Chapter 6
Microwave Resonant Circuit

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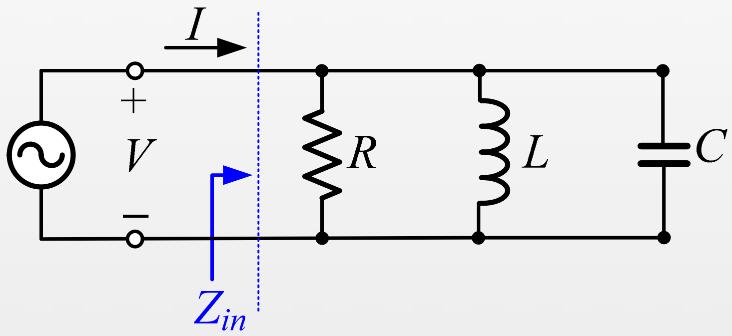
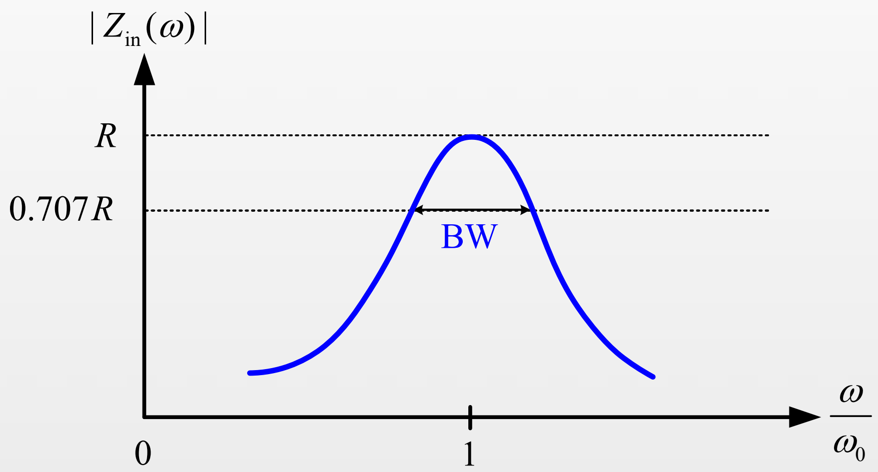




Learning Objectives

- Understanding about transmission line resonator
- Understanding about coaxial resonator
- Understand *Q*-factor of transmission line resonator

Learning contents

- Transmission Line Resonator
- Coaxial Resonator
- *Q* of transmission line resonator

1 Transmission Line Resonator

- **Shorted transmission line resonator**

  - Input impedance

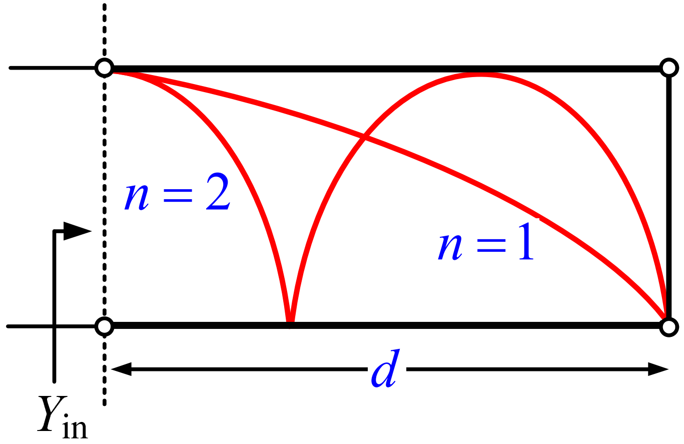
$$Y_{in} = -jY_0 \cot\left[\beta(\omega)d\right] \leftarrow \beta = \frac{2\pi}{\lambda} = \frac{\omega}{c} = f(\beta), @\beta(\omega_0)d = (2n-1)\cdot\frac{\pi}{2}$$

