Chapter 6
Microwave Resonator

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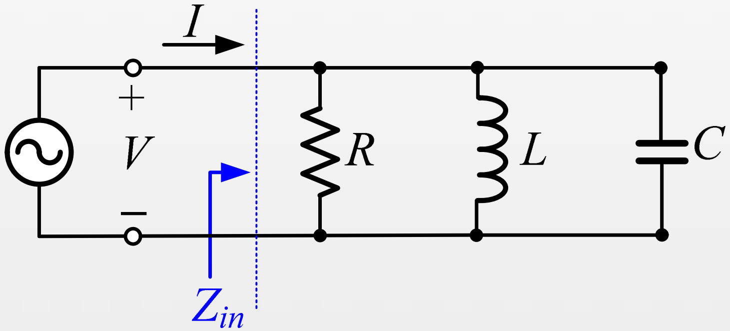
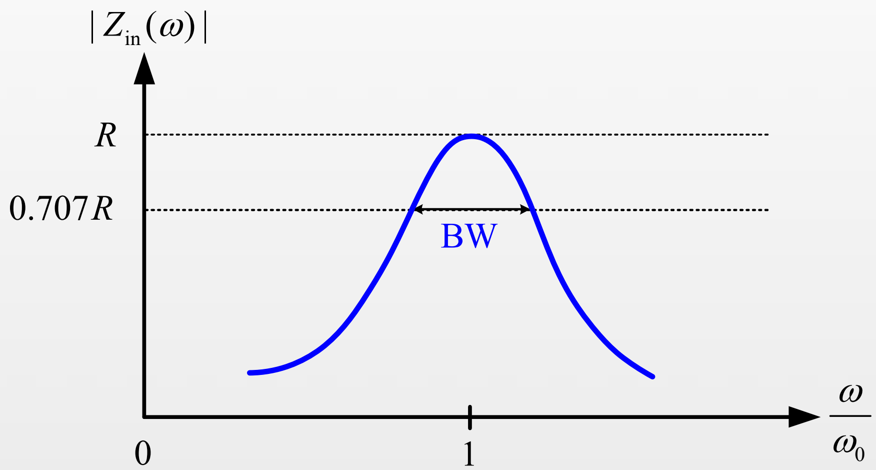




Learning Objectives

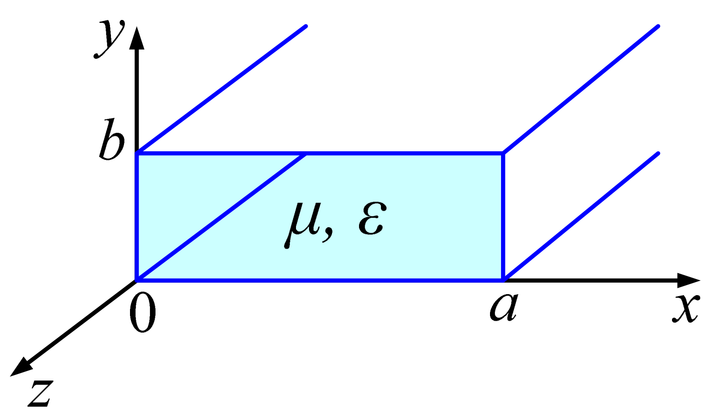
- Understanding characteristics of rectangular waveguide
- Understanding characteristics of rectangular waveguide cavity resonator

Learning contents

- Rectangular Waveguide
- Rectangular Waveguide Cavity Resonator

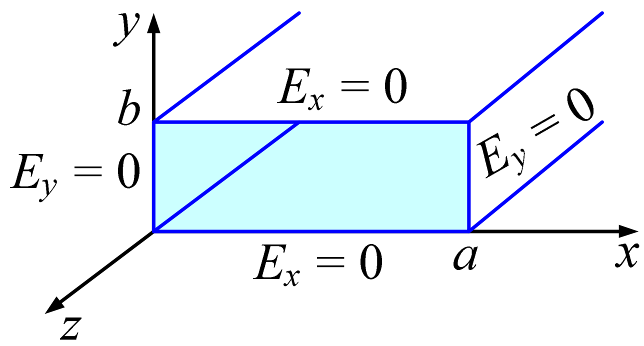
1 Rectangular Waveguide

- Metallic waveguides such as closed rectangular (cylindrical) conducting tubes support transverse electric (TE) or/and transverse magnetic (TM) modes.
- For rectangular waveguide, the radiation loss from an open-ended waveguide can be significant. Therefore, waveguide resonators are usually short circuited at both ends, thus forming a closed box or cavity.
- Electric and magnetic energies are stored within the cavity enclosure and the power is dissipated in the metallic walls of the cavity as well as in the dielectric material that may be filled the cavity.
- Typically, a rectangular waveguide consisted by a single conductor line → Cannot support TEM mode.
- For TE modes: $E_z = 0$, $H_z \neq 0$ → 5-field components
- For TM modes: $E_z \neq 0$, $H_z = 0$ → 5-field components



