Engineering Electromagnetics

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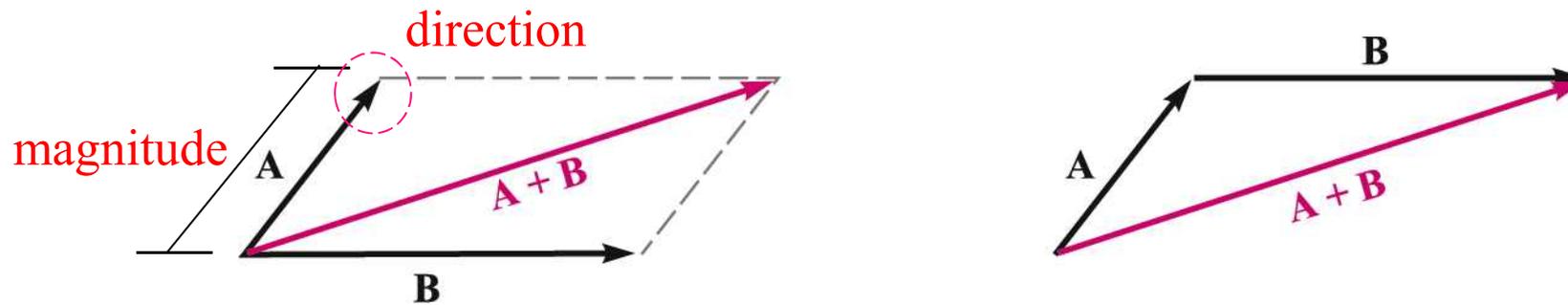
Chapter 1:

Vector Analysis

1.1 Scalars and Vectors

- Scalar: magnitude or quantity

- Complex scalar (or phasor): a set of scalar

  Ex.] $M \angle \theta, \ R + jX$

- Vector: identity having magnitudes and directions in the **n-dimensional spaces**

1.2 Vector Algebra

1.2.1 Addition and Subtraction

- The addition of vectors follows the parallelogram law.

Communicative Law: $\vec{A} + \vec{B} = \vec{B} + \vec{A}$

Associative Law: $\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C}$

Subtraction: $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$

1.2.2 Multiplication and Division

- Multiplication of vectors by scalar

  Distributive Law: $(r+s)(\mathbf{A}+\mathbf{B}) = r(\mathbf{A}+\mathbf{B}) + s(\mathbf{A}+\mathbf{B})$

  $$= r\vec{A} + r\vec{B} + s\vec{A} + s\vec{B}$$

- Division of vector by a scalar

  $$\frac{\vec{A}}{r} = \frac{1}{r}\vec{A}$$

- Equal: $\vec{A} = \vec{B}$ or $\vec{A} - \vec{B} = \mathbf{0}$

1.3 Rectangular Coordinate System

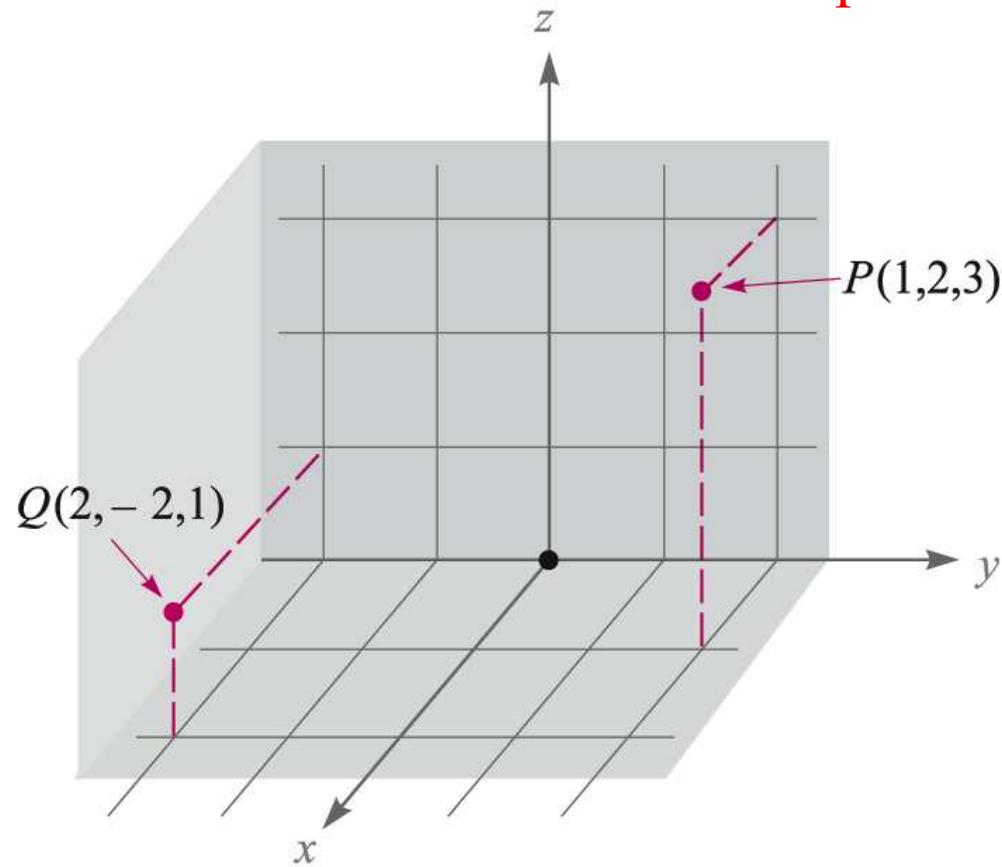▪ Right-handed coordinate system: Three coordinate axes are located mutually at right angle to each other.

