Chapter 4
Microwave Network Analysis

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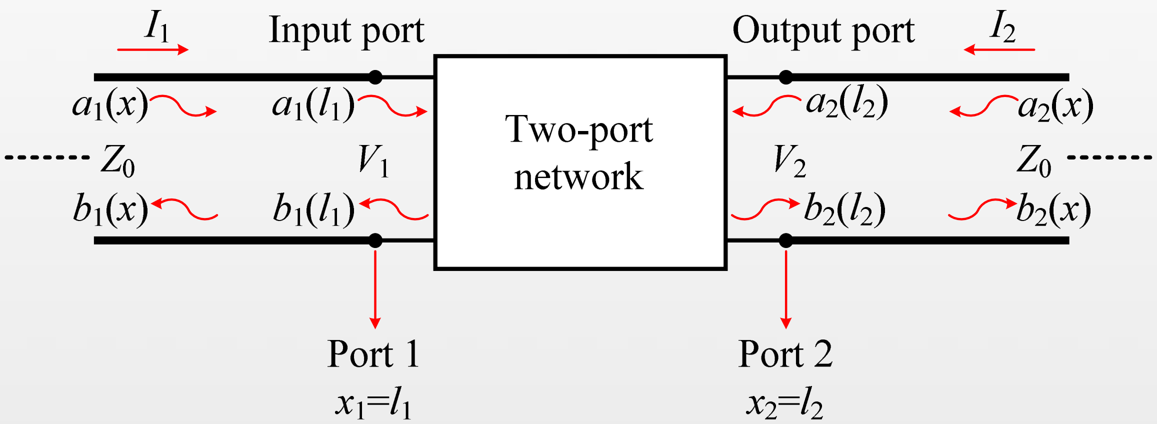


Learning Objectives

- Learn Scattering-parameters (*S*-parameters)
- Learn how to obtain *S*-parameters of 1-port network
- Learn how to obtain *S*-parameters of 2-port network
- Accustomed to parameter conversion

Learning contents

- Why *S*-parameters?
- *S*-parameters of 1-port network
- *S*-parameters of 2-port network
- Parameters Conversion and Examples

1 Why *S*-Parameters?

- **Scattering parameter** is also called ***S*-parameters**.

  - *S*-parameters denote the working of an electrical network or circuit when stimulated with signals.

  - The previous (*Z*-, *Y*-, *h*-, *g*-) parameters are useful for low frequency circuit designs.

  - At microwave frequency, it is not easy to measure the voltage and current.

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$

$$Z_{11} = \left. \frac{V_1}{I_1} \right|_{I_2=0} \quad \text{@ output port: open}$$

$$Z_{12} = \left. \frac{V_1}{I_2} \right|_{I_1=0} \quad \text{@ input port: open}$$

$$Z_{21} = \left. \frac{V_2}{I_1} \right|_{I_2=0} \quad \text{@ output port: open}$$

$$Z_{22} = \left. \frac{V_2}{I_2} \right|_{I_1=0} \quad \text{@ input port: open}$$

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$

$$Y_{11} = \left. \frac{I_1}{V_1} \right|_{V_2=0} \quad \text{@ output port: shorted}$$

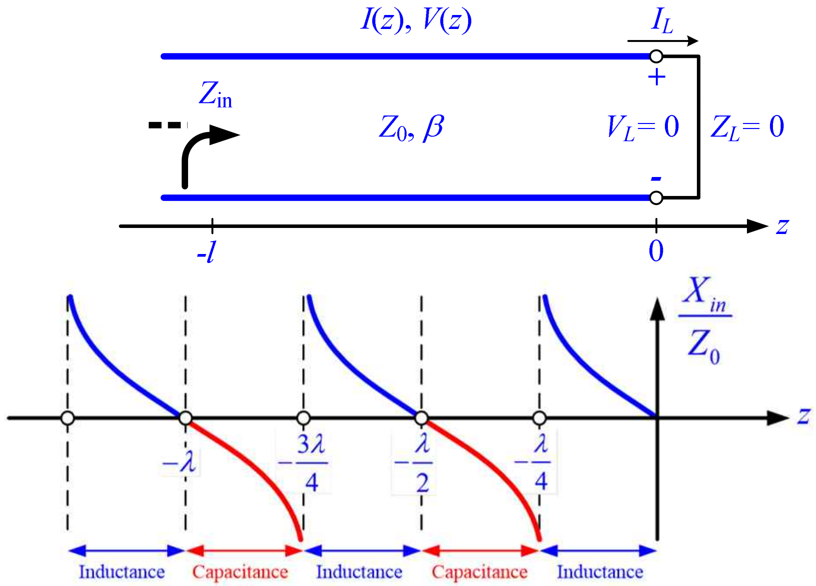
$$Y_{12} = \left. \frac{I_1}{V_2} \right|_{V_1=0} \quad \text{@ input port: shorted}$$

