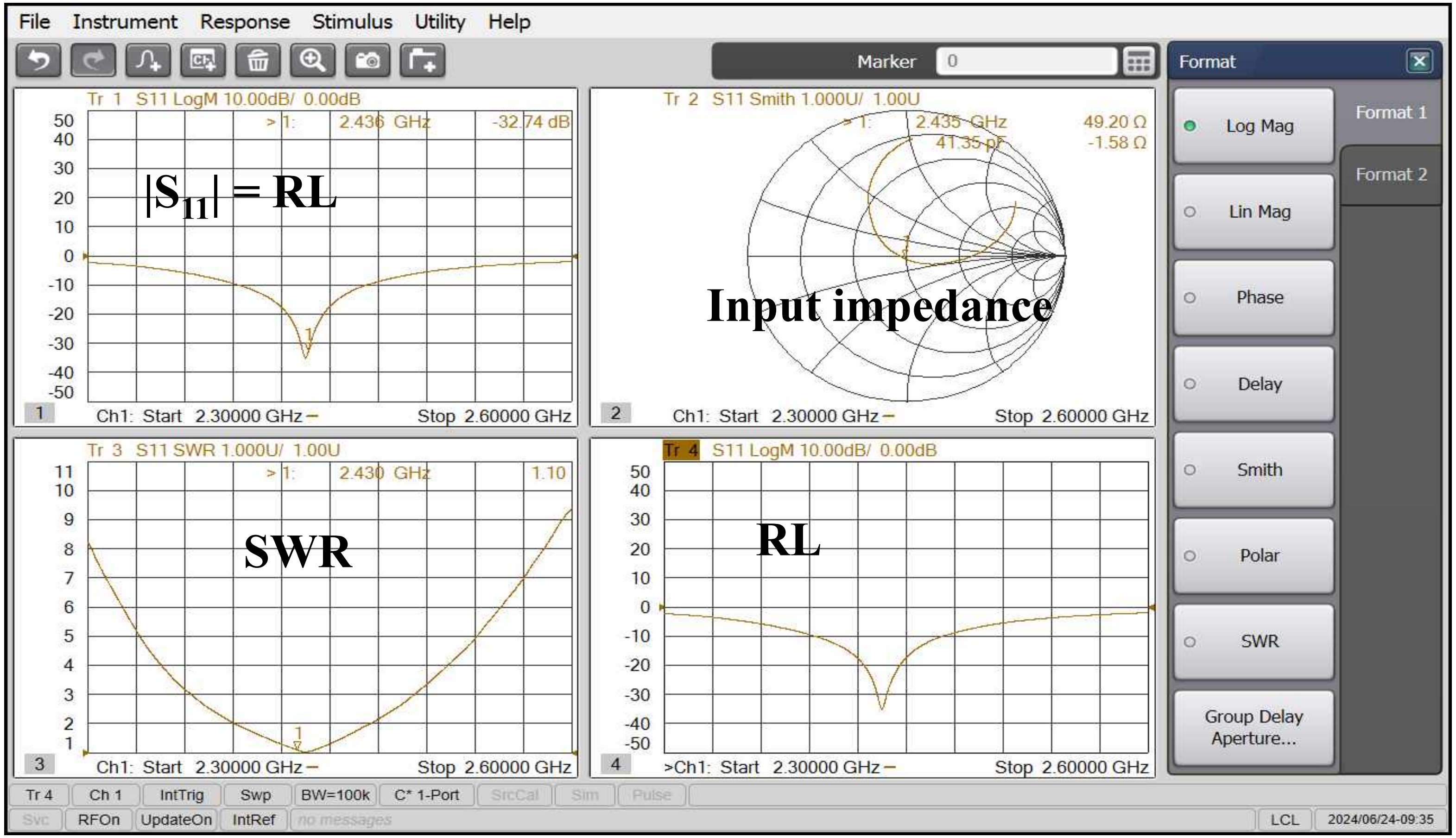
$$Z_{\text{in}} = Z_0\left(\frac{1+\Gamma}{1-\Gamma}\right) = Z_0\left(\frac{1+S_{11}}{1-S_{11}}\right)$$



1 Circuit Measurements: Vector Network Analyzer (VNA)

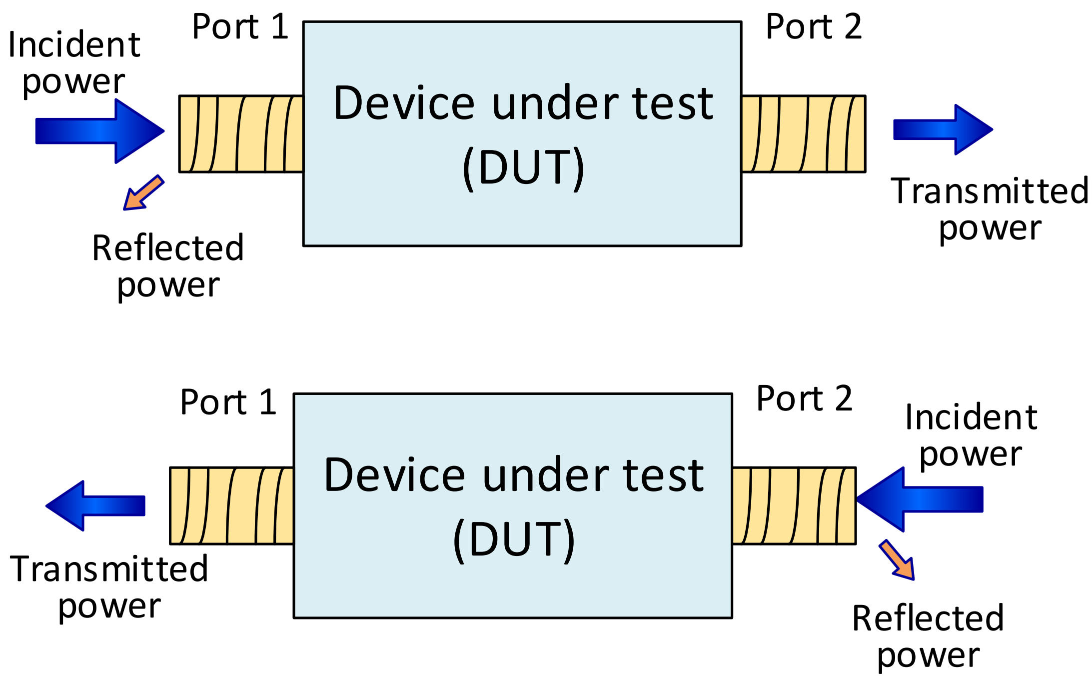
- Reflection measurement examples: 1-port device under test (DUT)

