Chapter 6
Microwave Resonators

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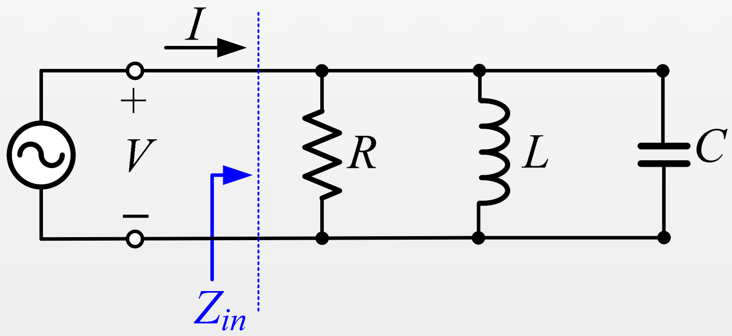
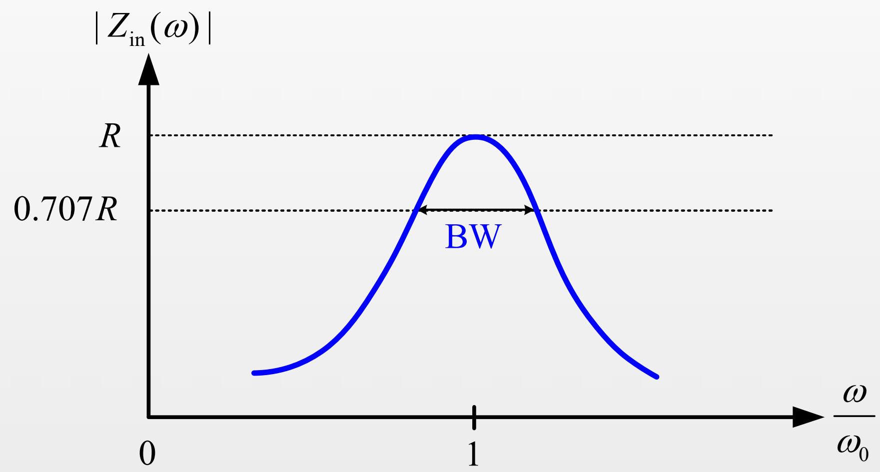




Learning Objectives

- Understanding overview of resonators
- Understanding parallel LC resonant circuit
- Understanding series LC resonant circuit

Learning contents

- Introduction to Microwave Resonators
- Parallel LC Resonant Circuit
- Series LC Resonant Circuit

1 Introduction To Microwave Resonators

- Microwave resonators are used in a variety of applications such as:

  - Filters

  - Oscillators

  - Frequency meters

  - Tuned amplifiers

  - etc.

- The operation of microwave resonators has a similar circuit theory to that of the lumped-element resonators, thus the basic of series and parallel RLC resonant circuits will be reviewed firstly and shortly.

- Types of resonator:

  - Lumped resonator or RLC resonator → Used for low frequency

  - Transmission line resonator → Used for microwave frequency

  - Cavity resonator → Used for RF frequency and high power handling application dye to high quality factor.

2 Parallel LC Resonant Circuit

- As its name, a parallel *LC* resonant circuit (or *LC*-tank) is an electrical circuit consisting of an inductor (*L*) and a capacitor (*C*) connected in parallel, which maintain the same voltage across the components.

- Input impedance

$$Z_{\text{in}} = \left(\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right)^{-1}$$

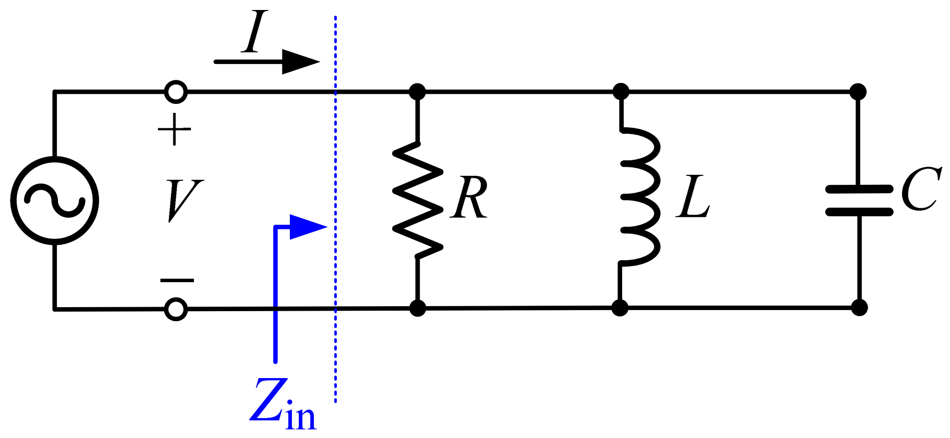
- Complex power delivered to resonator

$$P_{\text{in}} = \frac{1}{2}VI^* = \frac{1}{2}Z_{\text{in}}\,|I|^2 = \frac{1}{2}|V|^2\,\frac{1}{Z_{\text{in}}^*} = \frac{1}{2}|V|^2\,Y_{\text{in}}^*$$

