Chapter 5
Impedance Matching

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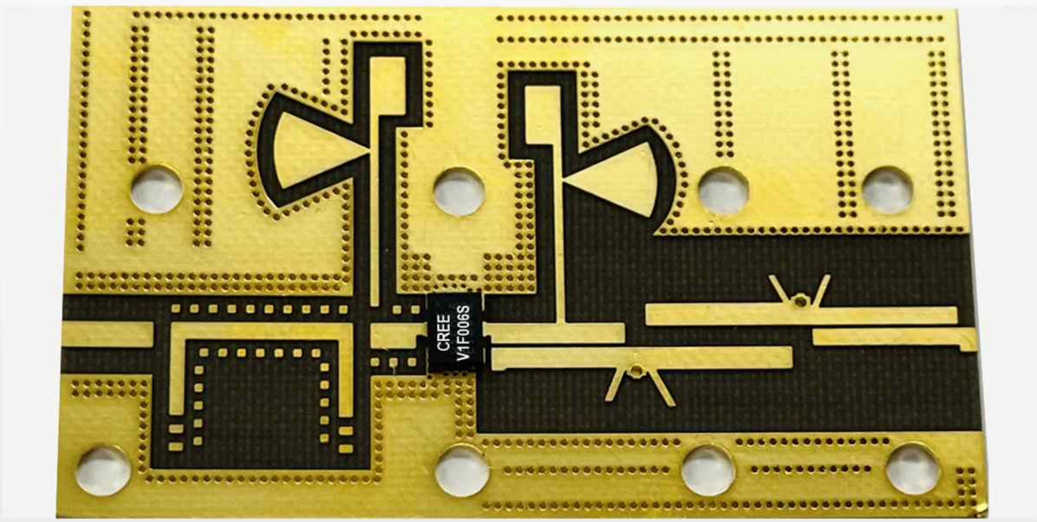


Learning Objectives

- Understanding what is theory of small reflection
- Learn about impedance transformation using single-section
- Learn about impedance transformation using multi-Section

Learning contents

- Introduction about Theory of Small Reflections
- Single-Section Transformer
- Multi-Section Transformer

1 Introduction about Theory of Small Reflection

- An important and useful approximation when considering multi-section matching networks is the '**Theory of Small Reflections'**.
- Theory of small reflections provides a simple mathematical form for analyzing the frequency response of many microwave circuits. → Total reflection coefficient can be obtained from the partial reflections from several small discontinuities.
- For applications requiring **more bandwidth** than single quarter-wave section, **multi-section transformers can be used**.
- Applications:

  - ***Engineering and Telecommunications***: improving the performances on transmission lines, circuits, and antennas by reducing signal reflections.
  - ***Optics***: designing anti-reflective coatings for lenses, glasses, and screens to reduce glare and increase transparency.
  - ***Acoustics***: developing materials and structures for soundproofing and acoustic treatment to minimize reflected sound waves and enhance sound quality.

2 Single-Section Transformer

- Derivation an approximate expression for overall reflection coefficient Γ

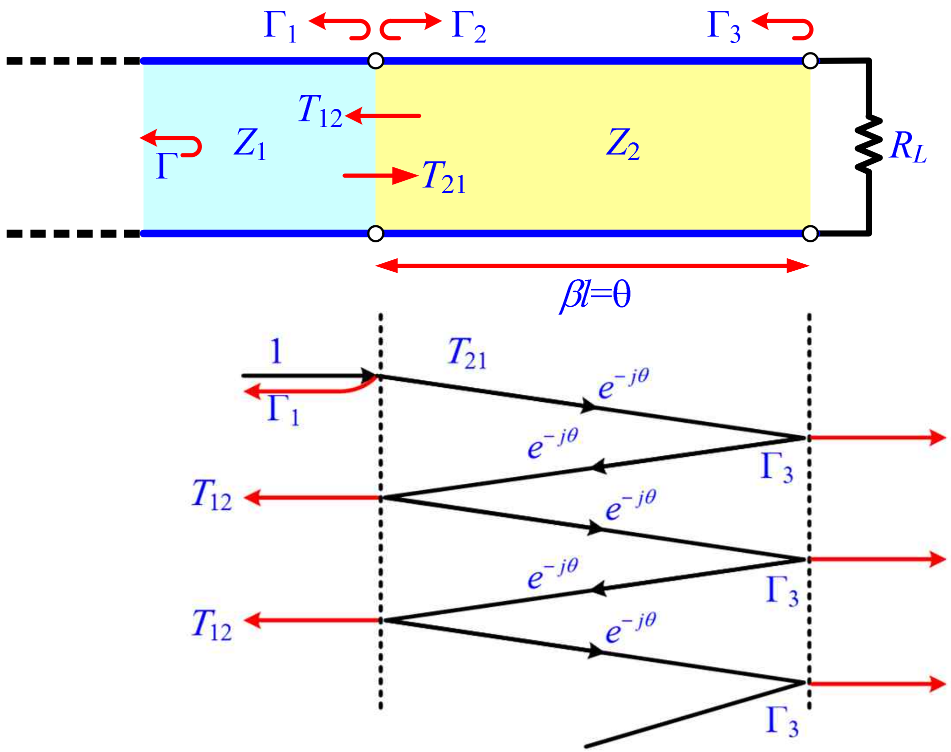
$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1}$$