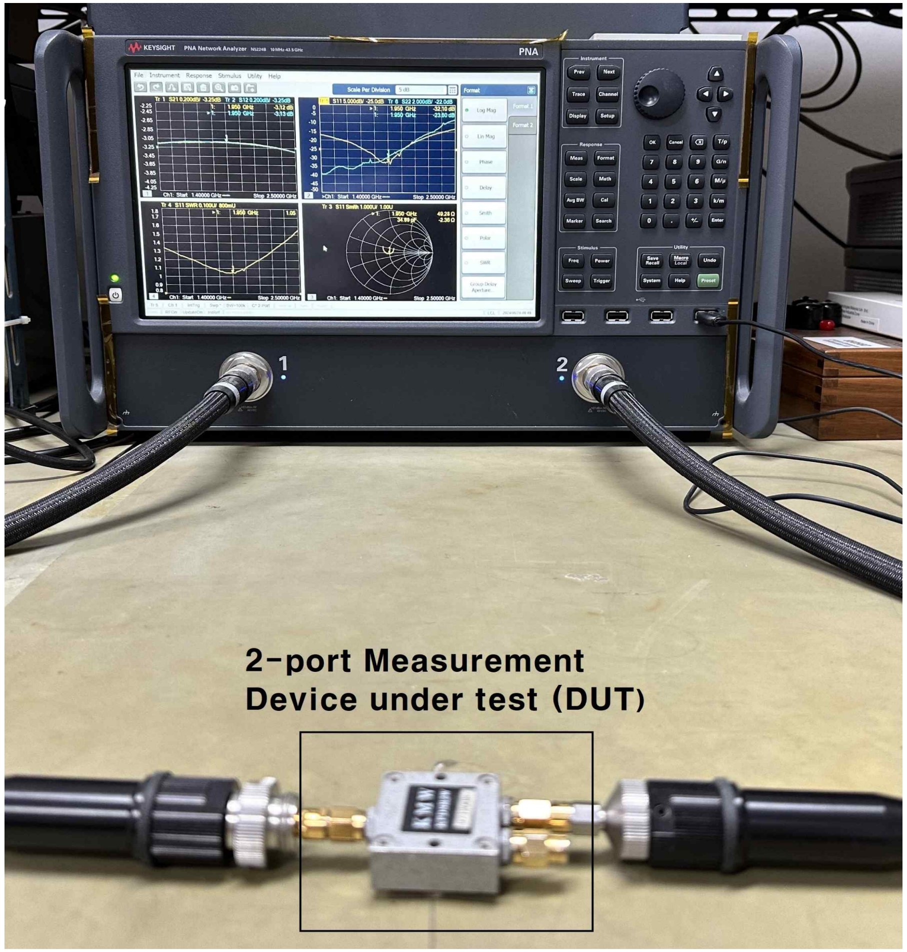




1 Circuit Measurements: Vector Network Analyzer (VNA)

- Transmission measurement examples on 2-port devices or circuits

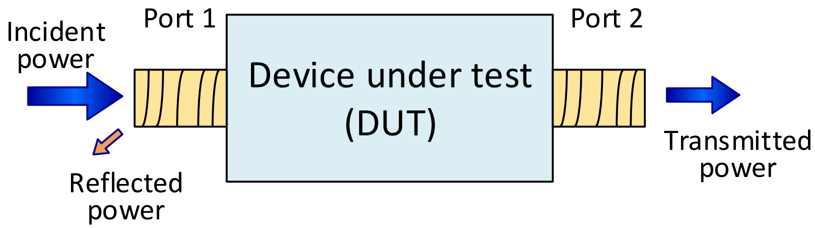


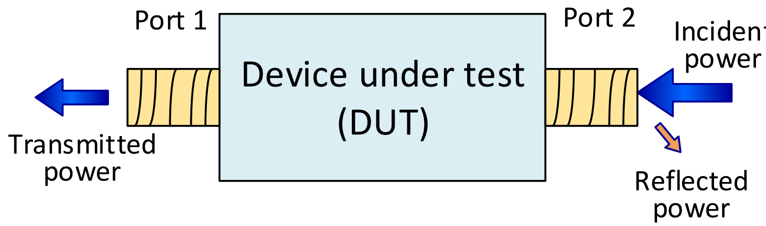


1 Circuit Measurements: Vector Network Analyzer (VNA)

- Transmission measurement: 2-port devices or circuits

  - S-parameters: S_{11}, S_{21}, S_{12}, S_{22}





$$S_{11} = \Gamma_{\text{in}} = \frac{V_{reflected}^{Port1}}{V_{reflected}^{Port1}} \qquad S_{21} = \frac{V_{trasmitted}^{Port2}}{V_{incident}^{Port1}}$$

$$S_{22} = \Gamma_{\text{out}} = \frac{V_{reflected}^{Port2}}{V_{reflected}^{Port2}} \qquad S_{12} = \frac{V_{trasmitted}^{Port1}}{V_{incident}^{Port2}}$$

- Input and output return losses (RLs):

$$RL_{\text{in}}\left(\text{dB}\right) = -20\log_{10}\left(|S_{11}|\right)$$

$$RL_{\text{out}}\left(\text{dB}\right) = -20\log_{10}\left(|S_{22}|\right)$$

- Input and output VSWRs:

$$\text{SWR}_{\text{in}} = \frac{1-\Gamma_{\text{in}}}{1+\Gamma_{\text{in}}}, \quad \text{SWR}_{\text{out}} = \frac{1-\Gamma_{\text{out}}}{1+\Gamma_{\text{out}}}$$