$$= \frac{1}{2}|V|^2\left(\frac{1}{R} - \frac{1}{j\omega L} - j\omega C\right)$$

$$= \frac{1}{2}\frac{|V|^2}{R} + j2\omega\left(\frac{1}{4}\frac{|V|^2}{\omega^2 L} - \frac{1}{4}|V|^2\,C\right)$$

$$= P_l + j2\omega\left(W_m - W_e\right)$$



2 Parallel LC Resonant Circuit

- Power dissipated by resistor R

$$P_l = \frac{1}{2}\frac{|V|^2}{R} = \frac{1}{2}G|V|^2$$

- Average electric energy stored in capacitor C

$$W_e = \frac{1}{4}|V|^2 C = \frac{1}{4}VV^* C$$

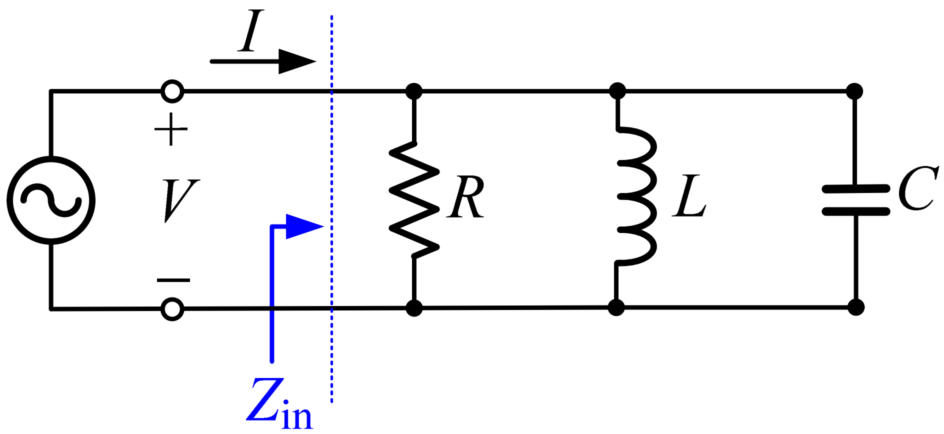
- Average magnetic energy stored in inductor L

$$W_m = \frac{1}{4}|I_L|^2 L = \frac{1}{4}I_L I_L^* L = \frac{1}{4}\left|\frac{V}{\omega L}\right|^2 L = \frac{1}{4\omega^2 L}VV^*$$

- Input impedance

$$P_{\text{in}} = \frac{1}{2}VI^* = \frac{1}{2}Z_{\text{in}}|I|^2$$

$$\rightarrow Z_{\text{in}} = \frac{P_{\text{in}}}{\frac{1}{2}|I|^2} = \frac{P_l + j2\omega\left(W_m - W_e\right)}{\frac{1}{2}|I|^2}$$

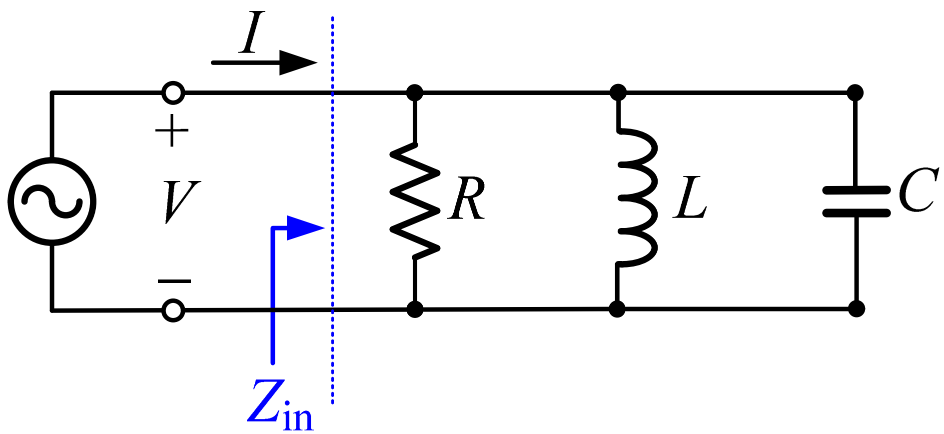


2 Parallel LC Resonant Circuit

- If $W_e = W_m$

$$\frac{1}{4}VV^*C = \frac{1}{4\omega_0^2 L}VV^* \rightarrow C = \frac{1}{\omega_0^2 L}$$

$$\therefore \ \omega = \omega_0 = \sqrt{\frac{1}{LC}}$$



- Quality factor (Q-factor)

$$Q = \omega_0 \frac{\text{average stored energy}}{\text{dissipated power loss}}$$