1 Rectangular Waveguide

- **Transverse electric (TE) mode in rectangular waveguide:**



- $E_z = 0$

- In free space,

$$\nabla \times \vec{E} = -j\omega \vec{B} \qquad \nabla \times \vec{H} = j\omega \vec{D}$$

$$\begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -j\omega(B_x\vec{a}_x + B_y\vec{a}_y + B_z\vec{a}_z) \qquad \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = -j\omega\varepsilon(E_x\vec{a}_x + E_y\vec{a}_y + E_z\vec{a}_z)$$

- 6 variables (E_x, E_y, E_z, H_x, H_y, H_z) must be defined to know electromagnetic wave operation. → *Difficult!!!*

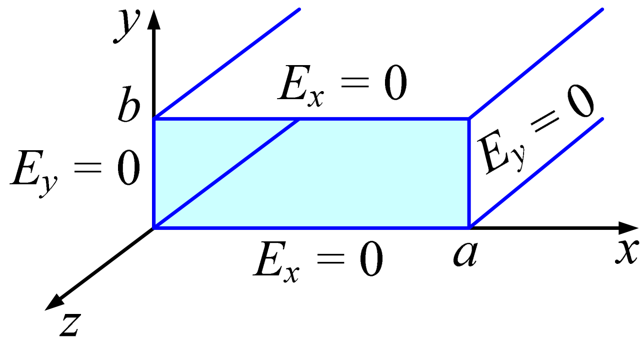
$$E_x = \frac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(-j\omega\mu\frac{\partial H_z}{\partial y} - \gamma\frac{\partial E_z}{\partial x}\right), \qquad H_x = \frac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(j\omega\varepsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x}\right)$$

$$E_y = \frac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(j\omega\mu\frac{\partial H_z}{\partial x} - \gamma\frac{\partial E_z}{\partial y}\right), \qquad H_y = \frac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(-j\omega\varepsilon\frac{\partial E_z}{\partial y} - \gamma\frac{\partial H_z}{\partial x}\right)$$

- If E_z and H_z are known, the remained 4 variables (E_x, E_y, H_x, H_y) can be easily obtained.

1 Rectangular Waveguide

- TE mode (continued)



- Helmholtz equation (or wave equation):

$$\nabla^2 \vec{H} + k^2 \vec{H} = 0, \qquad \frac{\partial^2 H_z}{\partial^2 x} + \frac{\partial^2 H_z}{\partial^2 y} + \frac{\partial^2 H_z}{\partial^2 z} + k^2 H_z = 0 \leftarrow k = \omega\sqrt{\mu\varepsilon}$$

$$\frac{\partial^2 H_Z}{\partial^2 x} + \frac{\partial^2 H_Z}{\partial^2 y} + \gamma^2 H_z + k^2 H_z = \frac{\partial^2 H_Z}{\partial^2 x} + \frac{\partial^2 H_Z}{\partial^2 y} + (\gamma^2 + k^2) H_z = 0$$

$$\frac{\partial^2 H_Z}{\partial^2 x} + \frac{\partial^2 H_Z}{\partial^2 y} + k_c^2 H_z = 0 \quad (1) \qquad \leftarrow k_c^2 = \gamma^2 + k^2 = (\alpha + j\beta)^2 + k^2 = -\beta^2 + k^2 \text{ (for lossless, } \alpha = 0)$$

- Separation of variables: $H_z = X(x)Y(y)e^{-j\beta z}$ (2)

- (1) ← (2): $Y(y)e^{-j\beta z}\frac{\partial^2 X}{\partial x^2} + X(x)e^{-j\beta z}\frac{\partial^2 Y}{\partial y^2} + k_c^2 X(x)Y(y)e^{-j\beta z} = 0$

$$\frac{1}{X(x)}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y(y)}\frac{\partial^2 Y}{\partial y^2} + k_c^2 = 0, \qquad -k_x^2 - k_y^2 + k_c^2 = 0 \quad (\text{or } k_c^2 = k_x^2 + k_y^2)$$