- Insertion phase: $\text{IL}_{\text{phase}} = \angle S_{21}$

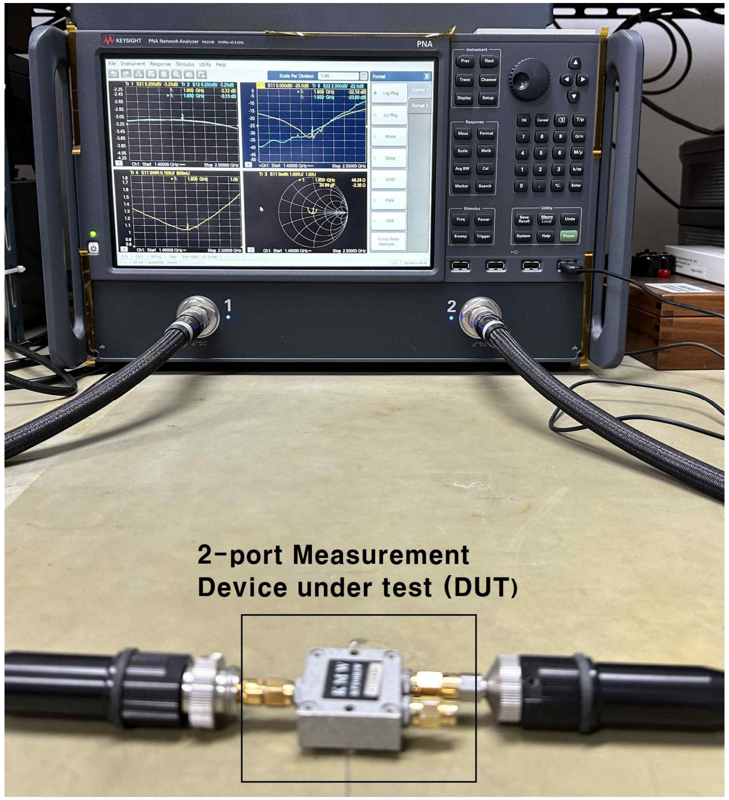
- Insertion loss (IL) and Gain:

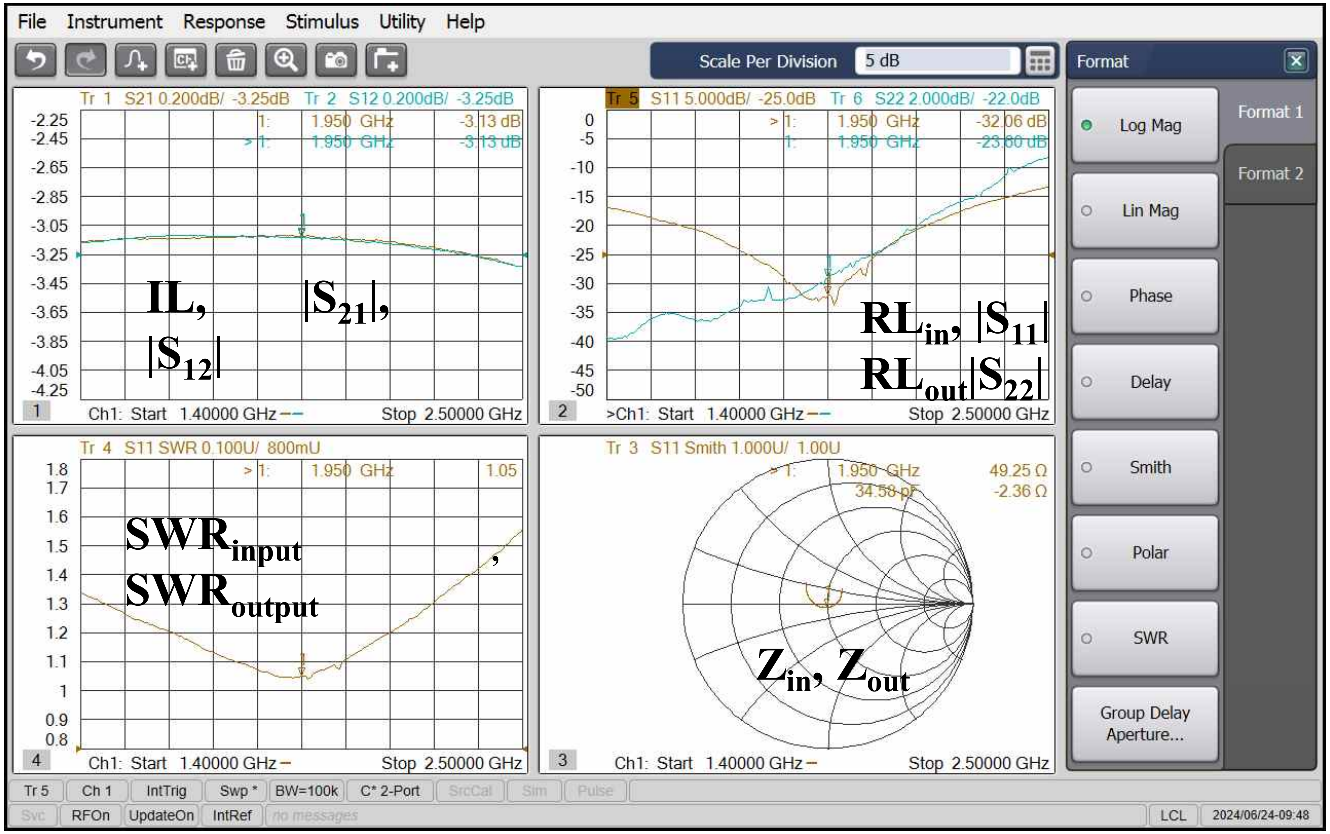
$$\text{IL}_{\text{dB}} = -20\log_{10}\left(|S_{11}|\right) \qquad \text{Gain (dB)} = 20\log_{10}\left(|S_{21}|\right)$$

- Group delay: $\tau = -\dfrac{d\angle S_{21}}{d\omega}$

1 Circuit Measurements: Vector Network Analyzer (VNA)

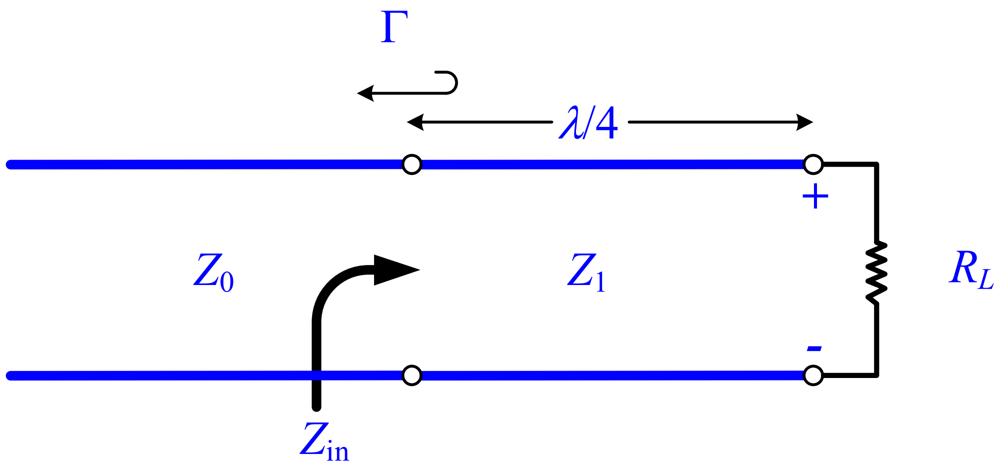
- Transmission examples: 2-port device under test (DUT)