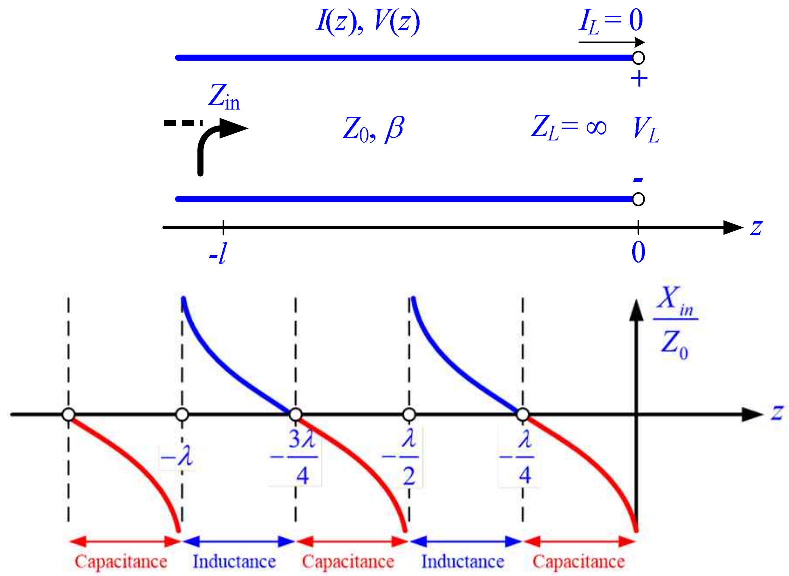
$$Y_{21} = \left. \frac{I_2}{V_1} \right|_{V_2=0} \quad \text{@ output port: shorted}$$

$$Y_{22} = \left. \frac{I_2}{V_2} \right|_{V_1=0} \quad \text{@ input port: shorted}$$

1 Why *S*-Parameters?

- Short ($Z = 0 + j0$) and open ($Z = \infty + j\,\infty$) conditions are also difficult to achieve at microwave frequency, whereas these conditions are easy at low frequency to realize due to long wavelength.





$$Z = 0 + j0 \neq 0 \pm jX \quad \text{: short condition}$$

$$Z = \infty + j\infty \neq 0 \pm jX \quad \text{: open condition}$$

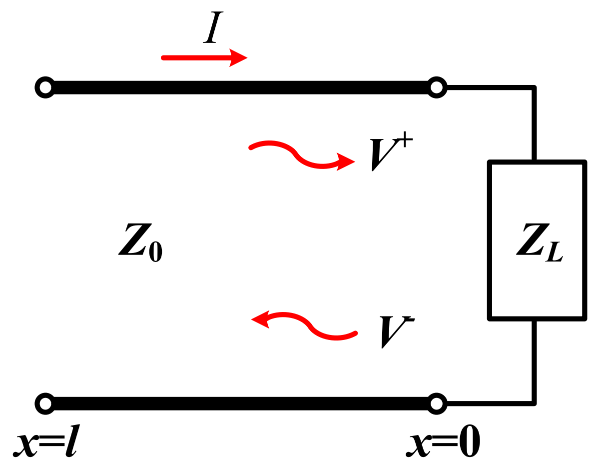
- Matching condition is easier to achieve than short- or open-circuited conditions.
  → **Define new parameter based on the reflection and transmission characteristics: *S*-parameters**