- This leads to $\frac{\partial^2 X}{\partial x^2} + k_x^2 X(x) = 0$ and $\frac{\partial^2 Y}{\partial y^2} + k_y^2 Y(y) = 0$

1 Rectangular Waveguide

- TE mode (continued)

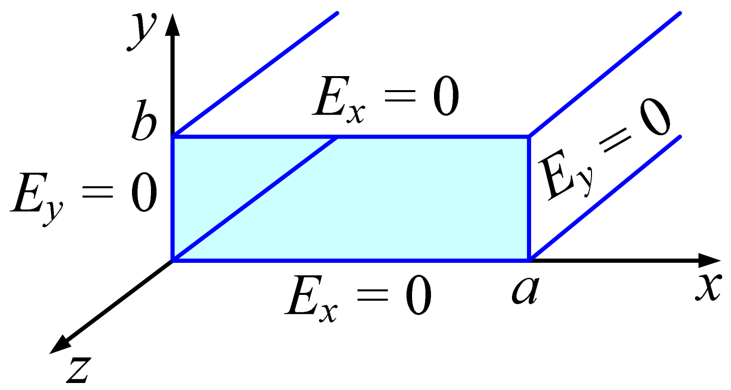
  - General solutions: $\begin{cases} X = A\cos k_x x + B\sin k_x x \\ Y = C\cos k_y y + D\sin k_y y \end{cases}$

  - Applying boundary conditions,

$$E_x\Big|_{E_z=0} = \frac{1}{\gamma^2 + \omega^2\mu\varepsilon}\left(-j\omega\mu\frac{\partial H_z}{\partial y} - \gamma\frac{\partial E_z}{\partial x}\right)\Big|_{E_z=0} = -\frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial y}$$

$$= -\frac{j\omega\mu}{k_c^2}X(x)\frac{\partial Y(y)}{\partial y} = 0 \quad @\, y = 0,\, b \quad (\because \text{ conducting wall})$$

$$\Rightarrow \frac{\partial Y}{\partial y} = 0 \quad @\, y = 0,\, b$$



$$\frac{\partial Y}{\partial y}\Big|_{y=0} = -k_y C\sin k_y y + k_y D\cos k_y y\Big|_{y=0} = k_y D = 0 \quad \Rightarrow D = 0$$

$$\frac{\partial Y}{\partial y}\Big|_{y=b} = -k_y C\sin k_y b = 0 \quad \Rightarrow k_y b = n\pi \qquad (n = 0, 1, 2, \cdots) \;\Rightarrow k_y = \frac{n\pi}{b}$$

1 Rectangular Waveguide

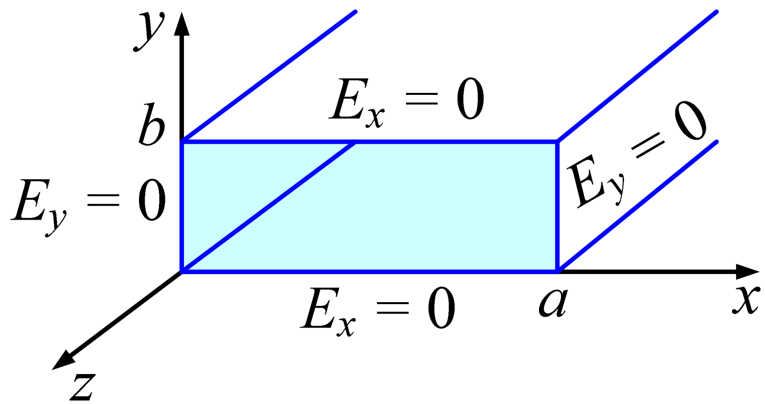
- TE mode (continued)

- Similarly,

$$E_y\Big|_{E_z=0} = \frac{1}{\gamma^2+\omega^2\mu\varepsilon}\left(j\omega\mu\frac{\partial H_z}{\partial x} - \gamma\frac{\partial E_z}{\partial y}\right)\Big|_{E_z=0} = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial x}$$

$$= \frac{j\omega\mu}{k_c^2}Y(y)\frac{\partial X(x)}{\partial x} = 0 \quad @\, x = 0,\ a$$

$$\Rightarrow \frac{\partial X}{\partial x} = -k_x A\sin k_x x + k_x B\cos k_x x = 0 \quad @\, x = 0,\ a$$



$$\frac{\partial X}{\partial x}\Big|_{x=0} = k_x B = 0 \ \Rightarrow B = 0$$

$$\frac{\partial X}{\partial x}\Big|_{x=a} = -k_x A\sin k_x a = 0 \ \Rightarrow k_x a = m\pi \qquad (m = 0, 1, 2, \cdots) \Rightarrow k_x a = m\pi \ \Rightarrow k_x = \frac{m\pi}{a}$$