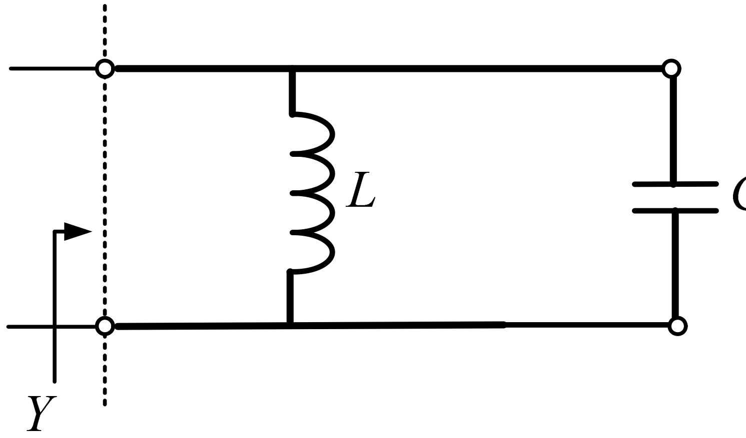
$$\Rightarrow Y_{in} = 0$$

  - For parallel LC resonant circuit:

$$Y = j\left(\omega C - \frac{1}{\omega L}\right) \leftarrow @\omega_0 = \frac{1}{\sqrt{LC}}$$

$$= 0$$





- In order to have the same frequency characteristics:

(1) $Y_{in} = Y = 0 \quad @\ \omega = \omega_0$

(2) $\left.\frac{\partial Y_{in}}{\partial \omega}\right|_{\omega_0} = \left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0}$

1 Transmission Line Resonator

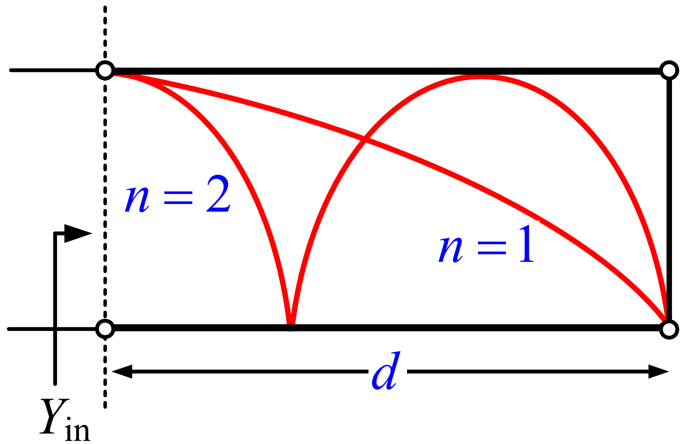
- In order to have the same frequency characteristics (continued)

$$\left.\frac{\partial Y_{in}}{\partial \omega}\right|_{\omega_0} = -jY_0 \frac{-1}{\sin^2 \beta(\omega)d} \cdot d \cdot \left.\frac{d\beta}{d\omega}\right|_{\omega_0} = jY_0 \frac{\beta'(\omega_0)}{\sin^2 \beta(\omega_0)d} \cdot d \quad \leftarrow Y_{in} = -jY_0 \cot\left[\beta(\omega)d\right],\ \beta(\omega_0)d = (2n-1)\cdot\frac{\pi}{2}$$

$$= jY_0\beta'(\omega_0)d = jY_0 \frac{\beta'(\omega_0)}{\beta(\omega_0)} \frac{(2n-1)\pi}{2}$$

$$\left.\frac{\partial Y}{\partial \omega}\right|_{\omega_0} = j\left.\left(C + \frac{1}{\omega^2 L}\right)\right|_{\omega_0} = j2C \quad \leftarrow Y = j\left(\omega C - \frac{1}{\omega L}\right),\ @\,\omega_0 = \frac{1}{\sqrt{LC}}$$

$$\Rightarrow C = \frac{Y_0\beta'(\omega_0)}{2\beta(\omega_0)} \frac{(2n-1)\pi}{2}, \qquad L = \frac{1}{\omega_0^2 C}$$



- For the coaxial transmission line: $\beta = \omega\sqrt{LC}$

$$C = \left.\frac{Y_0\beta'(\omega_0)}{2\beta(\omega_0)} \frac{(2n-1)\pi}{2}\right|_{\beta=\omega\sqrt{LC}} = \frac{Y_0\sqrt{LC}}{2\omega_0\sqrt{LC}} \frac{(2n-1)\pi}{2} = \frac{Y_0}{2\omega_0} \cdot \frac{(2n-1)\pi}{2}, \quad L = \frac{1}{\omega_0^2 C}$$