2 *S*-Parameters of 1-port Network

- A 1-port network has one *S*-parameter called S_{11}.
  - S_{11} is the ratio of the incident voltage wave to the reflected voltage wave at the input port of the network or circuit.
  - S_{11} is also known as return loss, a measure of the voltage reflection ratio at the input port.
  - Assuming:

$$V^+(x) = Ae^{-j\beta x} \quad \text{: incident voltage wave}$$

$$V^-(x) = Be^{j\beta x} \quad \text{: reflected voltage wave}$$



  - Total voltage and current waves:

$$V(x) = V^+(x) + V^-(x)$$

$$I(x) = I^+(x) + I^-(x) = \frac{V^+(x)}{Z_0} - \frac{V^-(x)}{Z_0}$$

  - Reflection coefficient:

$$\Gamma(x) = \frac{V^-(x)}{V^+(x)}$$

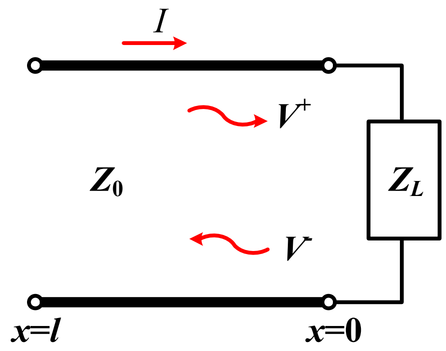
2 S-Parameters of 1-port Network

- Normalized voltage and current waves: $v(x) = \frac{V(x)}{\sqrt{Z_0}} = \frac{V^+(x)}{\sqrt{Z_0}} + \frac{V^-(x)}{\sqrt{Z_0}} = a(x) + b(x)$,

$$i(x) = \sqrt{Z_0} I(x) = \frac{V^+(x)}{\sqrt{Z_0}} - \frac{V^-(x)}{\sqrt{Z_0}} = a(x) - b(x)$$

- Normalized incident and reflected voltage waves:

$$a(x) = \frac{V^+(x)}{\sqrt{Z_0}}, \quad b(x) = \frac{V^-(x)}{\sqrt{Z_0}}$$



- Overall normalized voltage and current waves:

$$v(x) = a(x) + b(x), \quad i(x) = a(x) - b(x)$$

- Normalized incident and reflected voltage waves in terms of $V(x)$ and $I(x)$:

$$a(x) \equiv \frac{1}{2}\left[\frac{V(x)}{\sqrt{Z_0}} + \sqrt{Z_0} I(x)\right] = \frac{1}{2}\left[v(x) + i(x)\right] \ [\sqrt{\text{VA}}], \quad b(x) \equiv \frac{1}{2}\left[\frac{V(x)}{\sqrt{Z_0}} - \sqrt{Z_0} I(x)\right] = \frac{1}{2}\left[v(x) - i(x)\right] \ [\sqrt{\text{VA}}]$$

→ Conceptual unit $[\sqrt{\text{VA}}]$ is meaningless.

→ $a^2(x)$ and $b^2(x)$ have a unit of [VA] (or [W]), and it's power!!!

2 *S*-Parameters of 1-port Network

- Normalized network impedance:

$$z(x)=\frac{v(x)}{i(x)}=\frac{V(x)/\sqrt{Z_0}}{\sqrt{Z_0}I(x)}=\frac{1}{Z_0}\frac{V(x)}{I(x)}=\frac{Z_L}{Z_0}=z_L$$

- Power:

$$v(x)i(x)=\frac{V(x)}{\sqrt{Z_0}}\cdot\sqrt{Z_0}I=V(x)I(x)$$

- Definition of S_{11}:

$$S_{11}=\frac{b(x)}{a(x)}=\frac{v(x)-i(x)}{v(x)+i(x)}=\frac{v(x)/i(x)-1}{v(x)/i(x)+1}$$

$$=\frac{z-1}{z+1}=\frac{\frac{Z_L}{Z_0}-1}{\frac{Z_L}{Z_0}+1}=\frac{Z_L-Z_0}{Z_L+Z_0}=\Gamma_L$$

- Define power:

$$P(x)=\frac{1}{2}\mathrm{Re}\left[V(x)I^*(x)\right]=\frac{1}{2}\mathrm{Re}\left[v(x)i(x)^*\right]$$

$$=\frac{1}{2}\mathrm{Re}\left\{\left[a(x)+b(x)\right]\left[a(x)-b(x)^*\right]\right\}$$

$$=\frac{1}{2}\left[|a(x)|^2-|b(x)|^2\right]$$

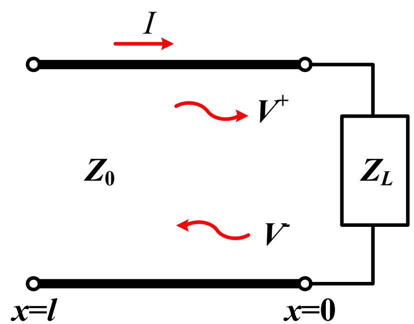
- If $Z_L = Z_0$, $S_{11} = 0$ and $b = 0$

$$P_i=\frac{1}{2}|a|^2$$: incident power

- If $b \neq 0$

$$P=\frac{1}{2}\left[|a|^2-|b|^2\right]=P_i-P_r$$

: real incident power



2 *S*-Parameters of 1-port Network

- **Example 1:** Calculate the S_{11} of following network.