1 Rectangular Waveguide

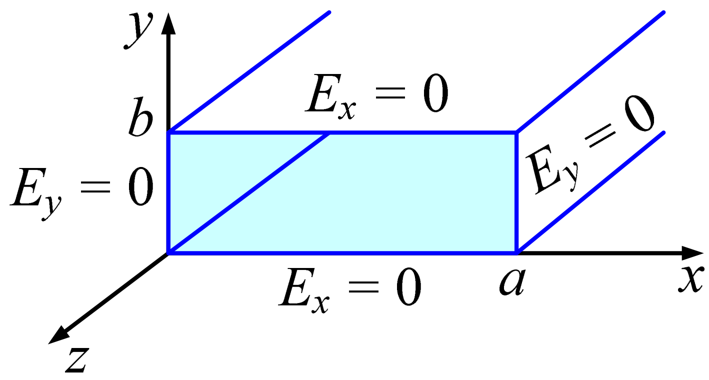
- TE mode (continued)

- Finally, $H_z = X(x)Y(y)e^{-j\beta z} = AC\cos k_x x \cos k_y y e^{-j\beta z}$

$$= H_{mn}\cos\frac{m\pi}{a}x\cos\frac{n\pi}{b}ye^{-j\beta z} \quad \leftarrow m = n = 0 \text{ excluded}$$

- If $m = n = 0$, $H_z = H_{mn}$ (constant)

Then $E_x = E_y = H_x = H_y = 0$ (No electromagnetic wave!)



- Field components

$$\begin{cases} E_x = -\dfrac{j\omega\mu}{k_c^2}\dfrac{\partial H_z}{\partial y} = \dfrac{j\omega\mu k_y}{k_c^2}H_{mn}\cos k_x x\cdot\sin k_y y e^{-j\beta_{mn}z} \\ E_y = \dfrac{j\omega\mu}{k_c^2}\dfrac{\partial H_z}{\partial x} = -\dfrac{j\omega\mu k_x}{k_c^2}H_{mn}\sin k_x x\cdot\cos k_y y e^{-j\beta_{mn}z} \\ H_x = -\dfrac{\gamma_{mn}}{k_c^2}\dfrac{\partial H_z}{\partial x} = \dfrac{\gamma_{mn}k_x}{k_c^2}H_{mn}\sin k_x x\cdot\cos k_y y e^{-j\beta_{mn}z} \\ H_y = -\dfrac{\gamma_{mn}}{k_c^2}\dfrac{\partial H_z}{\partial y} = \dfrac{\gamma_{mn}k_y}{k_c^2}H_{mn}\cos k_x x\cdot\sin k_y y e^{-j\beta_{mn}z} \end{cases}$$

where
$$\begin{cases} TE_{mn} \text{ mode: TE mode according to } m, n \\ \gamma_{mn}^2 = k_c^2 - k^2 = k_c^2 - \omega^2\mu\varepsilon, \quad \gamma_{mn} = -j\beta_{mn} \\ k_x = \dfrac{m\pi}{a}, \quad k_y = \dfrac{n\pi}{b} \\ k_c^2 = k_x^2 + k_y^2 \end{cases}$$

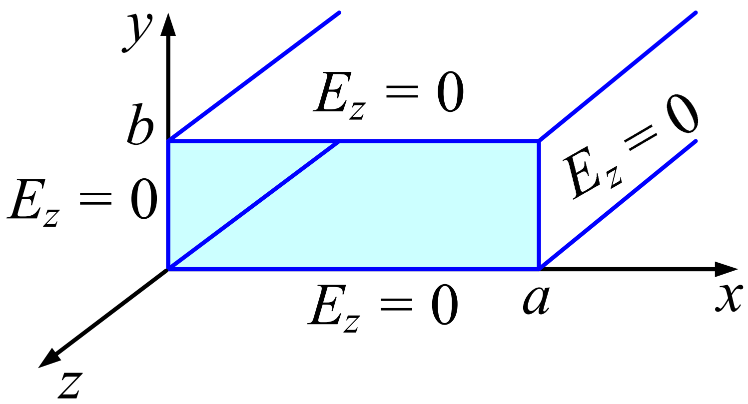
1 Rectangular Waveguide

- **TM mode in rectangular waveguide:**

  - $H_z = 0$

  - Helmholtz equation (or wave equation):

$$\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + k_c^2 E_z = 0 \qquad (3)$$