- Stored time average energy

$$W = W_m + W_e = 2W_m = 2W_e = \frac{1}{2}CVV^*$$

- Dissipated power: $P_l = \frac{1}{2}GVV^*$

$$\Rightarrow \ Q = \omega_0 \frac{\frac{1}{2}CVV^*}{\frac{1}{2}GVV^*} = \frac{\omega_0 C}{G} = \omega_0 RC = \frac{R}{\omega_0 L} \quad \text{(in parallel RLC network)}$$

2 Parallel LC Resonant Circuit

- Impedance at around resonate frequency (@ $\omega = \omega_0 \pm \Delta\omega$)

$$Z_{in} = \left[\frac{1}{R} + \frac{1}{j\omega L} + j\omega C\right]^{-1} = \frac{1}{\frac{1}{R} + \frac{1}{j\omega L} + j\omega C}$$

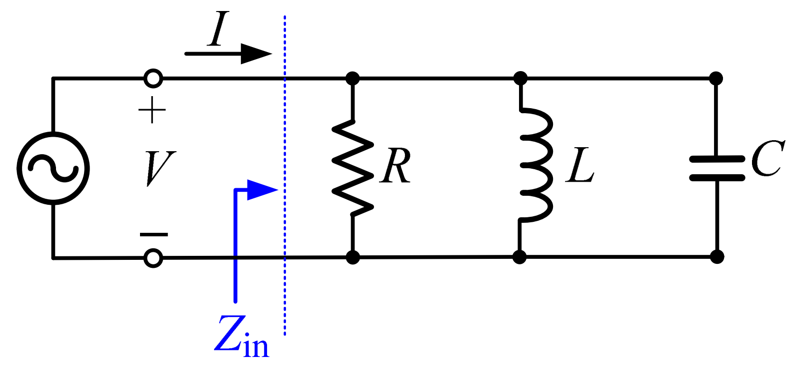
$$= \frac{R}{1 + j\omega_0 RC\left(\frac{\omega}{\omega_0} - \frac{1}{\omega_0 \omega LC}\right)} = \frac{R}{1 + jQ\left(\frac{\omega}{\omega_0} - \frac{\omega_0}{\omega}\right)}$$

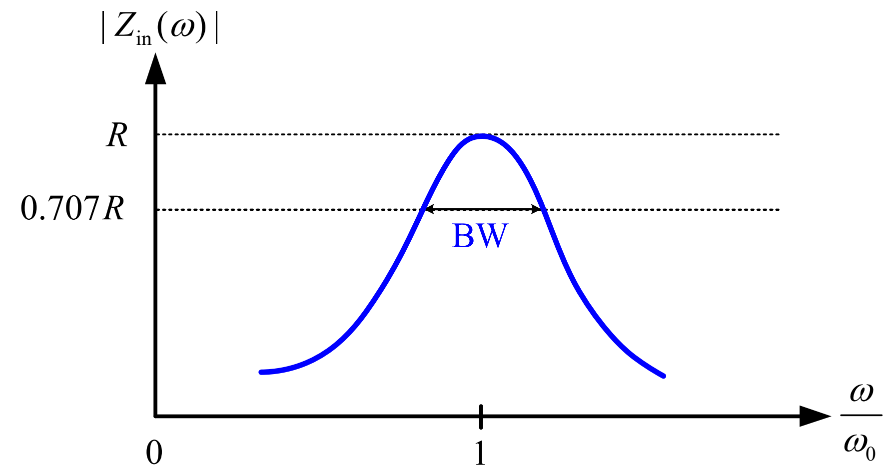
$$= \frac{R}{1 + jQ\frac{(\omega + \omega_0)(\omega - \omega_0)}{\omega\omega_0}} \approx \frac{R}{1 \pm j2Q\frac{\Delta\omega}{\omega_0}}$$

- $\left|Z_{in}\right|_{max} = R \quad @\ \omega = \omega_0$

- If $2Q\frac{\Delta\omega}{\omega_0} = 1,\ Q = \frac{\omega_0}{2\Delta\omega}$ and $\left|Z_{in}\right| = \frac{1}{\sqrt{2}}\left|Z_{in}\right|_{max}$

  where $2\Delta\omega$: bandwidth, $2\Delta\omega / \omega_0$: fractional bandwidth (FBW)