2 Quarter-wavelength (λ/4) Transformer

- Useful and practical circuit in microwave system
- Transmission line terminated with resistance load (R_L)



- quarter-wavelength transmission line of Z_1
- Feedline characteristic impedance: Z_0
- Input impedance:

$$Z_{in} = Z_1 \frac{R_L + jZ_1 \tan \beta l}{Z_1 + jR_L \tan \beta l} \qquad \leftarrow \beta l = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$$

$$= \frac{Z_1^2}{R_L}$$

- In order for $\Gamma = 0$, we must have $Z_{in} = Z_0$.

$$\frac{Z_1^2}{R_L} = Z_0 \quad \Leftrightarrow \quad Z_1 = \sqrt{Z_0 R_L}$$

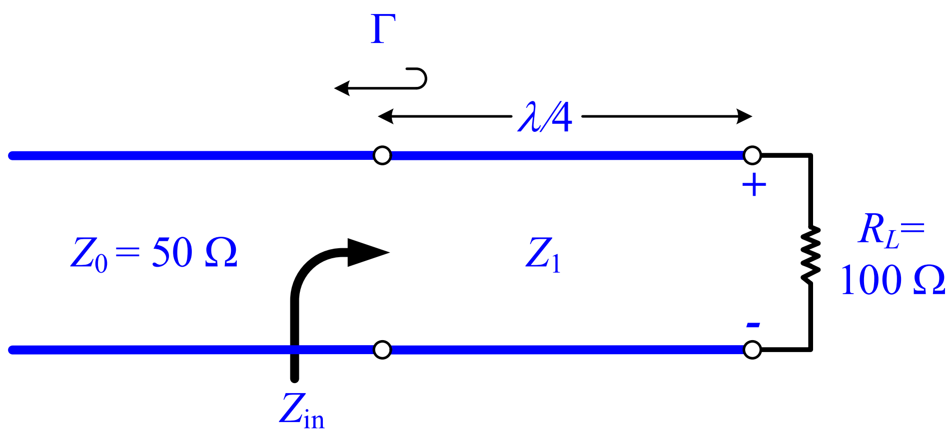
➔ Geometric mean of load and source impedances
➔ No standing waves on feedline (SWR = 1)
➔ Only when the length of the matching section is λ/4 or $(2n + 1)\lambda/4$
➔ ***A perfect match may be achieved at one frequency (or center frequency)***
➔ *Reflection tolerance* permits an operating bandwidth, not only center frequency

2 Quarter-wavelength (λ/4) Transformer

- Example quarter-wavelength impedance transformer

  - R_L = 100 Ω, Z_0 = 50 Ω,

  - Find characteristic impedance of transmission line of Z_1 and normalized frequency characteristics



Solution) Characteristic impedance of Z_1:

$$Z_1 = \sqrt{Z_0 R_L} = \sqrt{50 \times 100} = 70.71 \ [\Omega]$$

- Reflection coefficient magnitude:

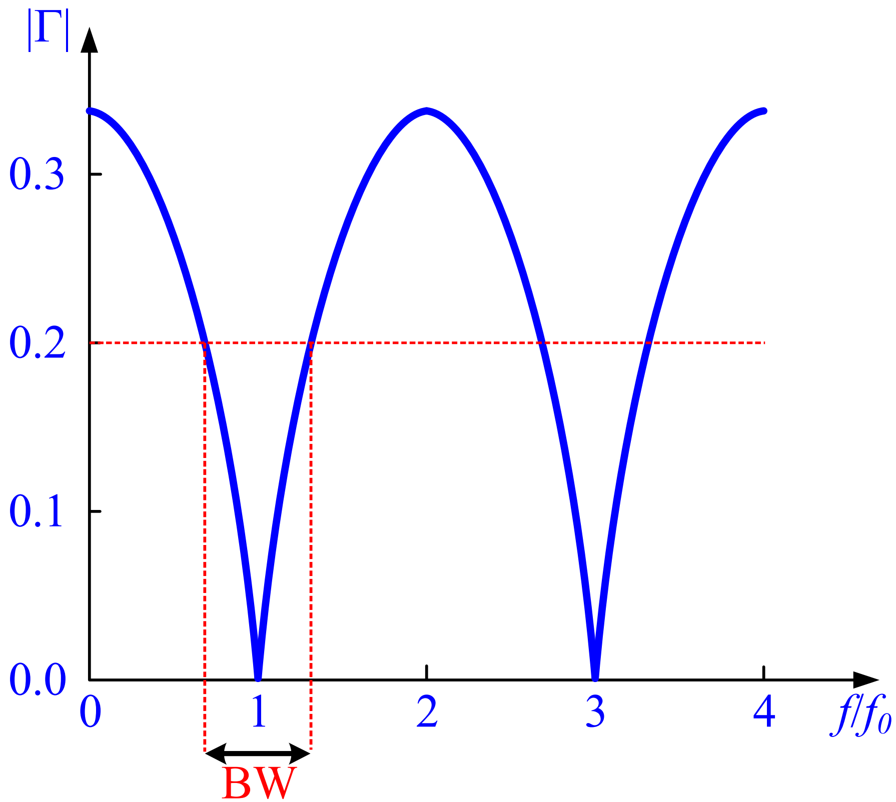
$$|\Gamma| = \left| \frac{Z_{in} - Z_0}{Z_{in} + Z_0} \right| \qquad \leftarrow Z_{in} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

$$\beta l = (\frac{2\pi}{\lambda})(\frac{\lambda_0}{4}) = (\frac{2\pi f}{v_p})(\frac{v_p}{4 f_0}) = \frac{\pi}{2}\frac{f}{f_0} \quad \leftarrow f\lambda = v_p \ \text{ or } \ \lambda = \frac{v_p}{f}$$