  - Separation of variables: $E_z(x, y, z) = X(x)Y(y)e^{-j\beta z}$

$$\frac{\partial^2 X}{\partial x^2} + k_x^2 X = 0 \quad \text{and} \quad \frac{\partial^2 Y}{\partial y^2} + k_y^2 Y(y) = 0$$

  - General solutions:

$$\begin{cases} X = A\cos k_x x + B\sin k_x x \\ Y = C\cos k_y y + D\sin k_y y \end{cases}$$

1 Rectangular Waveguide

- TM mode in rectangular waveguide (continued)

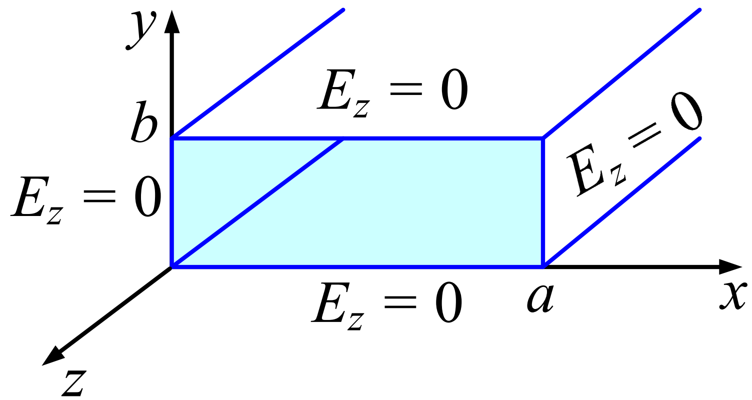
  - Wall boundary conditions:

$$E_z = 0 \quad \text{at} \quad x = 0,\ x = a \ \rightarrow X = 0 \text{ at } x = 0 \text{ and } a$$

$$A = 0,\ k_x a = m\pi,\ \ m = 1,\ 2, 3,\ \cdots$$

$$E_z = 0 \quad \text{at} \quad y = 0,\ y = a \ \rightarrow Y = 0 \text{ at } y = 0 \text{ and } b$$

$$C = 0,\ k_y b = n\pi,\ \ n = 1,\ 2, 3,\ \cdots$$



  - This leads to

$$E_z = E_{mn} \sin k_x x \sin k_y y e^{-j\beta_{mn} z}$$

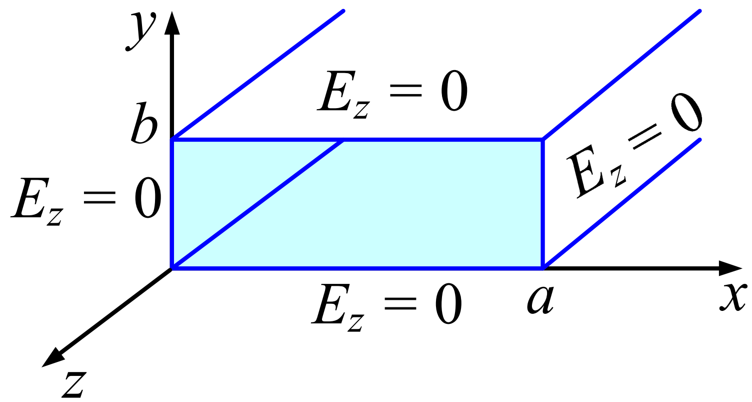
$$= BD \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y e^{-j\beta_{mn} z}$$

1 Rectangular Waveguide

- TM mode in rectangular waveguide (continued)

- Field components,

$$
\begin{cases}
E_x = -\dfrac{j\omega\mu}{k_c^2}\dfrac{\partial H_z}{\partial y} = \dfrac{j\omega\mu k_y}{k_c^2} H_{mn} \cos k_x x \cdot \sin k_y y e^{-j\beta_{mn} z} \\
E_y = \dfrac{j\omega\mu}{k_c^2}\dfrac{\partial H_z}{\partial x} = -\dfrac{j\omega\mu k_x}{k_c^2} H_{mn} \sin k_x x \cdot \cos k_y y e^{-j\beta_{mn} z} \\
H_x = -\dfrac{\gamma_{mn}}{k_c^2}\dfrac{\partial H_z}{\partial x} = \dfrac{\gamma_{mn} k_x}{k_c^2} H_{mn} \sin k_x x \cdot \cos k_y y e^{-j\beta_{mn} z} \\
H_y = -\dfrac{\gamma_{mn}}{k_c^2}\dfrac{\partial H_z}{\partial y} = \dfrac{\gamma_{mn} k_y}{k_c^2} H_{mn} \cos k_x x \cdot \sin k_y y e^{-j\beta_{mn} z}
\end{cases}
$$