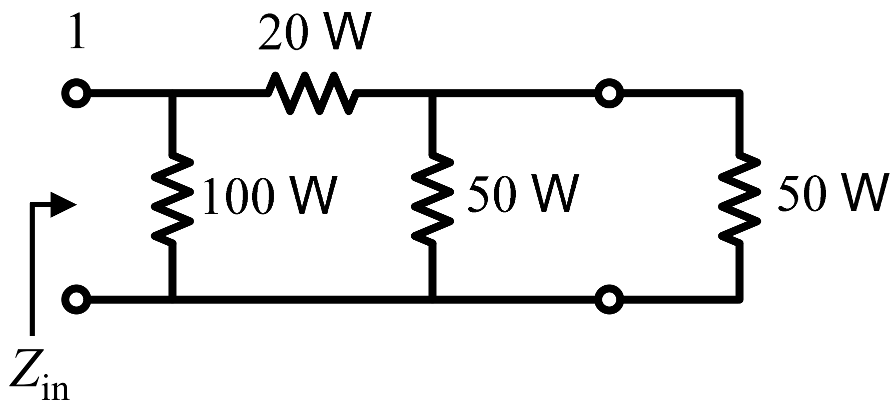
  - Input impedance:

$$
\begin{aligned}
Z_{\text{in}} &= 100 \,||\, [20 + (50 \,||\, 50)] \\
&= 100 \,||\, [20 + 25] \\
&= 100 \,||\, 45 = 31.03
\end{aligned}
$$

  - Reflection coefficient:

$$
S_{11} = \Gamma_{\text{IN}} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0}
$$

$$
= \frac{31.03 - 50}{31.03 + 50} = -0.23 \overset{\text{or}}{=} 0.23\angle\pi
$$

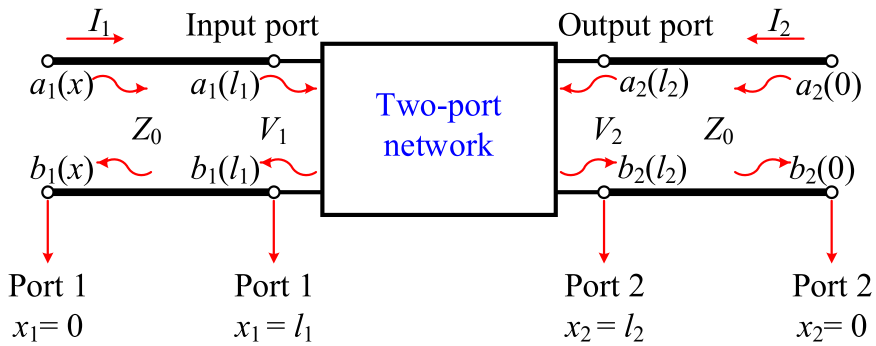


3 *S*-Parameters of 2-port Network

- 2-port network

- *S*-parameters of network in view of incident and reflected voltage waves

$$b_1(l_1) = S_{11}a_1(l_1) + S_{12}a_2(l_2)$$
$$b_2(l_2) = S_{21}a_1(l_1) + S_{22}a_2(l_2)$$



- In matrix form:

$$\begin{bmatrix} b_1(l_1) \\ b_2(l_2) \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} a_1(l_1) \\ a_2(l_2) \end{bmatrix} \Rightarrow [b] = [S][a]$$

where $S_{11}a_1(l_1)$: contribution to reflected wave $b_1(l_1)$ due to incident wave $a_1(l_1)$ at port 1.