2 Parallel LC Resonant Circuit

- Quality factor (Q-factor, unloaded Q)

$$Q = \frac{\omega_0}{2\Delta\omega} = \frac{1}{\text{FBW}}$$

- Q including load or source resistance: Q_L (loaded Q)

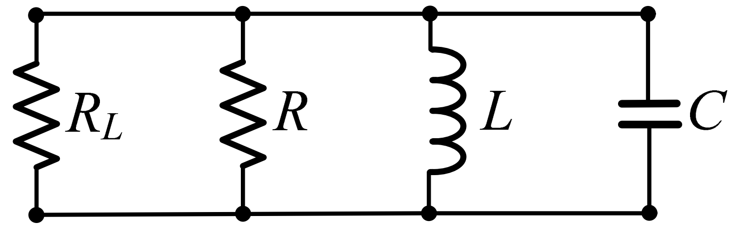
$$Q_L = \frac{R//R_L}{\omega_0 L} = \frac{RR_L/(R+R_L)}{\omega_0 L}$$

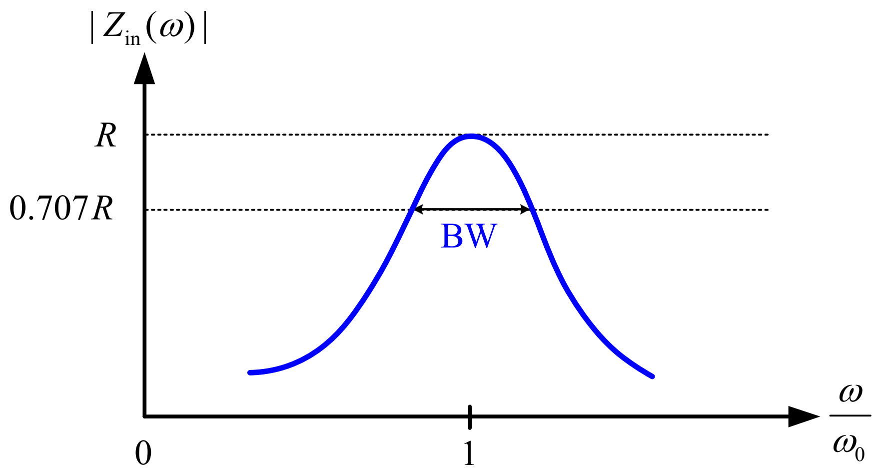
- If $R = \infty$, the external Q can be calculated as

$$Q_{ex} = \frac{R_L}{\omega_0 L}$$

- If $R_L = \infty$, the internal Q or unloaded Q is

$$Q_{\text{in}} = Q_u = \frac{R}{\omega_0 L}$$





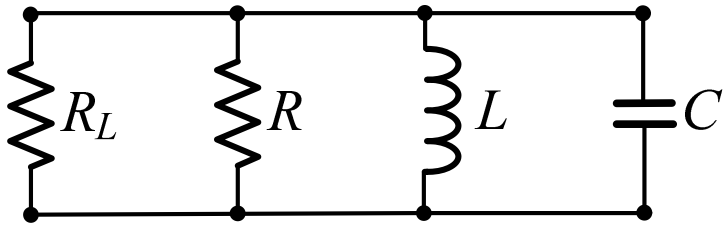
2 Parallel LC Resonant Circuit

- General relationship among Q-factors

$$Q_L = \frac{R//R_L}{\omega_0 L} = \frac{RR_L/(R+R_L)}{\omega_0 L}$$

$$\frac{1}{Q_L} = \frac{\omega_0 L}{\frac{RR_L}{R+R_L}} = \omega_0 L \frac{R+R_L}{RR_L} = \omega_0 L\left(\frac{1}{R}+\frac{1}{R_L}\right)$$

$$= \frac{\omega_0 L}{R} + \frac{\omega_0 L}{R_L} = \frac{1}{Q_u} + \frac{1}{Q_{ex}} \Rightarrow Q_L < Q_u \ \& \ Q_{ex}$$



- **Energy damping factor** (δ) : Decaying ratio of oscillation amplitude in case of drop of external energy

$$W = W_0 e^{-2\delta t}$$

$$-\frac{dW}{dt} = P_l = 2\delta W$$

$$\delta = \frac{P_l}{2W} = \frac{\omega_0}{2}\frac{P_l}{\omega_0 W} = \frac{\omega_0}{2Q}$$

$$W = W_0 e^{-\omega_0 t/Q} \quad \rightarrow \textit{Higher Q - factor guarantees longer energy convservation!}$$

3 Series LC Resonant Circuit

- In series LC circuit, the components share the same current but have different voltage across each, showing voltage summation.