1 Transmission Line Resonator

- **Loaded Transmission Line**

- Insert susceptance " jB " on **real** transmission line;

- Consider

$$\gamma = \alpha + j\beta = j(\beta - j\alpha)$$

where α: attenuation constant of transmission line

β: phase constant of transmission line

- Input Admittance

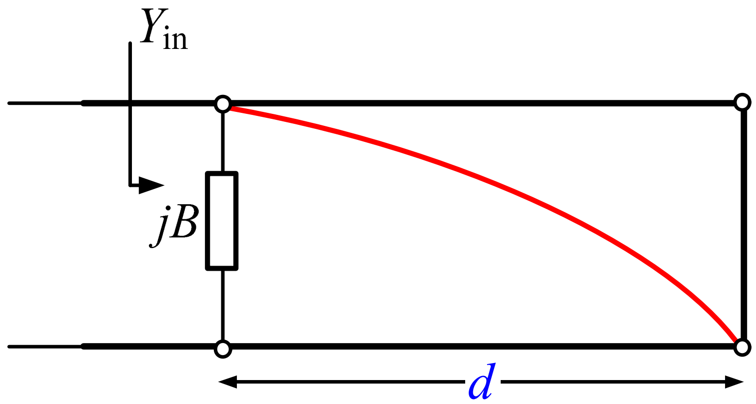
$$Y_{in} = jB - jY_0 \coth \gamma d = jB - jY_0 \cot(\beta - j\alpha)d$$

- At the resonant frequency ω_0 with small α, $Y_{in} = 0$

$$0 = jB - jY_0 \cot(\beta(\omega_0)d)$$

- For $|B| >> Y_0$

$$\therefore \beta(\omega_0)d = \cot^{-1}\left(\frac{B}{Y_0}\right) \approx n\pi$$



2 Coaxial Resonator

- Coaxial resonator

  - TEM mode resonator at VHF and UHF

$$l \approx \frac{n\lambda}{4} \quad , \quad n = 1, 2, 3, \ldots$$

  - @ $z = d_1$

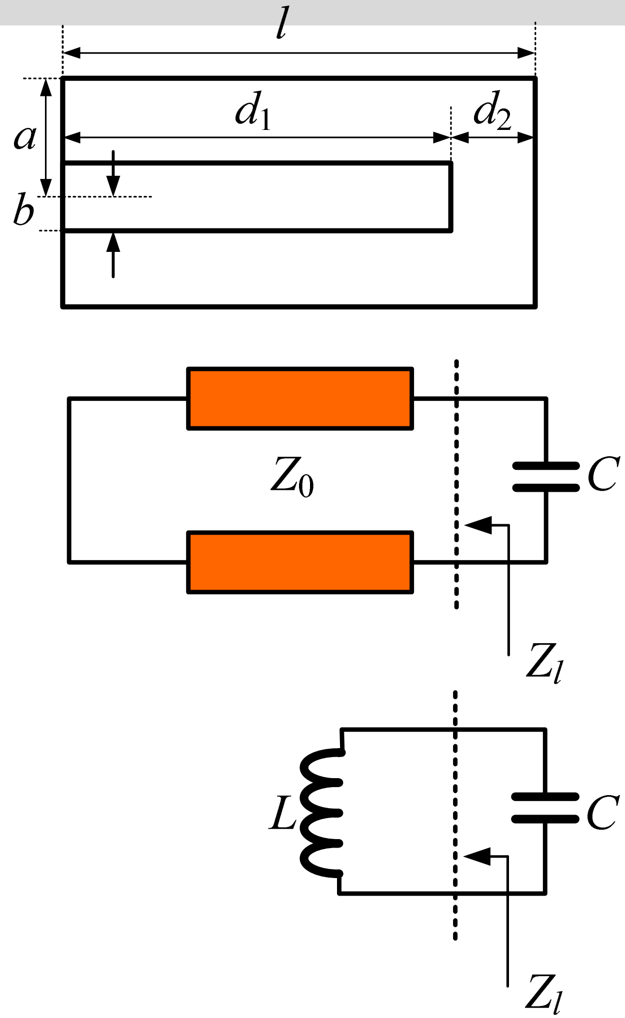
$$Z_l = jZ_0 \tan\left(\frac{2\pi}{\lambda} d_1\right)$$