$$\beta l \big|_{f=f_0} = \frac{\pi}{2}: \ \text{perfect matching}$$

2 Quarter-wavelength ($\lambda/4$) Transformer

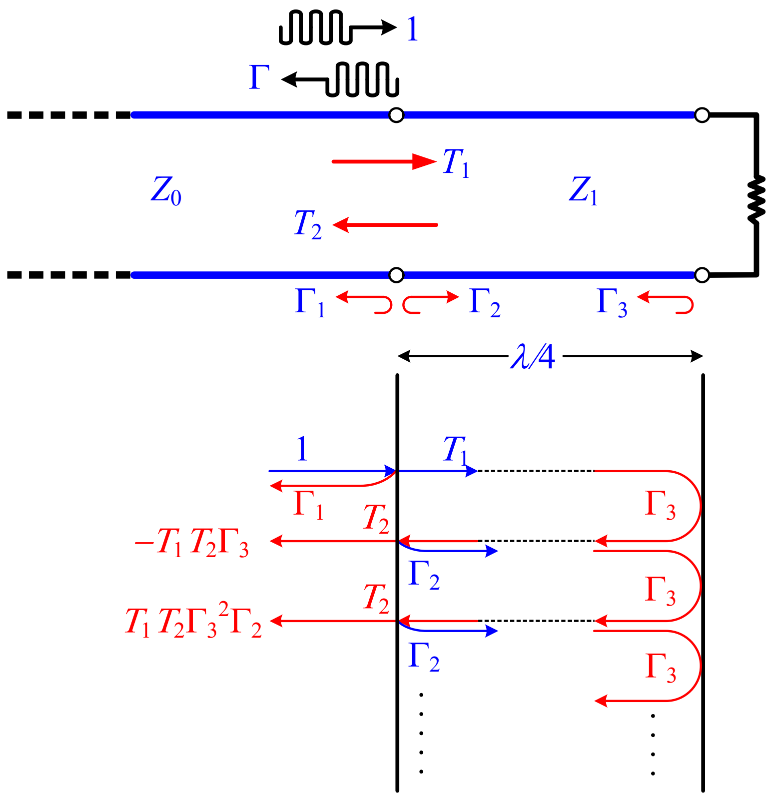
- Reflection coefficient characteristics according to normal frequency (f/f_0)



3 Multiple reflection Viewpoint for $\lambda/4$ Impedance Transformer

- Reflection and transmission for $\lambda/4$ impedance transformer

  - Γ: overall reflection coefficient of incident wave
  - Γ_1: partial reflection coefficient of wave incident on load Z_1 from Z_0 line
  - Γ_2: partial reflection coefficient of wave incident on load Z_0 from Z_1 line
  - Γ_3: partial reflection coefficient of wave incident on load R_L from Z_1 line
  - T_1: partial transmission coefficient of wave from Z_0 line into Z_1 line
  - T_2: partial transmission coefficient of a wave from Z_1 line into Z_0 line



- Individual coefficients

$$\Gamma_1 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \quad \Gamma_2 = \frac{Z_0 - Z_1}{Z_0 + Z_1} = -\Gamma_1, \quad \Gamma_3 = \frac{R_L - Z_1}{R_L + Z_1},$$

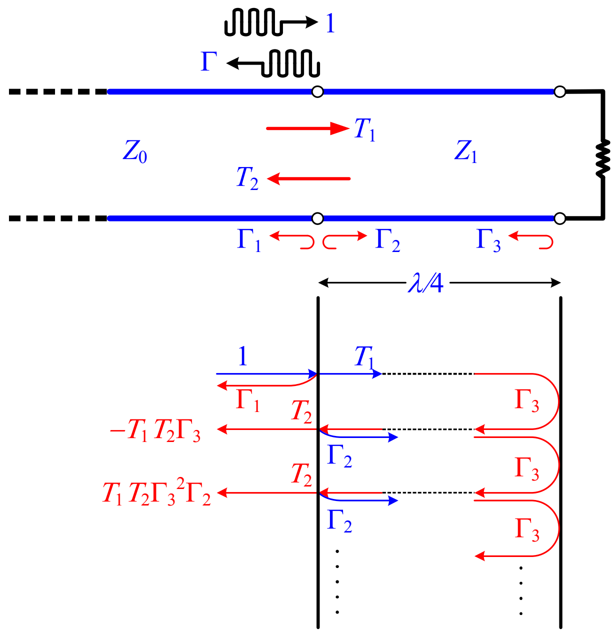
$$T_1 = 1 + \Gamma_1 = \frac{2Z_1}{Z_1 + Z_0}, \quad T_2 = 1 + \Gamma_2 = \frac{2Z_0}{Z_1 + Z_0}$$

3 Multiple reflection Viewpoint for $\lambda/4$ Impedance Transformer

- Total reflection coefficient

$$\Gamma = \Gamma_1 - T_1T_2\Gamma_3 + T_1T_2\Gamma_2\Gamma_3^2 - T_1T_2\Gamma_2^2\Gamma_3^3 + \cdots \quad \leftarrow 180^\circ \text{ phase shift}$$

$$= \Gamma_1 - T_1T_2\Gamma_3\sum_{n=0}^{\infty}(-\Gamma_2\Gamma_3)^n$$



- Since $|\Gamma_3| < 1$ and $|\Gamma_2| < 1$,

$$\Gamma = \Gamma_1 - \frac{T_1T_2\Gamma_3}{1+\Gamma_2\Gamma_3} \quad \leftarrow \sum_{n=0}^{\infty}x^n = \frac{1}{1-x} \quad \text{for } |x|<1$$

$$= \frac{\Gamma_1 + \Gamma_1\Gamma_2\Gamma_3 - T_1T_2\Gamma_3}{1+\Gamma_2\Gamma_3} = \frac{\Gamma_1 - \Gamma_1^2\Gamma_3 - T_1T_2\Gamma_3}{1+\Gamma_2\Gamma_3}$$

- Simplified numerator: $\Gamma_1 - \Gamma_3(\Gamma_1^2 + T_1T_2) = \Gamma_1 - \Gamma_3[\frac{(Z_1-Z_0)^2}{(Z_1+Z_0)^2} + \frac{4Z_1Z_0}{(Z_1+Z_0)^2}] = \Gamma_1 - \Gamma_3$

$$= \frac{(Z_1-Z_0)}{(Z_1+Z_0)} - \frac{(R_L-Z_1)}{(R_L+Z_1)} = \frac{(Z_1-Z_0)(R_L+Z_1)-(R_L-Z_1)(Z_1+Z_0)}{(Z_1+Z_0)(R_L+Z_1)}$$

$$= \frac{(Z_1R_L + Z_1^2 - Z_0R_L - Z_0Z_1) - (Z_1R_L + Z_0R_L - Z_1^2 - Z_0Z_1)}{(Z_1+Z_0)(R_L+Z_1)} = \frac{2(Z_1^2 - Z_0R_L)}{(Z_1+Z_0)(R_L+Z_1)}$$

➔ If $Z_1 = \sqrt{Z_0R_L}$, then $\Gamma = 0$ and the transmission line is matched.