- Input impedance

$$Z_{in} = R + j\omega L - j\frac{1}{\omega C}$$

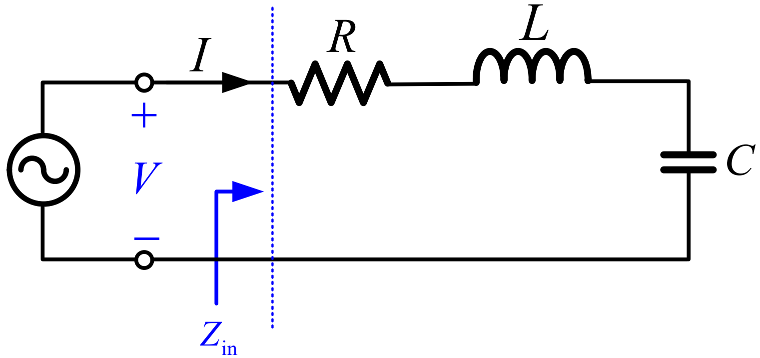
- Complex power delivered to the resonator

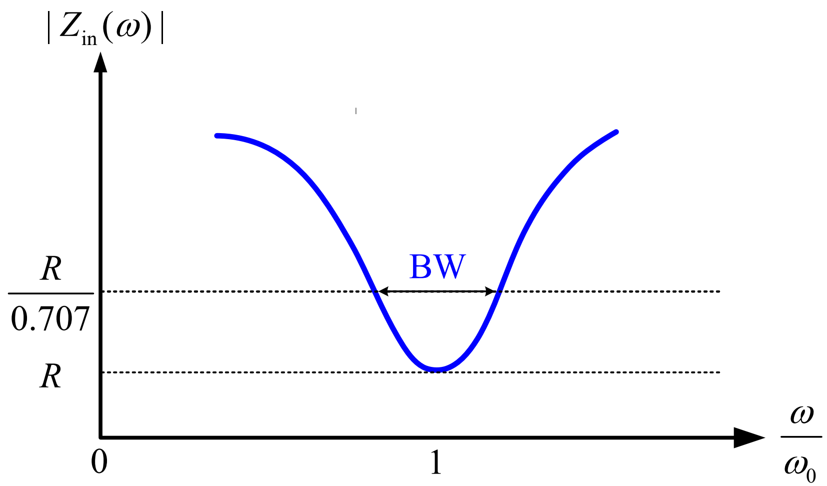
$$P_{in} = \frac{1}{2}VI^* = \frac{1}{2}Z_{in}|I|^2 = \frac{1}{2}|V|^2\frac{1}{Z_{in}^*}$$

$$= \frac{1}{2}|I|^2\left(R + j\omega L - j\frac{1}{\omega C}\right)$$

$$= \frac{1}{2}|I|^2 R + j2\omega\left(\frac{1}{4}|I|^2 L - \frac{1}{4}\frac{|I|^2}{\omega^2 C}\right)$$

$$= P_l + j2\omega\left(W_m - W_e\right)$$





3 Series LC Resonant Circuit

- Power dissipated by resistor (R)

$$P_l = \frac{1}{2}|I|^2 R$$

- Average electric energy stored in capacitor (C)

$$W_e = \frac{1}{4}|V_C|^2 C = \frac{1}{4}|I|^2 \frac{1}{\omega^2 C}$$

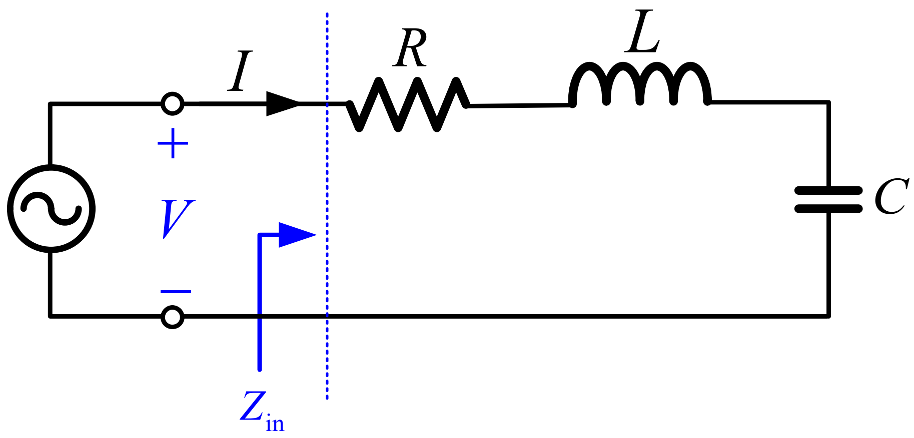
where V_C: voltage across capacitor

- Average magnetic energy stored in the inductor (L)

$$W_m = \frac{1}{4}|I|^2 L$$

- Input impedance

$$Z_{\text{in}} = \frac{P_{\text{in}}}{\frac{1}{2}|I|^2} = \frac{P_l + 2j\omega(W_m - W_e)}{\frac{1}{2}|I|^2}$$