$S_{12}a_2(l_2)$: contribution to reflected wave $b_1(l_1)$ due to incident wave $a_2(l_2)$ at port 2,

- *S*-parameter matrix of 2-port network measured at input and output ports:

$$[S] = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

3 *S*-Parameters of 2-port Network

- *S*-parameters are reflection and transmission coefficients.
- *S*-parameter at a specific port of network is measured when the other port is properly terminated or matched.

$$S_{11} = \left.\frac{b_1(l_1)}{a_1(l_1)}\right|_{a_2(l_2)=0}$$: input reflection coefficient with output port properly terminated

$$S_{21} = \left.\frac{b_2(l_2)}{a_1(l_1)}\right|_{a_2(l_2)=0}$$: forward transmission coefficient with output port properly terminated

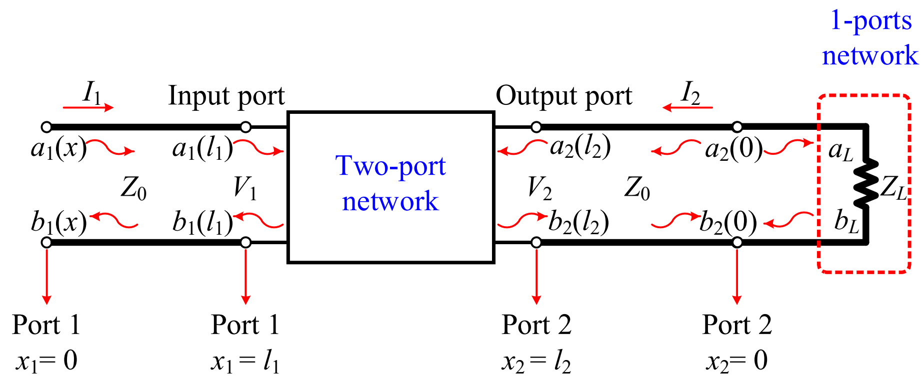
$$S_{22} = \left.\frac{b_2(l_2)}{a_2(l_2)}\right|_{a_1(l_1)=0}$$: output reflection coefficient with input port properly terminated

$$S_{12} = \left.\frac{b_1(l_1)}{a_2(l_2)}\right|_{a_1(l_1)=0}$$: reverse transmission coefficient with input port properly terminated

- What are $a_2(l_2) = 0$ and $a_1(l_1) = 0$?

- 2-port network combined with 1-port load
  - At port 2, $a_L = b_2(0)$ and $b_L = a_2(0)$
  - Reflection condition in cases of $Z_L = Z_0 = Z_s$

$$\Gamma_L = \left.\frac{Z_L - Z_0}{Z_L + Z_0}\right|_{Z_L = Z_0} = 0 = \frac{b_L}{a_L} = \frac{a_2(0)}{b_2(0)} \rightarrow b_L = a_2(0) = a_2(l_2) = 0$$



- *Source and load matched conditions guarantee* $a_2(l_2) = 0$ *and* $a_1(l_1) = 0$!!!
- *Calibration of network analyzer*

3 Parameters Conversion and Examples

- **Relation between [*S*] and [*Z*]**

$$V_1(x_1) = \sqrt{Z_0}v_1(x_1), \qquad V_2(x_2) = \sqrt{Z_0}v_2(x_2)$$

$$I_1(x_1) = i_1(x_1)/\sqrt{Z_0}, \qquad I_2(x_2) = i_2(x_2)/\sqrt{Z_0}$$

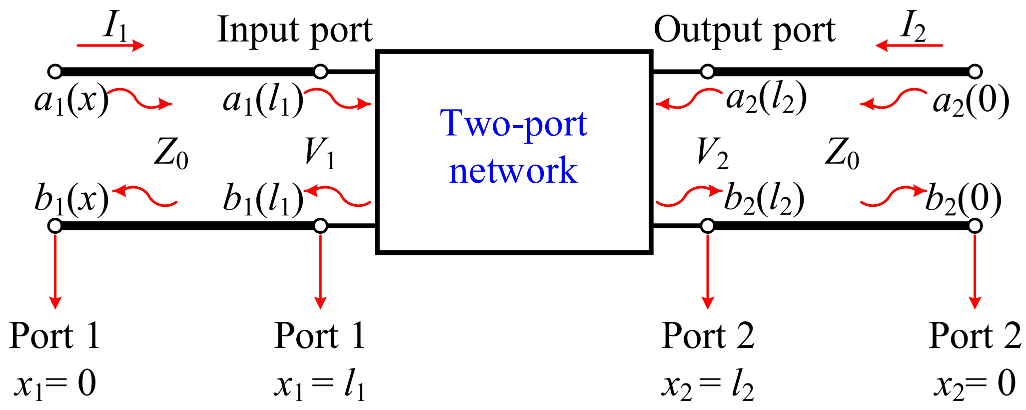
$$\begin{bmatrix} V_1(x_1) \\ V_2(x_2) \end{bmatrix} = \begin{bmatrix} \sqrt{Z_0}v_1(x_1) \\ \sqrt{Z_0}v_2(x_2) \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I_1(x_1) \\ I_2(x_2) \end{bmatrix}$$