4 Impedance Matchings

- Lossless transmission line circuit with arbitrary generator and load impedances

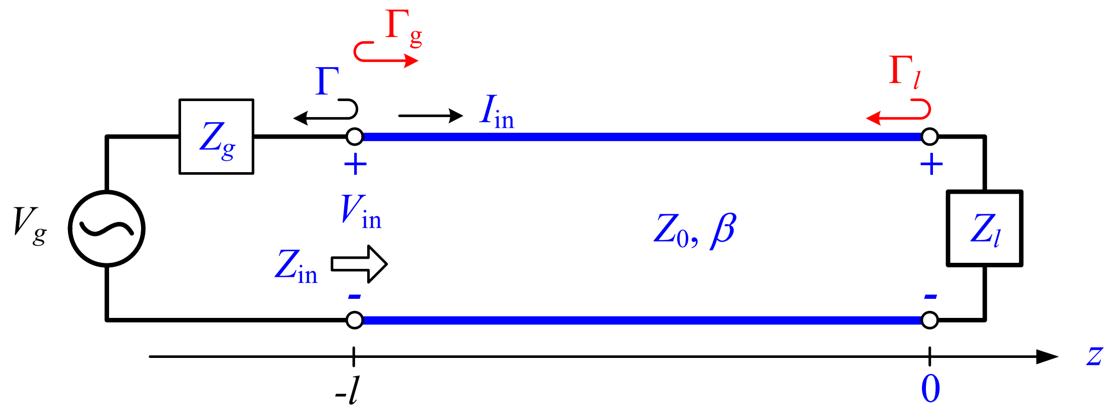
 - Input impedance looking into terminated transmission line from generator:

$$Z_{in} = \frac{V_{in}}{I_{in}} = \frac{(V_0^+ e^{-j\beta l} + V_0^- e^{-j\beta l})}{(V_0^+ e^{-j\beta l} + V_0^- e^{-j\beta l})/Z_0} = Z_0 \frac{1+\Gamma_l e^{-j2\beta l}}{1-\Gamma_l e^{-j2\beta l}}$$

$$= Z_0 \frac{Z_l + jZ_0 \tan \beta l}{Z_0 + jZ_l \tan \beta l}$$

where Γ_l: reflection coefficient at load

l: length from load



 - Voltage wave at arbitrary location of transmission line:

$$V(z) = (V_0^+ e^{-j\beta z} + V_0^- e^{-j\beta z}) = V_0^+ (e^{-j\beta z} + \Gamma_l e^{j\beta z}) \quad \leftarrow \quad \Gamma_l = \frac{Z_l - Z_0}{Z_l + Z_0}$$

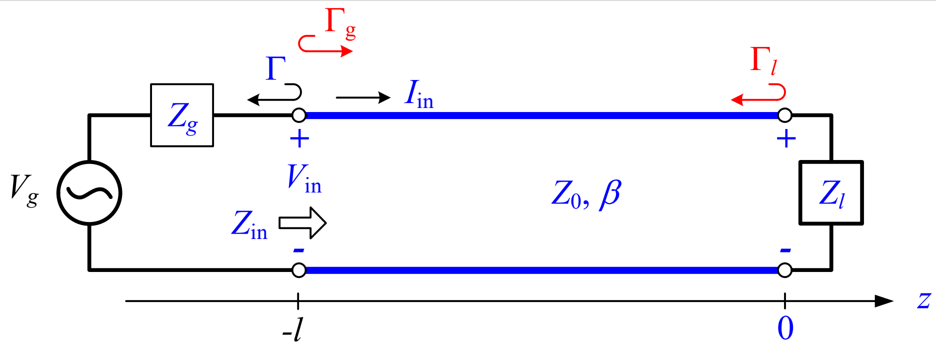
 - Voltage at generator end (@ $z = -l$)

$$V(-l) = V_g \frac{Z_{in}}{Z_g + Z_{in}} = V_0^+ (e^{j\beta l} + \Gamma_l e^{-j\beta l})$$

$$V_0^+ = V_g \frac{Z_{in}}{Z_{in} + Z_g} \frac{1}{(e^{j\beta l} + \Gamma_l e^{-j\beta l})}$$

4 Impedance Matchings

- Voltage at generator end (@ $z = -l$ (V_0^+))



$$V_0^+ = V_g \frac{Z_{in}}{Z_{in}+Z_g}\frac{1}{(e^{j\beta l}+\Gamma_l e^{-j\beta l})} \leftarrow Z_{in} = Z_0\frac{1+\Gamma_l e^{-j2\beta l}}{1-\Gamma_l e^{-j2\beta l}}$$

$$= V_g \frac{Z_0\dfrac{1+\Gamma_l e^{-2j\beta l}}{1-\Gamma_l e^{-2j\beta l}}}{Z_0\dfrac{1+\Gamma_l e^{-2j\beta l}}{1-\Gamma_l e^{-2j\beta l}}+Z_g}\frac{1}{e^{j\beta l}+\Gamma_l e^{-j\beta l}}$$

$$= V_g \frac{Z_0}{Z_0+Z_g\dfrac{1-\Gamma_l e^{-2j\beta l}}{1+\Gamma_l e^{-2j\beta l}}}\frac{1}{e^{j\beta l}+\Gamma_l e^{-j\beta l}} = V_g \frac{Z_0}{Z_0+Z_g\dfrac{e^{j\beta l}-\Gamma_l e^{-j\beta l}}{e^{j\beta l}+\Gamma_l e^{-j\beta l}}}\frac{1}{e^{j\beta l}+\Gamma_l e^{-j\beta l}} = V_g\frac{Z_0}{Z_0(e^{j\beta l}+\Gamma_l e^{-j\beta l})+Z_g(e^{j\beta l}-\Gamma_l e^{-j\beta l})}$$

$$= V_g\frac{Z_0}{e^{j\beta l}(Z_0+Z_g)+\Gamma_l e^{-j\beta l}(Z_0-Z_g)} = V_g\frac{Z_0}{Z_0+Z_g}\frac{1}{e^{j\beta l}+\Gamma_l e^{-j\beta l}\dfrac{Z_0-Z_g}{Z_0+Z_g}} = V_g\frac{Z_0}{Z_0+Z_g}\frac{1}{e^{j\beta l}-\Gamma_l e^{-j\beta l}\dfrac{Z_g-Z_0}{Z_g+Z_0}}$$

$$= V_g\frac{Z_0}{Z_0+Z_g}\frac{1}{e^{j\beta l}-\Gamma_l\Gamma_g e^{-j\beta l}} = V_g\frac{Z_0}{Z_0+Z_g}\frac{e^{-j\beta l}}{1-\Gamma_l\Gamma_g e^{-2j\beta l}}$$