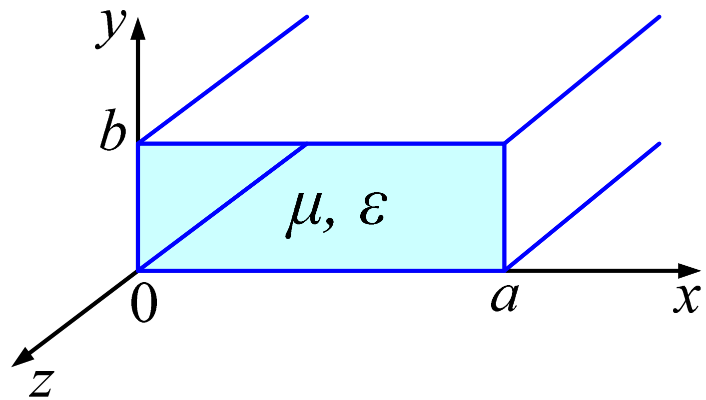
$$
\text{where} \quad
\begin{cases}
\gamma_{mn} = -j\beta_{mn} \\
k_x = \dfrac{m\pi}{a},\ k_y = \dfrac{n\pi}{b} \\
m = 0 \text{ or } n = 0 \ \rightarrow TM_{10} \text{ or } TM_{01} \text{ mode.}
\end{cases}
$$

- If m = 1 and n = 0 (or m = 0, n = 1), then $E_z = 0$ and no electomagnetic waves.

1 Rectangular Waveguide

Mode characteristics:

- **Orthogonality:** Power can't be coupled between two different modes, among TE and TM modes. Coupling may exist between the same TM and TE modes.



- **Cutoff frequency:**

$$k_c^2 = k^2 + \gamma^2 \quad \leftarrow \quad k_c^2 = k_x^2 + k_y^2$$

$$\gamma^2 = k_c^2 - k^2 = k_x^2 + k_y^2 - k^2$$

$$\rightarrow \gamma = \alpha + j\beta = \sqrt{k_c^2 - k^2} = \sqrt{\omega_c^2 \mu\varepsilon - \omega^2 \mu\varepsilon} = \sqrt{\mu\varepsilon}\sqrt{\omega_c^2 - \omega^2}$$

i) $\omega > \omega_c \quad \gamma = j\beta$: Propagation mode, $\alpha = 0$

ii) $\omega < \omega_c \quad \gamma = \alpha$: Evanescent mode, $\beta = 0$

iii) $\omega = \omega_c \quad \gamma = 0$: Cutoff

$$k_c = k = \omega_c\sqrt{\mu\varepsilon} = \left[\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2\right]^{\frac{1}{2}} \Rightarrow \omega_c = \frac{1}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}$$

$$\lambda_c = \frac{c}{f_c} = \frac{\omega_c / k_c}{f_c} = \frac{2\pi}{k_c},$$

where $k = \omega\sqrt{\mu\varepsilon} = \frac{\omega}{c}$, $c = \frac{\omega}{k}$

λ_c: cutoff wavelength

➔ 1. *Rectangular waveguide is a high-pass filtering device.* 2. *Cutoff frequency is depended on modes* (m, n).

1 Rectangular Waveguide

- **Mode characteristics:**

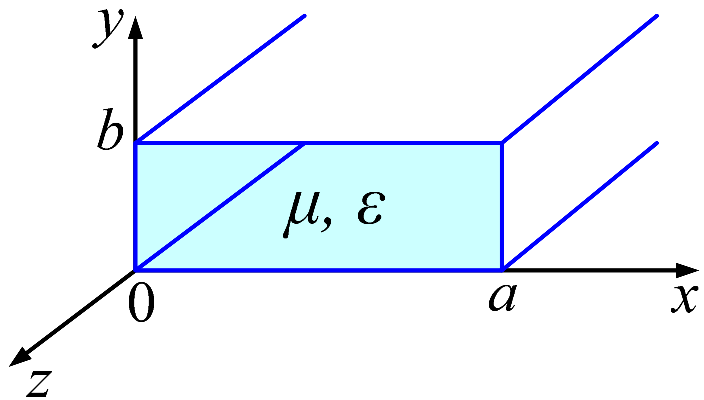
- Dominant mode: mode with the lowest cutoff frequency.