$$= \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} i_1(x_1)/\sqrt{Z_0} \\ i_2(x_2)/\sqrt{Z_0} \end{bmatrix}$$

$$\Rightarrow \sqrt{Z_0} \begin{bmatrix} v_1(x_1) \\ v_2(x_2) \end{bmatrix} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} i_1(x_1) \\ i_2(x_2) \end{bmatrix} / \sqrt{Z_0}$$

$$\Leftrightarrow Z_0[v(x)] = [Z][i(x)]$$

$$\Leftrightarrow [\zeta] = \frac{[Z]}{Z_0}$$: normalized impedance matrix



- $[v(x)] = [a(x)] + [b(x)] = [a(x)] + [S][a(x)]$
  $= ([U] + [S])[a(x)]$
- $[i(x)] = [a(x)] - [b(x)] = [a(x)] - [S][a(x)]$
  $= ([U] - [S])[a(x)]$

3 Parameters Conversion and Examples

- Since $[v(x)] = [\zeta][i(x)]$, the following equation can be obtained.

$$[a(x)] + [b(x)] = [\zeta]([a(x)] - [b(x)])$$

$$([\zeta] + [U])[b(x)] = ([\zeta] - [U])[a(x)]$$

$$[b(x)] = ([\zeta] + [U])^{-1}([\zeta] - [U])[a(x)] = [S][a(x)]$$

$$\therefore [S] = ([\zeta] + [U])^{-1}([\zeta] - [U]) \rightarrow [S] = f([\zeta])$$

or

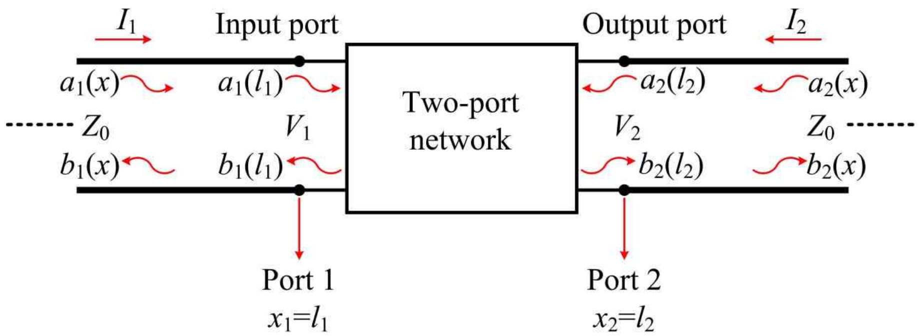
$$[v(x)] = [\zeta][i(x)]$$

$$[a(x)] + [b(x)] = [\zeta]([a(x)] - [b(x)])$$

$$([U] + [S])[a(x)] = [\zeta]([U] - [S])[a(x)]$$

$$\therefore [\zeta] = ([U] + [S])([U] - [S])^{-1}$$

$$\rightarrow [\zeta] = g([S])$$



○ For $[b(x)] = [S][a(x)]$, $[v(x)] = [\zeta][i(x)]$,

$[b(x)] = ([v] - [i(x)])/2$, and

$[a(x)] = ([v(x)] + [i(x)])/2$

$\rightarrow ([\zeta] - [U])[i(x)] = [S]([\zeta] + [U])[i(x)]$

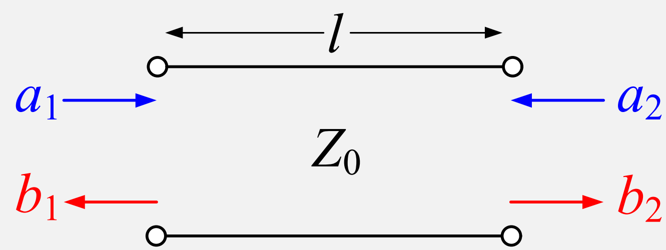
$\therefore [S] = ([\zeta] - [U])([\zeta] + [U])^{-1}$

or $[\zeta] = ([U] - [S])^{-1}([U] + [S])$

3 Parameters Conversion and Examples

- **Example 2:** Determine [Z] of transmission line from [S] of transmission line