  - @ resonant frequency (ω_0)

$$Z_0 \tan\left(\frac{2\pi}{\lambda} d_1\right) - \frac{1}{\omega_0 C} = 0$$

  - Resonant frequency is determined by Z_0, d_1, and C.

  - Low Q–factor but simple and single mode (TEM mode) operation for variable resonant frequency.



3 Q of Transmission Line Resonator

- Real transmission line resonator

  - Q-factor of real ($\gamma = \alpha + j\beta$) transmission line resonator

  - Length of transmission line: $d = \dfrac{\lambda}{4}$ at f_0

  - Near the resonant frequency ($\omega = \omega_0 + \Delta\omega$)

$$\beta d = \omega\sqrt{LC}d = \omega_0\sqrt{LC}d + \Delta\omega\sqrt{LC}d$$

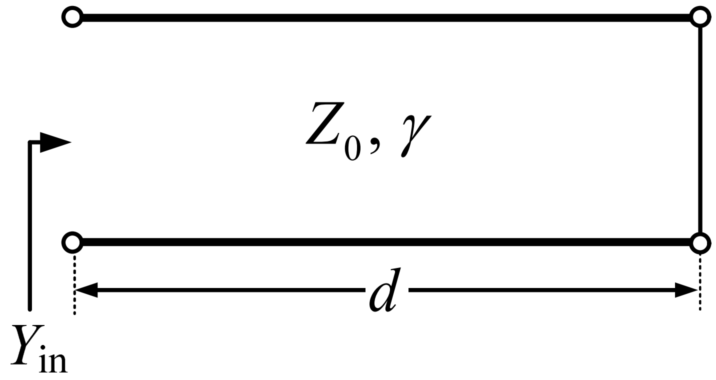
  - Since

$$\tan \beta d = \tan\left(\omega_0\sqrt{LC}d + \Delta\omega\sqrt{LC}d\right) = \tan\left(\frac{\pi}{2} + \Delta\omega\sqrt{LC}d\right)$$

$$= -\cot(\Delta\omega\sqrt{LC}d)$$

$$\approx \frac{-1}{\Delta\omega\sqrt{LC}d}$$

$$\tanh \alpha d \approx \alpha d \quad (@\,|\alpha d| \ll 1),$$



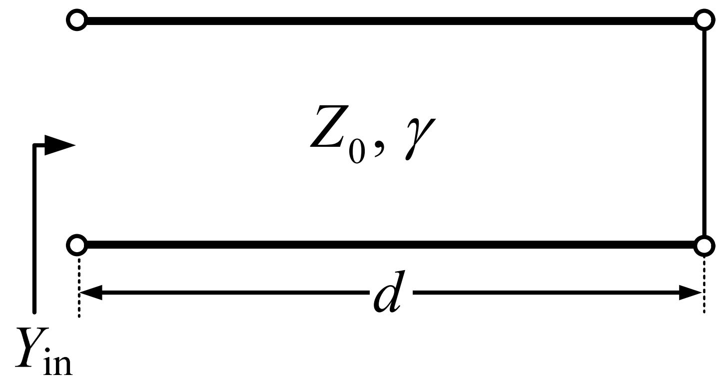
3 Q of Transmission Line Resonator

- Normalized input admittance at around resonant frequency

$$\overline{Y}_{in} = \frac{Y_{in}}{Y_0} = \coth \gamma d = \coth(\alpha + j\beta)d = \frac{1}{\tanh(\alpha + j\beta)d}$$

$$= \frac{1 + j\tan\beta d \tanh\alpha d}{\tanh\alpha d + j\tan\beta d} \approx \frac{1 - j\alpha d / \Delta\omega\sqrt{LC}d}{\alpha d - j / \Delta\omega\sqrt{LC}d}$$

$$= \frac{\Delta\omega\sqrt{LC}d - j\alpha d}{\Delta\omega\sqrt{LC}\alpha d^2 - j} = \frac{\alpha d + j\Delta\omega\sqrt{LC}d}{1 + j\Delta\omega\sqrt{LC}\alpha d^2}$$