$$k_c^2 = \omega_c^2 \mu\varepsilon = k_x^2 + k_y^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2$$

$$\therefore\ f_c = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}, \quad a > b$$

$$\Rightarrow f_{c\_TE10} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\Bigg|_{\substack{m=1\\n=0}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\frac{\pi}{a} = \frac{c}{2a} \quad \leftarrow c = \frac{1}{\sqrt{\mu\varepsilon}}$$

$$\lambda_{c\_TE10} = \frac{c}{f_c} = 2a$$



- Higher order modes: other modes except the dominant mode.

2 Rectangular Waveguide Cavity Resonator

- Propagation constant of TE_{mn} or TM_{mn} mode of rectangular waveguide cavity

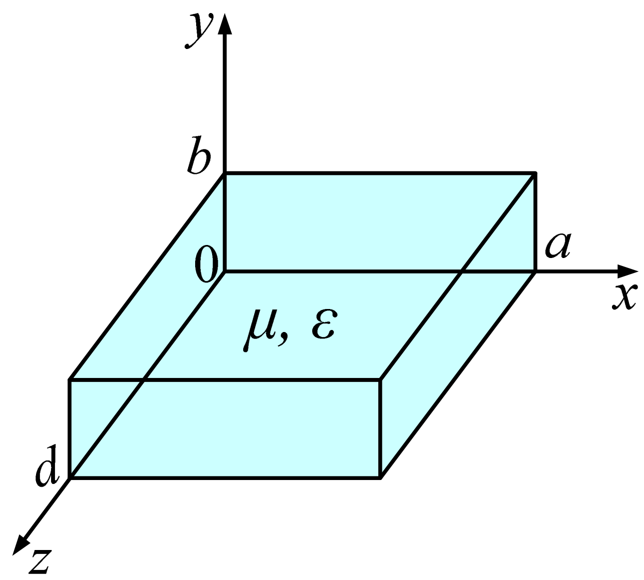
$$\beta_{mn} = \sqrt{k^2 - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}$$

- Resonant wavenumber for rectangular cavity

$$k_{mnl} = \sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$$

- Resonant frequency of cavity

$$f_c = \frac{ck_{mnl}}{2\pi\sqrt{\mu\varepsilon}} = \frac{1}{2\pi\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$$



2 Rectangular Waveguide Cavity Resonator

- **Unloaded Q of the TE_{10l} mode:**

  - Unloaded Q with lossy dielectric material filling but perfectly conductor wall

$$Q_d = \frac{1}{\tan\delta},$$

where $\tan\delta$: loss tangent of dielectric material

  - Unloaded Q with lossy conducting walls but lossless dielectric

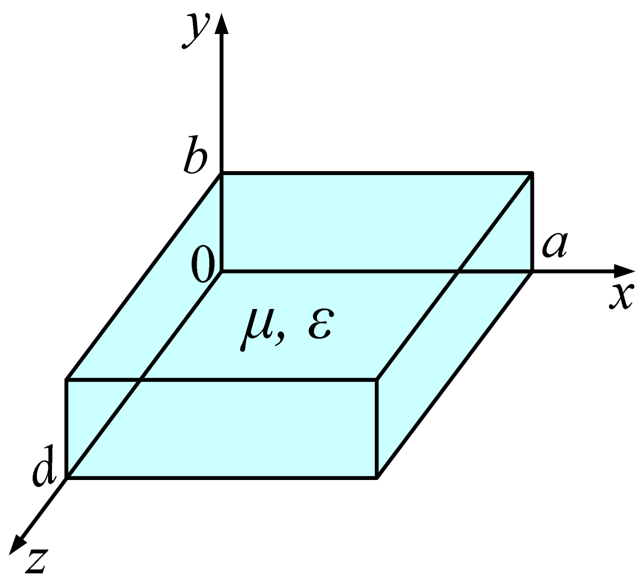
$$Q_c = \frac{(kad)^3 b\eta}{2\pi^2 R_S} \frac{1}{2l^2a^3b + 2bd^3 + l^2a^3d + ad^3},$$

where $k = \omega\sqrt{\mu\varepsilon}$: wave number

$R_S = \sqrt{\omega\mu_0 / 2\sigma}$: surface resistivity of metallic wall

$\eta = \sqrt{\mu / \varepsilon}$: intrinsic impedance

  - Total unloaded Q: $Q_u = \left(\frac{1}{Q_c} + \frac{1}{Q_d} \right)^{-1}$



2 Rectangular Waveguide Cavity Resonator

Example:

- Rectangular waveguide cavity made with copper WR−187 for H-band waveguide (a = 4.7 cm, b = 2.2 cm)
- Filled with polyethylene ($\epsilon_r = 2.25$, $\tan\delta = 0.0004$)
- Assumptions: f_0 = 5 GHz, (surface resistivity) $R_S = 0.1\ \Omega$, and (wavenumber) k = 157/m.
- Find required length d and resulting unloaded Q for m =1, n = 0, and l = 1 resonant modes.