$$\Gamma_2 = -\Gamma_1$$

$$\Gamma_3 = \frac{R_L - Z_2}{R_L + Z_2}$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2}$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2}$$



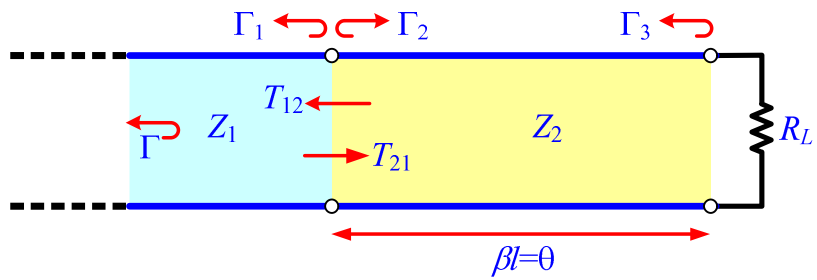
Partial reflections and transmissions on a single-section matching transformer.

2 Single-Section Transformer

- Expression of total reflection as an infinite sum of partial reflections and transmissions:

$$\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-j4\theta} + \ldots$$

$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-j2\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-j2n\theta}$$

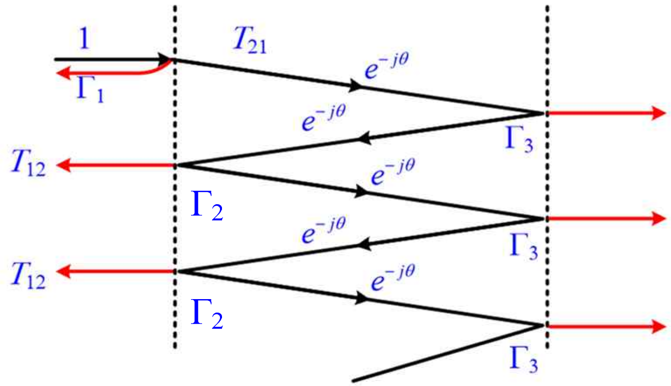


- Using the geometric series and small reflection condition,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \text{for} \left|x\right| < 1$$

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-j2\theta}}{1-\Gamma_2\Gamma_3 e^{-j2\theta}} \leftarrow \ \Gamma_2 = -\Gamma_1,\ T_{21} = 1+\Gamma_1,\ T_{12} = 1+\Gamma_2 = 1-\Gamma_1$$

$$= \frac{\Gamma_1 - \Gamma_1\Gamma_2\Gamma_3 e^{-j2\theta} + (1-\Gamma_1)(1+\Gamma_1)\Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}}$$



2 Single-Section Transformer

- (continue)

$$= \frac{\Gamma_1 - \Gamma_1\Gamma_2\Gamma_3 e^{-j2\theta} + (1-\Gamma_1^2)\Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}}$$

$$\cong \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}} \qquad (1)$$

$\Gamma_1\Gamma_2\Gamma_3 e^{-2j\theta} \approx 0$ because Γ_1, Γ_2, Γ_3 are small

$(1-\Gamma_1^2) \approx 1$ because $\Gamma_1^2 \approx 0$

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-j2\theta} \qquad (2)$$

$1+\Gamma_1\Gamma_3 e^{-2j\theta} \approx 1$ because $\Gamma_1\Gamma_3 \approx 0$