- Further approximation:

$$\overline{Y}_{in} = \frac{\alpha d + j\Delta\omega\sqrt{LC}d}{1 + j\Delta\omega\alpha d^2\sqrt{LC}} \approx \alpha d + j\Delta\omega\sqrt{LC}d \leftarrow \alpha \text{ and } \Delta\omega\text{: too small values}$$

- Using $Y_0 = \sqrt{\frac{C}{L}}$ and $\alpha = \frac{Y_0 R}{2}$,

$$Y_{in} = \overline{Y}_{in} Y_0 = (\alpha d + j\Delta\omega\sqrt{LC}d)Y_0 = \frac{R}{2}Y_0^2 d + j\Delta\omega\sqrt{LC}dY_0 = \frac{RC}{2L}d + j\Delta\omega Cd$$

3 Q of Transmission Line Resonator

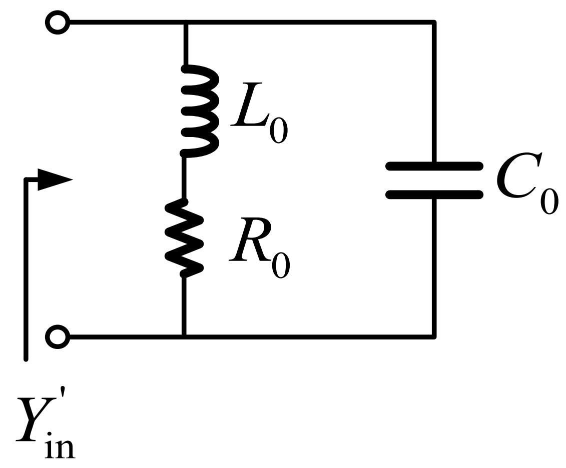
- Equivalent circuit of real transmission line resonator

- Input admittance:

$$Y_{in}' = \frac{1}{R_0 + j\omega L_0} + j\omega C_0 \quad \leftarrow R_0\text{: parasitic resistance of inductor}$$

$$= \frac{j\omega C_0\left(R_0 + j\omega L_0\right) + 1}{R_0 + j\omega L_0}$$

$$\approx \frac{j\omega C_0 R_0 - \omega^2 L_0 C_0 + 1}{j\omega L_0} \qquad \leftarrow R_0 << \omega L_0 \text{ assumed}$$



- @ resonant frequency, $\omega_0^2 L_0 C_0 = 1$

$$Y_{in}' = \frac{C_0 R_0}{L_0} + \frac{\omega_0^2 L_0 C_0 - \omega^2 L_0 C_0}{j\omega L_0}$$

$$= R_0 \frac{C_0}{L_0} - jL_0 C_0 \frac{\left(\omega_0 - \omega\right)\left(\omega_0 + \omega\right)}{\omega L_0} \approx R_0 \frac{C_0}{L_0} + jC_0 2\Delta\omega$$

3 Q of Transmission Line Resonator

- Comparison to obtain equivalent circuit element values

$$Y_{\text{in}} = \frac{RC}{2L}d + j\Delta\omega Cd$$

$$Y_{\text{in}}' = R_0\frac{C_0}{L_0} + j2\Delta\omega C_0$$

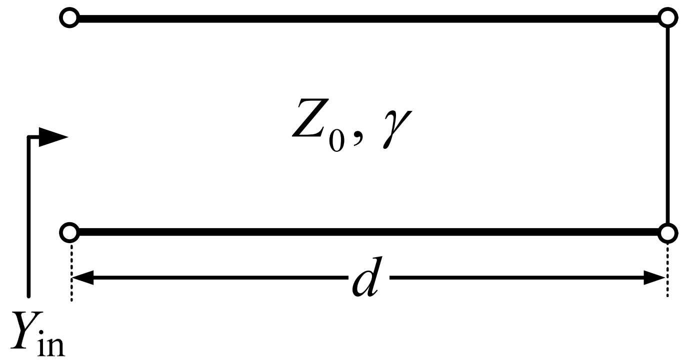
$$\Rightarrow R_0\frac{C_0}{L_0} = \frac{RC}{2L}d = \frac{RC_0}{L} \leftarrow Cd = 2C_0$$