Solution:



$$[S]=\begin{bmatrix}0 & e^{-j\beta l}\\ e^{-j\beta l} & 0\end{bmatrix}$$

$$[\zeta]=\left([U]+[S]\right)\left([U]-[S]\right)^{-1}$$

$$=\begin{bmatrix}1 & e^{-j\beta l}\\ e^{-j\beta l} & 1\end{bmatrix}\begin{bmatrix}1 & -e^{-j\beta l}\\ -e^{-j\beta l} & 1\end{bmatrix}^{-1}$$

$$=\frac{1}{1-e^{-j2\beta l}}\begin{bmatrix}1 & e^{-j\beta l}\\ e^{-j\beta l} & 1\end{bmatrix}\begin{bmatrix}1 & e^{-j\beta l}\\ e^{-j\beta l} & 1\end{bmatrix}=\begin{bmatrix}\dfrac{1+e^{-j2\beta l}}{1-e^{-j2\beta 1}} & \dfrac{2e^{-j\beta l}}{1-e^{-j2\beta 1}}\\ \dfrac{2e^{-j\beta l}}{1-e^{-j2\beta l}} & \dfrac{1+e^{-j2\beta l}}{1-e^{-j2\beta l}}\end{bmatrix}$$

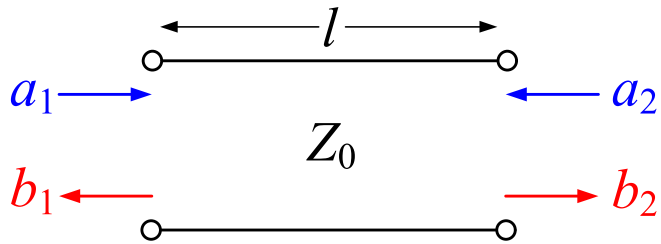
$$=\begin{bmatrix}-j\dfrac{\left(e^{j\beta l}+e^{-j\beta l}\right)/2}{\left(e^{j\beta l}-e^{-j\beta l}\right)/2j} & -j\dfrac{1}{\left(e^{j\beta l}-e^{-j\beta l}\right)/2j}\\ -j\dfrac{1}{\left(e^{j\beta l}-e^{-j\beta l}\right)/2j} & -j\dfrac{\left(e^{j\beta l}+e^{-j\beta l}\right)/2}{\left(e^{j\beta l}-e^{-j\beta l}\right)/2j}\end{bmatrix}=\begin{bmatrix}-j\cot\beta l & -j\csc\beta l\\ -j\csc\beta l & -j\cot\beta l\end{bmatrix}\Rightarrow[Z]=\begin{bmatrix}-jZ_0\cot\beta l & -jZ_0\csc\beta l\\ -jZ_0\csc\beta l & -jZ_0\cot\beta l\end{bmatrix}$$

3 Parameters Conversion and Examples

- **Example 3:** Determine the Z-parameters of transmission line with open- and short-circuited conditions.

Solution:

$$\begin{cases} V_1 = Z_{11}I_1 + Z_{12}I_2 \\ V_2 = Z_{12}I_1 + Z_{11}I_2 \end{cases} \quad \text{@symmetric network}$$



- Output open : $I_2 = 0,\ Z_{11} = \dfrac{V_1}{I_1} = Z_0 \dfrac{Z_L + jZ_0 \tan\theta}{Z_0 + jZ_L \tan\theta}$

$$= -jZ_0 \cot \beta l$$

- Output short : $V_2 = 0, 0 = Z_{12}I_1 + Z_{11}I_2 \Rightarrow I_2 = -\dfrac{Z_{12}}{Z_{11}} I_1$

$$V_1 = Z_{11}I_1 + Z_{12}I_2 = \left(Z_{11} - \frac{Z_{12}^2}{Z_{11}} \right) I_1$$

$$Z_{in} = \frac{V_1}{I_1} = Z_{11} - \frac{Z_{12}^2}{Z_{11}} = Z_0 \frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l}$$

$$= jZ_0 \tan \beta l \quad \text{: short-circuited}$$

$$Z_{12}^2 = Z_{11}^2 - Z_{11}\left(jZ_0 \tan \beta l \right) = -Z_0^2 \cot^2 \beta l - Z_0^2$$