3 Lumped Resonant Circuit: LC Series Resonant Circuit

- At the resonant frequency, $W_m = W_e$

$$\omega = \omega_0 = \frac{1}{\sqrt{LC}}$$

- Quality factor (Q-factor)

$$Q = \omega_0 \frac{\text{average stored energy}}{\text{dissipated power loss}} = \omega_0 \frac{W_m + W_e}{P_l}$$

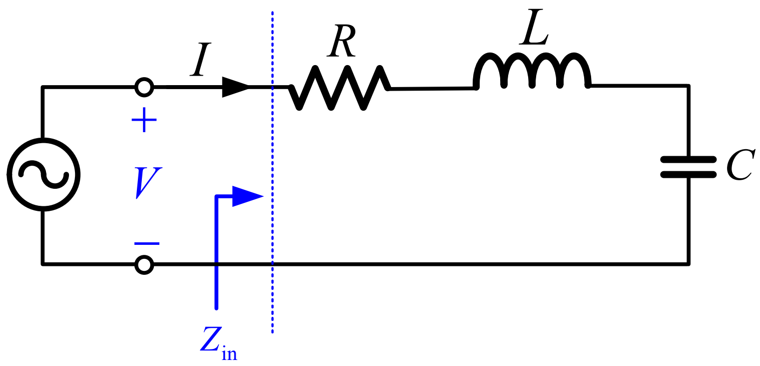
- Unloaded Q can be evaluated at resonance: $W_m = W_e$

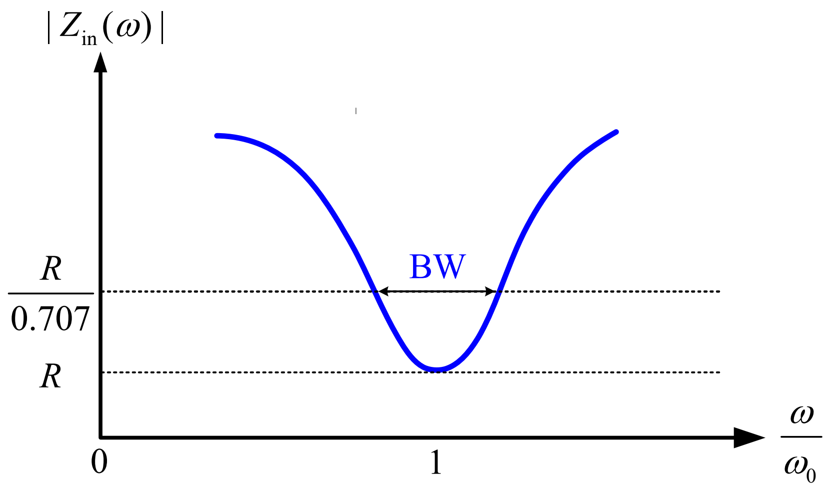
$$Q_u = \omega_0 \frac{2W_m}{P_l} = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$

→ Q is increased as R decreases

- Impedance near the resonate frequency: $\omega = \omega_0 \pm \Delta\omega$

$$Z_{\text{in}} = R + j\omega_0 L\left(\frac{\omega}{\omega_0} - \frac{1}{\omega\omega_0 LC}\right) = R + j\omega_0 L\left(\frac{\omega^2 - \omega_0^2}{\omega_0 \omega}\right)$$