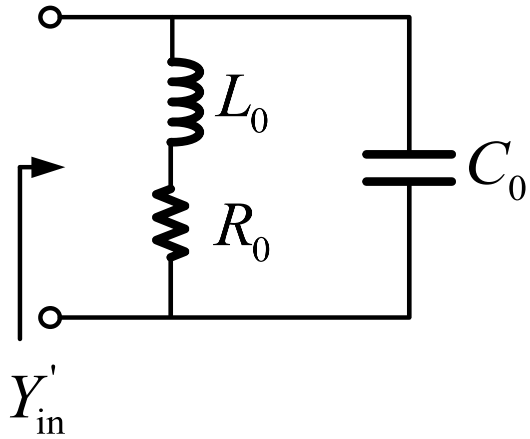
$$\Rightarrow \frac{R_0}{L_0} = \frac{R}{L}$$

- Q-factor of equivalent circuit

$$Q = \frac{\omega L_0}{R_0} \leftarrow \text{ Assuning lossless } C_0$$

$$= \frac{\omega L}{R} = \frac{\omega L}{\frac{2\alpha}{Y_0}} = \frac{\omega L}{2\alpha}\sqrt{\frac{C}{L}} = \frac{\omega\sqrt{LC}}{2\alpha} = \frac{\beta}{2\alpha}$$





3 Q of Transmission Line Resonator

- **Example 1**
  - Open circuited $\lambda/2$ microstrip line resonator of 50 Ω
  - Teflon substrate (ε_r = 2.08, tan δ = 0.0004, and h(thickness) = 0.159 cm) with copper conductors
  - Ignore fringing fields at the end of the line.
  - W = 0.508 cm (50 Ω line width), ε_e = 1.8 (effective permittivity), and R_S = 1.84 pΩ (surface resistivity)
  - Compute required length of line for resonance at 5 GHz and unloaded Q of resonator.

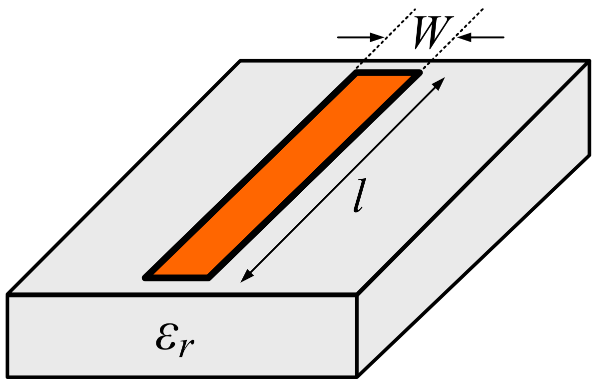
Solution:

- Resonant length with effective dielectric constant (ε_e)

$$l = \frac{\lambda}{2} = \frac{v_p}{2f} = \frac{c}{2f\sqrt{\varepsilon_e}} = \frac{3\times10^8}{2\left(5\times10^9\right)\sqrt{1.8}} = 2.24 \quad \text{cm}$$

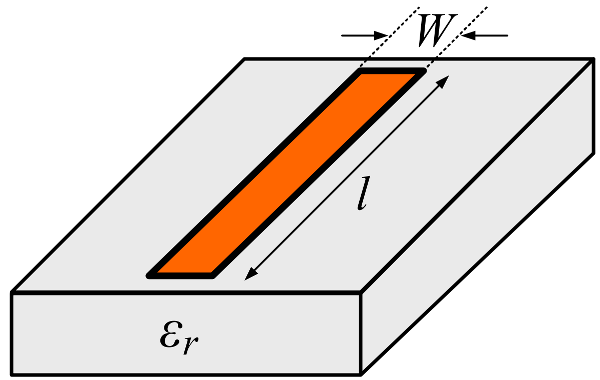
- Propagation constant

$$\beta = \frac{2\pi f}{v_p} = \frac{2\pi f\sqrt{\varepsilon_e}}{c} = \frac{2\pi\left(5\times10^9\right)\sqrt{1.8}}{3\times10^8} = 151 \quad \text{rad/m}$$