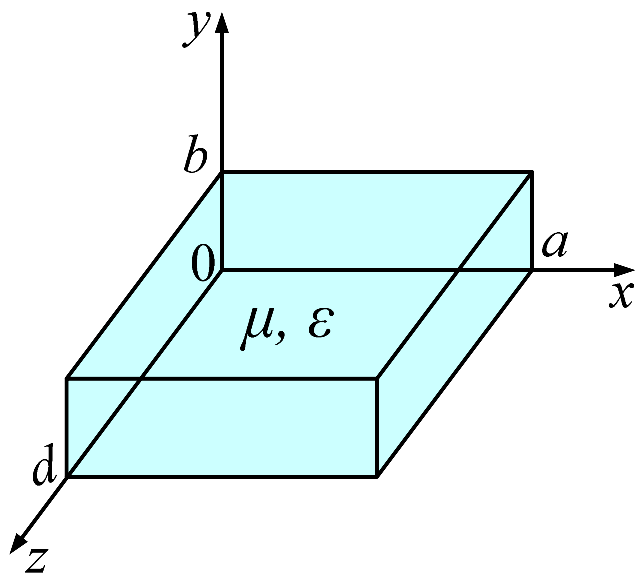
Solution:

- From dominant mode (TE_{101}),

$$k_{101} = \sqrt{\left(\frac{\pi}{a}\right)^2 + (0)^2 + \left(\frac{\pi}{d}\right)^2}$$

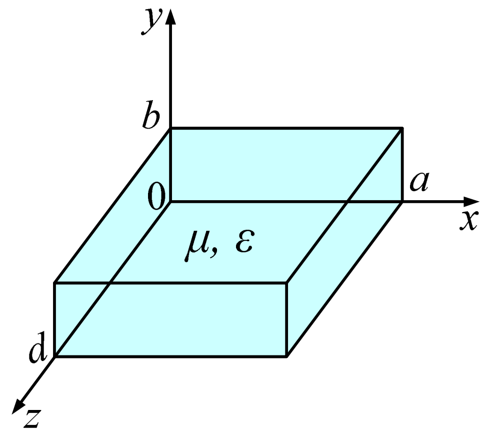
$$\Leftrightarrow \left(\frac{\pi}{d}\right)^2 = k_{101}^2 - \left(\frac{\pi}{a}\right)^2$$

$$\Rightarrow d = \frac{\pi}{\sqrt{k_{101}^2 - \left(\frac{\pi}{a}\right)^2}} = \frac{\pi}{\sqrt{(157)^2 - \left(\frac{\pi}{0.047}\right)^2}} = 2.21 \text{ cm}$$



2 Rectangular Waveguide Cavity Resonator

Solution (continued)



- Q due to conductor loss

$$Q_c = \frac{(kad)^3 b\eta}{2\pi^2 R_S} \frac{1}{2l^2a^3b + 2bd^3 + l^2a^3d + ad^3}$$

where $\eta = \frac{377}{\sqrt{\varepsilon_r}} = 251.3\ \Omega$ for polyethylene.

$$\Rightarrow Q_c = \frac{0.024}{1.9739} \frac{1}{(4.5682\times10^{-6}) + (4.7492\times10^{-7}) + (2.2944\times10^{-6}) + (5.0731\times10^{-7})}$$

$$= \frac{12.1586\times10^{-3}}{2.8017\times10^{-6}} = 4339.74$$

- Q due to dielectric loss

$$Q_d = \frac{1}{\tan\delta} = \frac{1}{0.0004} = 2500$$

- Unloaded Q of cavity resonator

$$Q_0 = \left(\frac{1}{Q_c} + \frac{1}{Q_d}\right)^{-1} = \left(\frac{1}{4339.74} + \frac{1}{2500}\right)^{-1} = 1586.22$$

3 Review

- Rectangular waveguide
- Rectangular waveguide cavity resonator