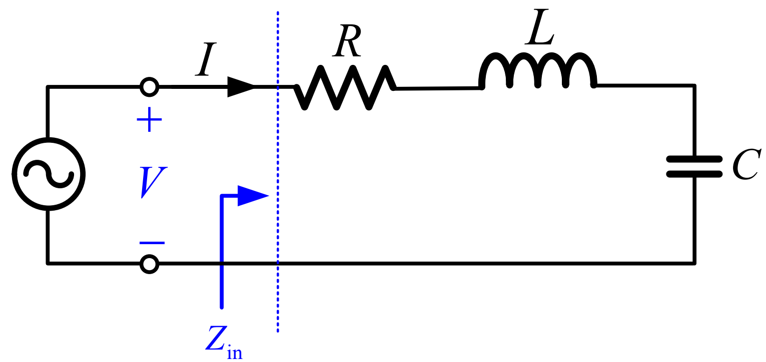


3 Lumped Resonant Circuit: LC Series Resonant Circuit

- Due to $\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0)$

$$= \Delta\omega(2\omega - \Delta\omega)$$

$$\simeq 2\omega\Delta\omega \quad \text{for small } \Delta\omega$$



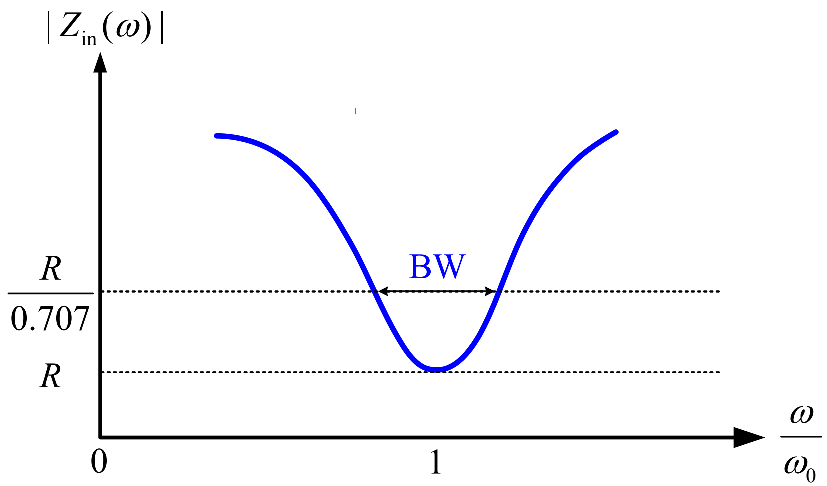
- Relation between Z_{in} and Q_u

$$Z_{in} = R + j\omega_0 L\left(\frac{\omega^2 - \omega_0^2}{\omega_0\omega}\right)$$

$$\simeq R\left\{1 + j\frac{\omega_0 L}{R}\left(\frac{2\omega\Delta\omega}{\omega_0\omega}\right)\right\}$$

$$= R\left(1 + j2Q\frac{\Delta\omega}{\omega_0}\right)$$

(in series RLC network)



3 Lumped Resonant Circuit: LC Series Resonant Circuit

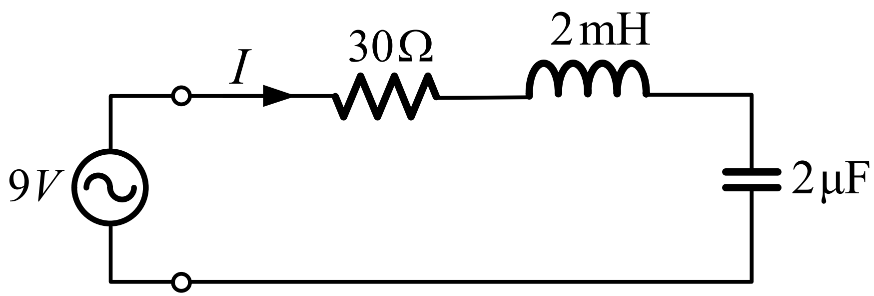
- **Example:** A series resonant network consisting of a resistor of 30 Ω, a capacitor of 2 μF, and an inductor of 2 mH is connected across a sinusoidal supply voltage which has a constant output of 9 V at all frequencies.

  a) Determine resonant frequency

  b) Determine current at resonance

  c) Determine voltage magnitude across inductor

  d) Determine unloaded Q of circuit at resonance



Solution:

a) Resonant frequency:

$$f_r = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{0.002\times 2\times 10^{-6}}} = 2{,}516 \text{ Hz}$$

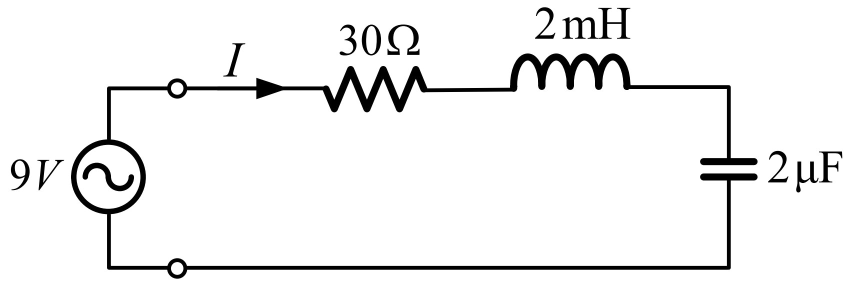
3 Lumped Resonant Circuit: LC Series Resonant Circuit

b) Current at resonant

$$I = \frac{V}{R} = \frac{9}{30} = 0.3\,\text{A}$$

c) Voltage across inductor

$$|V_L| = I \times X_L = 0.3 \times (2\pi f L)$$
$$= 0.3 \times (2\pi \times 2516 \times 0.002) = 9.48\ \text{V}$$