3 Q of Transmission Line Resonator

Solution (continued)



- Approximate attenuation due to conductor loss

$$\alpha_c = \frac{Y_0}{2}\frac{R_S}{W} = \frac{R_S}{Z_0 W} = \frac{1.84\times10^{-12}}{2\times50\times0.00508} = 0.0362 \quad \text{Np/m}$$

$$\leftarrow R_S[\Omega/\text{m}^2],\ \text{m}^2 = \text{width}\times\text{length}\ [1\,\text{m}] = \text{width}(W)$$

- Approximate attenuation due to dielectric loss

$$\alpha_d = \frac{k_0\varepsilon_r(\varepsilon_e - 1)\tan\delta}{2\sqrt{\varepsilon_e}(\varepsilon_r - 1)} = \frac{104.7\times2.08\times(1.8-1)\times0.0004}{2\sqrt{1.8}\times(2.08-1)} = 0.024 \quad \text{Np/m}$$

- Total attenuation with ignoring radiation loss

$$\alpha = \alpha_c + \alpha_d = 0.0362 + 0.024 = 0.0604 \quad \text{Np/m}$$

- Unloaded Q of the microstrip resonator

$$Q_0 = \frac{\beta}{2\alpha} = \frac{151}{2\times0.0604} = 1,250$$

3 Q of Transmission Line Resonator

- **Example 2**

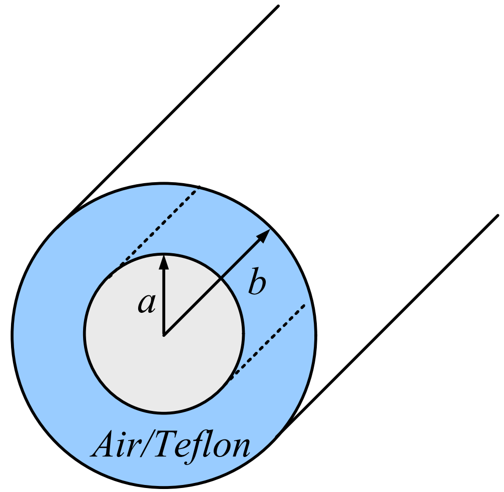
  - A $\lambda/2$ copper coaxial line with $a = 1$ mm , $b = 4$ mm, $\varepsilon_r = 2.08$, and tan $\delta = 0.0004$.
  - Conductivity of copper $\sigma = 5.813 \times 10^7$ S/m
  - Intrinsic impedance of electric material filled coaxial line: $\eta = 377$ and surface resistivity $R_S = 1.84 \times 10^{-2}\ \Omega$.
  - Resonant frequency: 5 GHz
  - Compare the unloaded Q of an air-filled coaxial line resonator to Teflon-filled coaxial line resonator.

Solution:

- Propagation constants of each dielectric material (symmetrical & homogenous materials)

$$\beta_{\text{air}} = \frac{2\pi f}{v_p} = \frac{2\pi f\sqrt{\varepsilon_0}}{c} = \frac{2\pi\left(5\times10^9\right)\sqrt{1}}{3\times10^8} = 104.7 \quad \text{rad/m}$$

$$\beta_{\text{Teflon}} = \frac{2\pi f}{v_p} = \frac{2\pi f\sqrt{\varepsilon_r}}{c} = \frac{2\pi\left(5\times10^9\right)\sqrt{2.08}}{3\times10^8} = 151.02 \quad \text{rad/m}$$