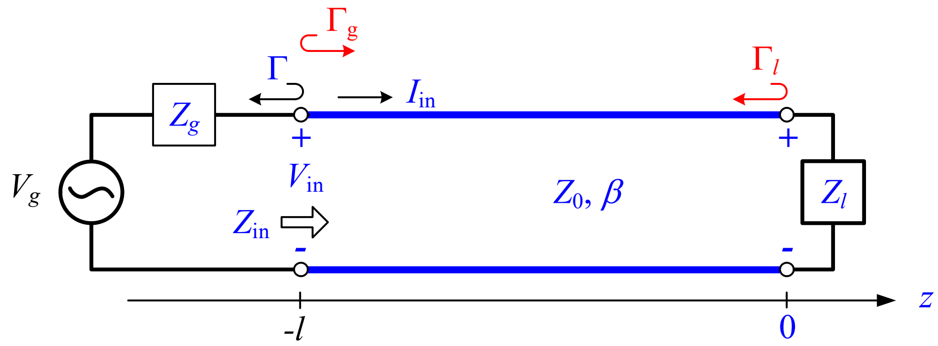
where Γ_g: reflection coefficient seen looking into generator $\Gamma_g = \dfrac{Z_g-Z_0}{Z_g+Z_0}$

4 Impedance Matchings

- (Averaging) Power delivered to load:

$$P = \mathrm{Re}[\frac{V_{\text{in}}}{\sqrt{2}}\frac{I_{\text{in}}^*}{\sqrt{2}}] = \frac{1}{2}\mathrm{Re}[V_{\text{in}} I_{\text{in}}^*] = \frac{1}{2}|V_{\text{in}}|^2 \mathrm{Re}[\frac{1}{Z_{\text{in}}^*}]$$

$$= \frac{1}{2}|V_g|^2 \left|\frac{Z_{\text{in}}}{Z_{\text{in}} + Z_g}\right|^2 \mathrm{Re}[\frac{1}{Z_{\text{in}}^*}]$$



- Let assume general source and load conditions: $Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}}$ and $Z_g = R_g + jX_g$

$$P = \frac{1}{2}|V_g|^2 \frac{|Z_{\text{in}}|^2}{(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2} \mathrm{Re}[\frac{1}{Z_{\text{in}}^*}] = \frac{1}{2}|V_g|^2 \frac{|Z_{\text{in}}|^2}{(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2} \mathrm{Re}\left[\frac{Z_{\text{in}}}{|Z_{\text{in}}|^2}\right]$$

$$= \frac{1}{2}|V_g|^2 \frac{|Z_{\text{in}}|^2}{(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2} \frac{\mathrm{Re}(Z_{\text{in}})}{|Z_{\text{in}}|^2} = \frac{1}{2}|V_g|^2 \frac{R_{\text{in}}}{(R_{\text{in}} + R_g)^2 + (X_{\text{in}} + X_g)^2}$$

- Assume that the generator impedance (Z_g) is fixed and consider three cases according to load impedance.

4 Impedance Matchings

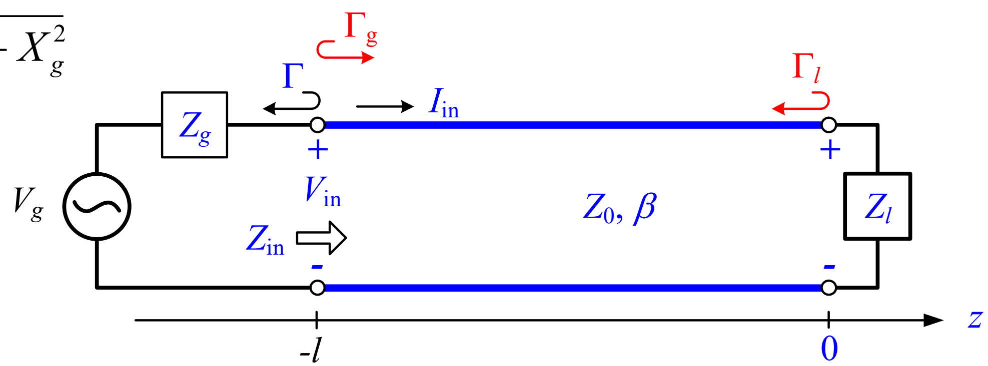
- Case study 1: load matched to transmission line

  - $Z_l = Z_0 \rightarrow \Gamma_l = 0$ & (V)SWR = 1.

  - $Z_{in} = Z_0 \neq Z_g$

  - Power delivered to load:

$$P = \frac{1}{2}|V_g|^2 \frac{R_{in}}{(R_{in}+R_g)^2+(X_{in}+X_g)^2}\Bigg|_{Z_l=Z_0=R_{in}} = \frac{1}{2}|V_g|^2 \frac{Z_0}{(Z_0+R_g)^2+X_g^2}$$



- Case study 2: generator matched to transmission line

  - $Z_{in} = Z_g \rightarrow \Gamma_l = 0$ & (V)SWR = 1.