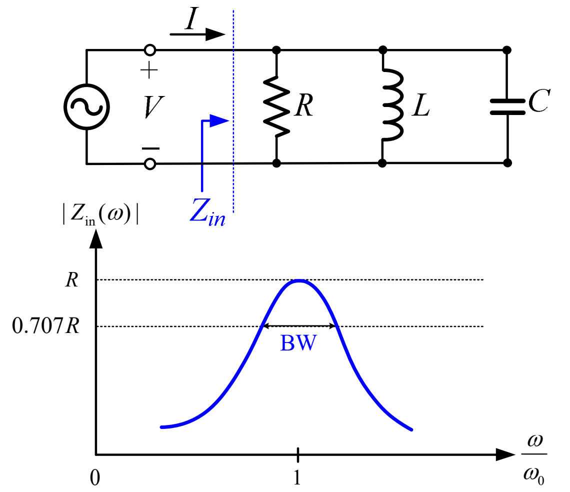


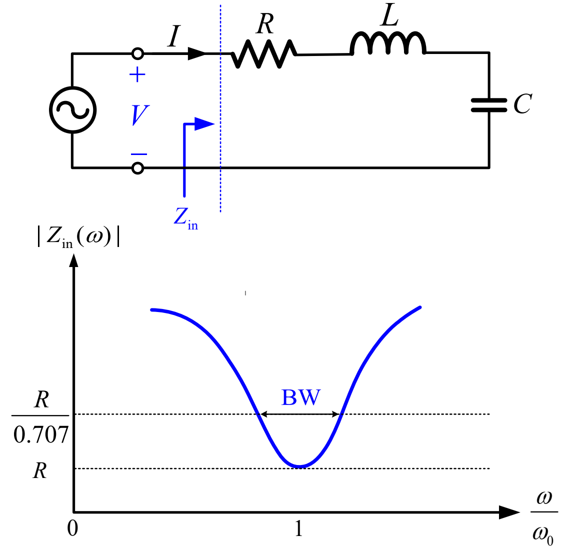
d) Determine unloaded Q:

$$Q_u = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$$
$$= \frac{2\pi f L}{R} = \frac{2\pi \times 2516 \times 0.002}{30} = 1.05$$: not good Q-factor

4 Review

Quantity	Parallel Resonator	Series Resonator
Input impedance/admittance	$Y_{in} = \frac{1}{R} + j\omega C - j\frac{1}{\omega L} \approx \frac{1}{R} + j\frac{2Q_u \Delta\omega}{R\omega_0}$	$Z_{in} = R + j\omega L - j\frac{1}{\omega C} \approx R + j\frac{2RQ_u \Delta\omega}{\omega_0}$
Power loss	$P_l = \frac{1}{2}\frac{\lvert V\rvert^2}{R}$	$P_l = \frac{1}{2}\lvert I\rvert^2 R$
Stored magnetic energy	$W_m = \frac{1}{4}\lvert V\rvert^2 \frac{1}{\omega^2 L}$	$W_m = \frac{1}{4}\lvert I\rvert^2 L$
Stored electric energy	$W_e = \frac{1}{4}\lvert V\rvert^2 C$	$W_e = \frac{1}{4}\lvert I\rvert^2 \frac{1}{\omega^2 C}$
Resonant frequency	$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \frac{1}{\sqrt{LC}}$
Unloaded Q	$Q_u = \omega_0 RC = \frac{R}{\omega_0 L}$	$Q_u = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 RC}$
External Q	$Q_{ex} = \frac{R_L}{\omega_0 L}$	$Q_{ex} = \frac{\omega_0 L}{R_L}$





4 Review

Why the *LC* circuit (or *LC*-tank) is called a tuned circuit or tank circuit?

- The charge flows back and forth between the plates of the capacitor and through the inductor. The energy oscillates between a capacitor and an inductor until the internal resistance of the components and connecting wires makes the oscillations to die-out.
- The LC circuit behaves like a harmonic oscillator, akin to a pendulum swinging or water sloshing in a tank, which is why it's called a tuned or tank circuit.
- The LC circuit can act as an electrical resonator and storing energy oscillates between the electric field and magnetic field at the frequency called a resonant frequency.