3 Q of Transmission Line Resonator

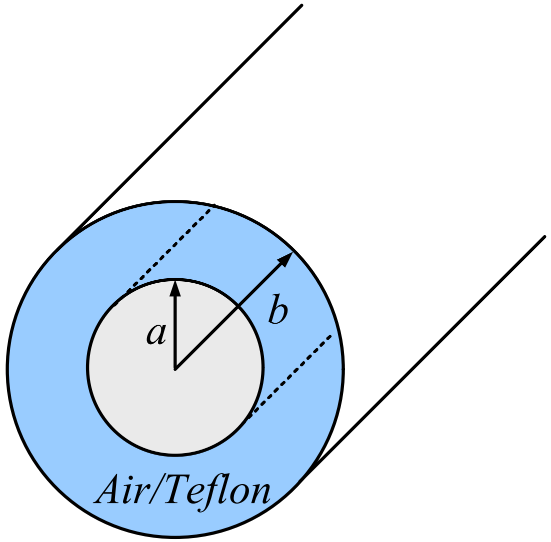
Solution (continued)

- Attenuation due to conductor loss for the air-filled line

$$\alpha_{c\_\text{air}} = \frac{R_S}{2\eta \ln\left(\frac{b}{a}\right)}\left(\frac{1}{a}+\frac{1}{b}\right) = \frac{1.84\times10^{-2}}{2\times377\times\ln\left(\frac{0.004}{0.001}\right)}\left(\frac{1}{0.001}+\frac{1}{0.004}\right) = 0.022 \quad \text{Np/m}$$

- Attenuation due to conductor loss for teflon-filled line

$$\alpha_{c\_\text{Teflon}} = \frac{R_S\sqrt{\varepsilon_r}}{2\eta \ln\left(\frac{b}{a}\right)} = \frac{1.84\times10^{-12}\times\sqrt{2.08}}{2\times\ln\left(\frac{0.004}{0.001}\right)}\left(\frac{1}{0.001}+\frac{1}{0.004}\right) = 0.032 \quad \text{Np/m}$$



- Dielectric loss of air-filled line: $\alpha_{c\_\text{air}} = 0$.

- Dielectric loss of teflon-filled line

$$\alpha_{d\_\text{Teflon}} = k_0\frac{\sqrt{\varepsilon_r}}{2}\tan\delta = \frac{104.7\times\sqrt{2.08}\times0.0004}{2} = 0.03 \quad \text{Np/m}$$

3 Q of Transmission Line Resonator

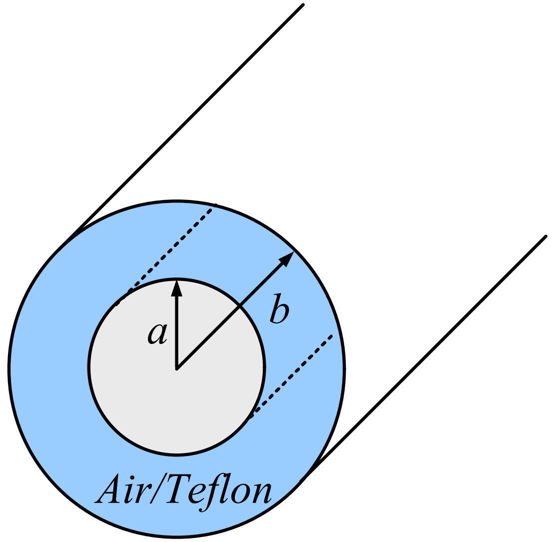
Solution (continued)

- Unloaded Q of air-filled coaxial transmission line resonator

$$Q_{\text{air}} = \frac{\beta_{\text{air}}}{2\left(\alpha_{c\_air} + \alpha_{d\_air}\right)} = \frac{104.7}{2\times\left(0.022+0\right)} = 2{,}379.5$$

- Unloaded Q of teflon-filled coaxial transmission line resonator

$$Q_{\text{Teflon}} = \frac{\beta_{\text{Teflon}}}{2\left(\alpha_{c\_\text{Teflon}} + \alpha_{d\_\text{Teflon}}\right)} = \frac{151.02}{2\times\left(0.032+0.03\right)} = 1{,}218$$



- Thus, it is seen that the Q of the teflon-filled line is almost half that of the air-filled line .

4 Review

- Transmission line resonator
- Coaxial resonator
- Unloaded Q of transmission line resonator