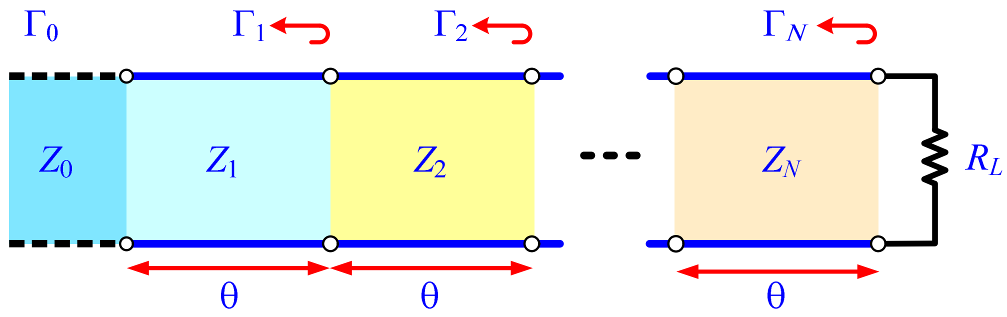
- Intuitive ideas:

1) The total reflection is dominated by the reflection from the initial discontinuity between Z_1 and Z_2 (**Γ_1**), and the first reflection from the discontinuity between Z_2 and Z_L (**Γ_3**).

2) The $e^{-j2\theta}$ term accounts for the round-trip phase delay when the incident wave travels up and down the line.

3 Multi-Section Transformer

- Consider a sequence of N transmission line sections and each section has **equal length** θ, but **dissimilar characteristic impedances**:



Partial reflection coefficients for a multisection matching transformer

- Derivation an approximate expression for the total reflection coefficient Γ.

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \cdots, \Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n} \; (@1 < n < N), \cdots, \Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}$$

- Assumptions:

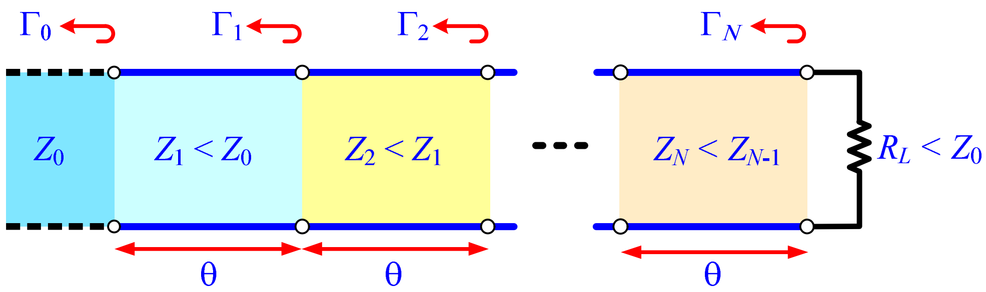
1) All Z_n increase or decrease monotonically across the transformer.

2) R_L is real.

3 Multi-Section Transformer

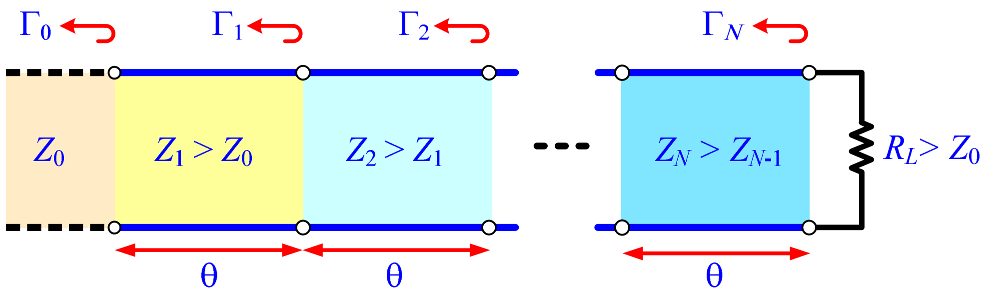
- If $R_L < Z_0$,

$$Z_0 > Z_1 > Z_2 \ldots > Z_N > R_L \qquad \Rightarrow \qquad \Gamma_n < 0$$



- If $R_L > Z_0$,

$$Z_0 < Z_1 < Z_2 \ldots < Z_N < R_L \qquad \Rightarrow \qquad \Gamma_n > 0$$



3 Multi-Section Transformer

- Since R_L is real and lossless transmission lines are assumed, all Γ_n will be real.