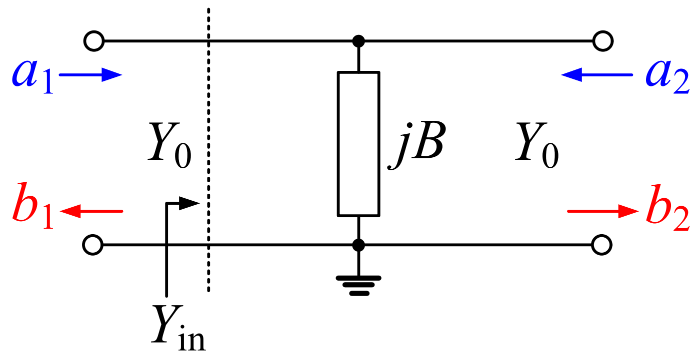
$$= -Z_0^2 \csc^2 \beta l$$

$$\therefore \quad Z_{12} = \pm jZ_0 \csc \beta l$$

3 Parameters Conversion and Examples

- **Example 4:** Determine the S-parameter of the given network.

Solution:



- $\begin{cases} b_1 = S_{11}a_1 + S_{12}a_2 \\ b_2 = S_{21}a_1 + S_{22}a_2 \end{cases}$

- $Y_{\text{in}} = Y_0 + jB$ (In case port 2 is matched)

- $S_{11} = \left.\dfrac{b_1}{a_1}\right|_{a_2=0}$ ← Reflection coefficient

$$= \frac{Y_0 - Y_{\text{in}}}{Y_0 + Y_{\text{in}}} = \frac{-jB}{2Y_0 + jB}$$

- $S_{21} = 1 + S_{11} = \dfrac{2Y_0}{2Y_0 + jB}$

- From symmetry, $S_{11} = S_{22}$ and $S_{12} = S_{21}$.

- **Example 5:** Determine the S-parameter of the given network.

  Solution:

  - reciprocal network.

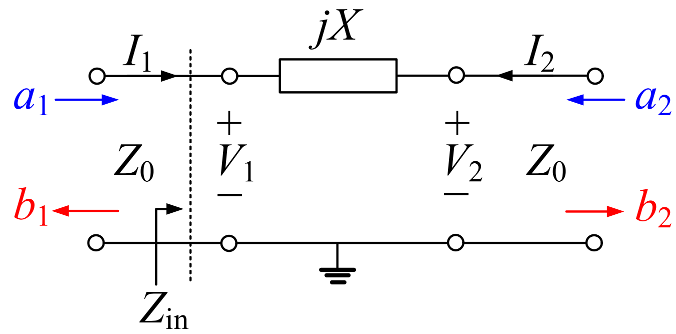
  $Z_{\text{in}} = Z_0 + jX$ (In case port 2 is matched)

  - Use normalized voltage:

$$\begin{cases} a_1 = \overline{V_1^+} = V_1^+ / \sqrt{Z_c} \\ a_2 = \overline{V_2^+} = V_2^+ / \sqrt{Z_c} \\ b_1 = \overline{V_1^-} = V_1^- / \sqrt{Z_c} \\ b_2 = \overline{V_2^-} = V_2^- / \sqrt{Z_c} \end{cases}$$

  - Overall input voltage and current:

$$\begin{cases} V_1 = V_1^+ + V_1^- = V_1^+\left(1 + \dfrac{V_1^-}{V_1^+}\right) = V_1^+\left(1 + S_{11}\right) \\ I_1 = Y_0\left(V_1^+ - V_1^-\right) = Y_0 V_1^+\left(1 - \dfrac{V_1^-}{V_1^+}\right) = Y_0 V_1^+\left(1 - S_{11}\right) \end{cases}$$



- Apply KCL for incidence from port 1:

$$-I_2 = I_2^- = I_1$$

$$= Y_0 V_1^+\left(1 - S_{11}\right)$$

$$\Rightarrow\ I_2^- = Y_0 V_2^- = Y_0 V_1^+ S_{21} = Y_0 V_1^+\left(1 - S_{11}\right)$$

$$S_{11} = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \frac{jX}{2Z_0 + jX}$$

$$S_{21} = \frac{b_2}{a_1} = \frac{\overline{V_2^-}}{\overline{V_1^+}} = \frac{V_2^- / \sqrt{Z_c}}{V_1^+ / \sqrt{Z_c}} = \frac{V_2^-}{V_1^+}$$

$$= 1 - S_{11} = \frac{2Z_0}{2Z_0 + jX}$$

4 Review

- Why S-parameter?
- S-parameter of 1-port network
- S-parameter of 2-port network
- Relation between $[S]$ and $[Z]$