  - Overall reflection coefficient: $\Gamma = \frac{Z_{in}-Z_g}{Z_{in}+Z_g} = 0$

  - Power delivered to load: $P = \frac{1}{2}|V_g|^2 \frac{R_{in}}{(R_{in}+R_g)^2+(X_{in}+X_g)^2}\Bigg|_{Z_g=Z_{in}} = \frac{1}{2}|V_g|^2 \frac{R_g}{(R_g+R_g)^2+(X_g+X_g)^2}$

$$= \frac{1}{2}|V_g|^2 \frac{R_g}{4(R_g^2+X_g^2)} = \frac{1}{8}|V_g|^2 \frac{R_g}{R_g^2+X_g^2}$$

4 Impedance Matchings

- Case study 3: ***conjugate matching***

 - Assumption: fixed generator impedance (Z_g)

 - Variable input impedance (Z_{in}) due to unknown load impedance (Z_l) and length of transmission line

 - Conditions for maximum power delivered to load:

Condition 1: $$\frac{\partial P}{\partial R_{\text{in}}} = \frac{\partial}{\partial R_{\text{in}}}\left[\frac{1}{2}\left|V_g\right|^2 \frac{R_{\text{in}}}{(R_{\text{in}}+R_g)^2+(X_{\text{in}}+X_g)^2}\right] = 0 \leftarrow \frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'g - fg'}{g^2}$$

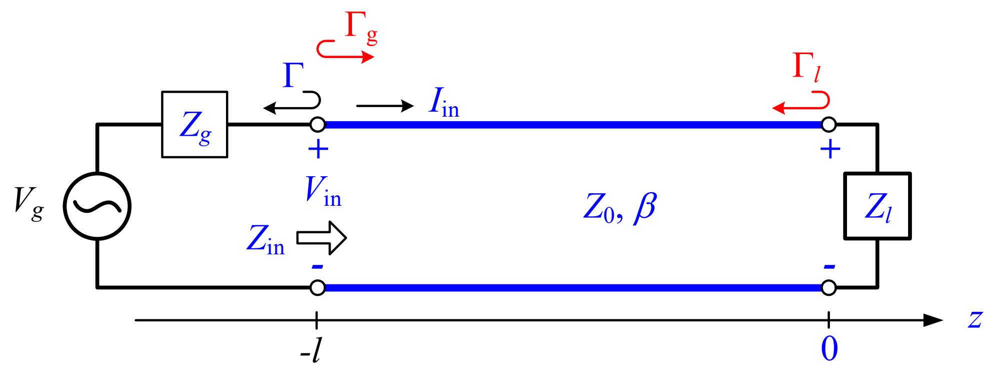
$$\rightarrow \frac{1}{(R_{\text{in}}+R_g)^2+(X_{\text{in}}+X_g)^2} + \frac{-2R_{\text{in}}(R_{\text{in}}+R_g)}{[(R_{\text{in}}+R_g)^2+(X_{\text{in}}+X_g)^2]^2} = 0$$

$$(R_{\text{in}}+R_g)^2+(X_{\text{in}}+X_g)^2 - 2R_{\text{in}}(R_{\text{in}}+R_g) = 0$$

$$R_g^{\ 2} - R_{\text{in}}^2 + (X_{\text{in}}+X_g)^2 = 0 \qquad (*)$$

Condition 2: $$\frac{\partial P}{\partial X_{\text{in}}} = 0 \rightarrow \frac{-2R_{\text{in}}(X_{\text{in}}+X_g)}{[(R_{\text{in}}+R_g)^2+(X_{\text{in}}+X_g)^2]^2} = 0$$

$$R_{\text{in}}(X_{\text{in}}+X_g) = 0 \qquad (**)$$



4 Impedance Matchings

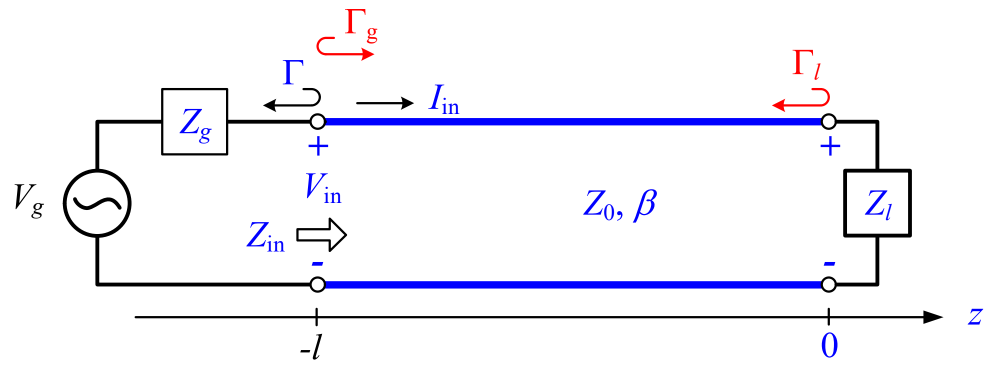
- Solving (*) and (**) simultaneously for R_{in} ($\neq 0$) and X_{in}

  ($R_{\text{in}} = R_g$ and $X_{\text{in}} = -X_g$) $\leftrightarrow$ $Z_{\text{in}} = Z_g^*$

- Maximum power delivered to load:

$$P = \frac{1}{2}|V_g|^2 \frac{R_{\text{in}}}{(R_{\text{in}}+R_g)^2 + (X_{\text{in}}+X_g)^2}\bigg|_{R_{\text{in}}=R_g \ \& \ X_{\text{in}}=-X_g}$$

$$= \frac{1}{2}|V_g|^2 \frac{R_g}{(R_g+R_g)^2 + (-X_g+X_g)^2} = \frac{1}{2}|V_g|^2 \frac{1}{4R_g}$$



➔ **Conjugate matching**

➔ **Maximum power transfer** to load for fixed generator impedance

cf.) $$P = \frac{1}{2}|V_g|^2 \frac{R_{\text{in}}}{(R_{\text{in}}+R_g)^2 + (X_{\text{in}}+X_g)^2}\bigg|_{Z_l=Z_0=R_{\text{in}}} = \frac{1}{2}|V_g|^2 \frac{Z_0}{(Z_0+R_g)^2 + X_g^2}$$ (case study #1)

$$P = \frac{1}{2}|V_g|^2 \frac{R_{\text{in}}}{(R_{\text{in}}+R_g)^2 + (X_{\text{in}}+X_g)^2}\bigg|_{Z_g=Z_{\text{in}}} = \frac{1}{2}|V_g|^2 \frac{R_g}{4(R_g^2 + X_g^2)}$$ (case study #2)

5 Review

- Network analyzer

  - General and widely used circuit measurement system

  - 1 ~ *n* ports network measurement

  - Magnitude and phase

  - *S*-parameters, RL, IL, GD, etc

- $\lambda/4$ Impedance Transformer

- Impedance matching

  - $Z_{\text{in}} = Z_g^*$ or $Z_g = Z_{\text{in}}^*$

    $\Leftrightarrow \quad Z_{\text{in}} = R_{\text{in}} + jX_{\text{in}} = R_g - jX_g^*$

  - At any reference plane