  - Due to gradually impedance transition from one section to another, '$Z_{n+1} - Z_n$' will be **small.**

    $\Rightarrow$ Each marginal reflection coefficient Γ_n will be **real** and have a **small** magnitude.

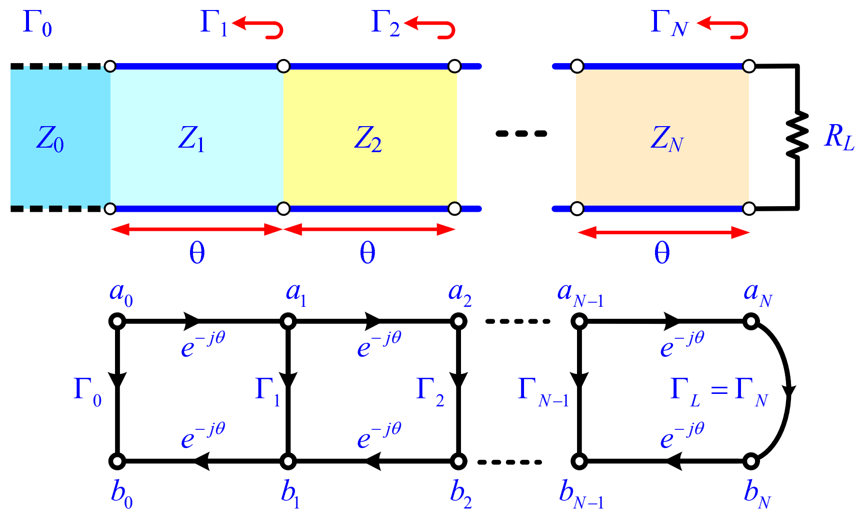
    $\Rightarrow$ **Theory of Small Reflections**

  - Input reflection coefficient of multi-section impedance transformer

$$\frac{b_0}{a_0} = \Gamma(\theta)$$

$$\Gamma_{in}(\theta) \cong \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-j2N\theta} \quad (3)$$

$$= \sum_{n=0}^{N} \Gamma_n e^{-j2N\theta}$$



3 Multi-Section Transformer

- To simplify this process, transformer is **symmetrical**,

$$\Gamma_0 = \Gamma_N,\ \Gamma_1 = \Gamma_{N-1},\ \Gamma_2 = \Gamma_{N-2},\ \cdots.$$

$\rightarrow$ This does not mean that $Z_0 = Z_N$, $Z_1 = Z_{N-1}$, $Z_2 = Z_{N-2}, \cdots$

- (3) can be re-written as:

$$\Gamma_{in}(\theta) \cong \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \cdots + \Gamma_N e^{-j2N\theta} \qquad (2)$$

$$= \left(\Gamma_0 + \Gamma_N e^{-j2N\theta}\right) + \left(\Gamma_1 e^{-j2\theta} + \Gamma_{N-1} e^{-j2(N-1)\theta}\right) + \left(\Gamma_2 e^{-j4\theta} + \Gamma_{N-2} e^{-j2(N-2)\theta}\right) + \cdots +$$

$$= e^{-jN\theta}\left\{\Gamma_0[e^{jN\theta} + e^{-jN\theta}] + \Gamma_1[e^{j(N-2)} + e^{-j(N-2)\theta}] + \cdots\right\}$$

$$= 2e^{-jN\theta}\left\{\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \cdots\right\} \leftarrow e^{jx} + e^{-jx} = 2\cos(x)$$

$\Gamma_0 = \Gamma_N,\ \Gamma_1 = \Gamma_{N-1},\ \Gamma_2 = \Gamma_{N-2},$ etc.

- For **odd** ***N***, the last term is $\Gamma_{(N-1)/2}(e^{j\theta} + e^{-j\theta})$ as

$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots + \Gamma_n \cos(N-2n)\theta + \cdots + \Gamma_{(N-1)/2} \cos\theta]$$

3 Multi-Section Transformer

- For **even *N***, the last term is $\Gamma_{N/2}$ as

$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots + \Gamma_n \cos(N-2n)\theta + \cdots + \frac{1}{2}\Gamma_{N/2}]$$

➔ ***Above result imply that the desired reflection coefficient responds a function of θ by using enough sections N and properly choosing Γ_n.***

- ***Questions***

- How to determine the necessary number of sections N ?

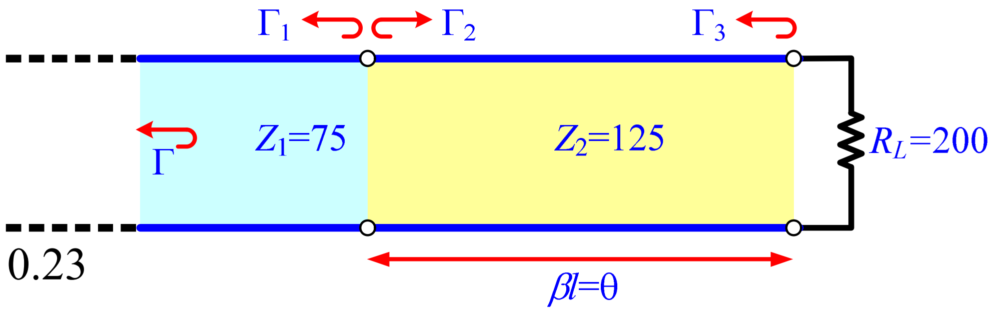
- How do determine the values of all reflection coefficients Γ_n ?

Answer: Answers will be clear if the design multi-section transformers for a small reflection is designed with two of the most commonly used passband responses:

- Binomial (maximally flat) response
- Chebyshev (equal-ripple) response.

4 Example

- Ex.] Consider the 2-stage quarter-wave transformer of below figure with Z_1 = 75 Ω, Z_2 = 125 Ω, and R_L =200 Ω. Evaluate the worst-case percent error in computing |Γ| from the approximate small reflection theory and the almost exact result.



Solution

- Reflection coefficient between transmission lines

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} = \frac{125-75}{125+75} = 0.25,\ \Gamma_3 = \frac{R_L - Z_2}{R_L + Z_2} = \frac{200-125}{200+125} = 0.23$$

- Compare Γ the approximate and exact expressions

$$\Gamma = \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}} \cong \Gamma_1 + \Gamma_3 e^{-j2\theta}$$

→ The greatest error will occur depend on denominator.
The worst error will be occurred in conditions of $\theta = 0^{o}$ or $\theta = 180^{o}$.

- Almost exact Γ

$$\Gamma \cong \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1+\Gamma_1\Gamma_3 e^{-j2\theta}} = \frac{0.25+0.23e^{j0}}{1+0.25\times 0.23e^{j0}} = 0.454$$

- Γ using small reflection theory

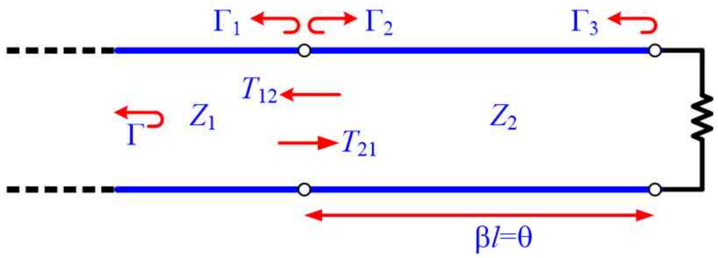
$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-j2\theta} = 0.25 + 0.23e^{j0} = 0.48$$

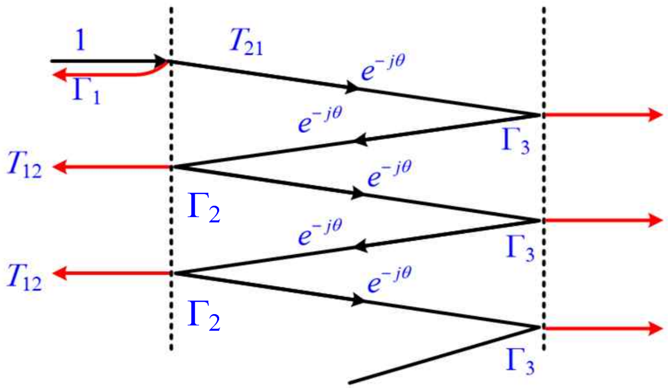
- Thus, the worst-case percent error is about 0.026 or 5.7% .

5 Review

- Single-section transformer

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-2j\theta}$$





- Multi-section transformer

$$\Gamma_{in}(\theta) \cong \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_N e^{-2jN\theta}$$