$(x \rightarrow y \rightarrow z \rightarrow x \rightarrow \cdots)$

(Ex. Fleming's right-hand rule)
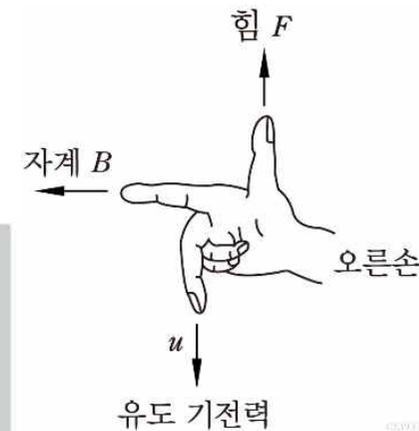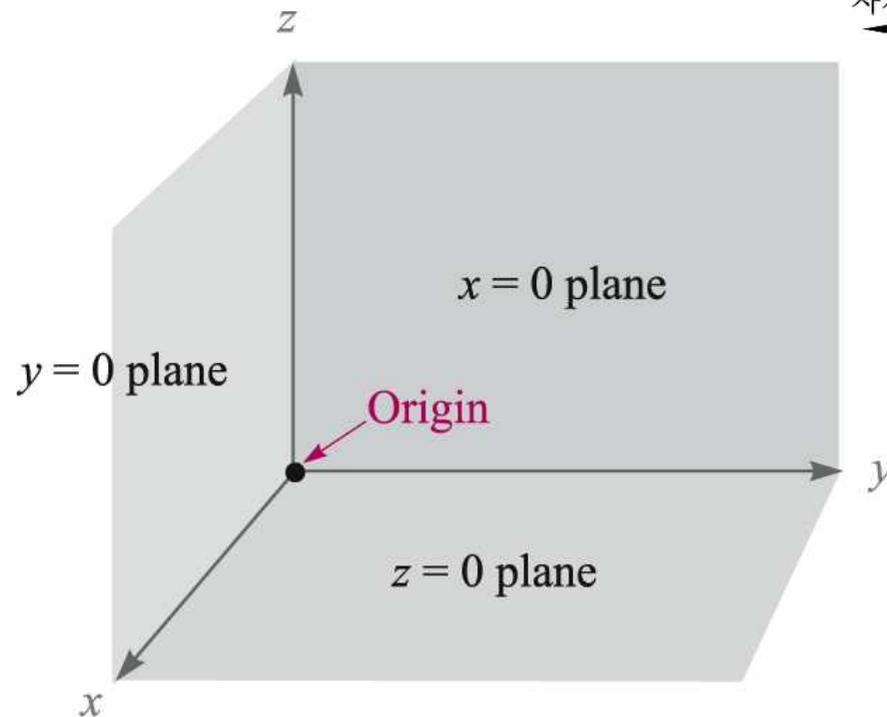
Point Location in Rectangular Coordinates

Ex.] (x_1, y_1, z_1) ➔ $x = x_1, y = y_1, z = z_1$ → intersection of orthogonal three planes
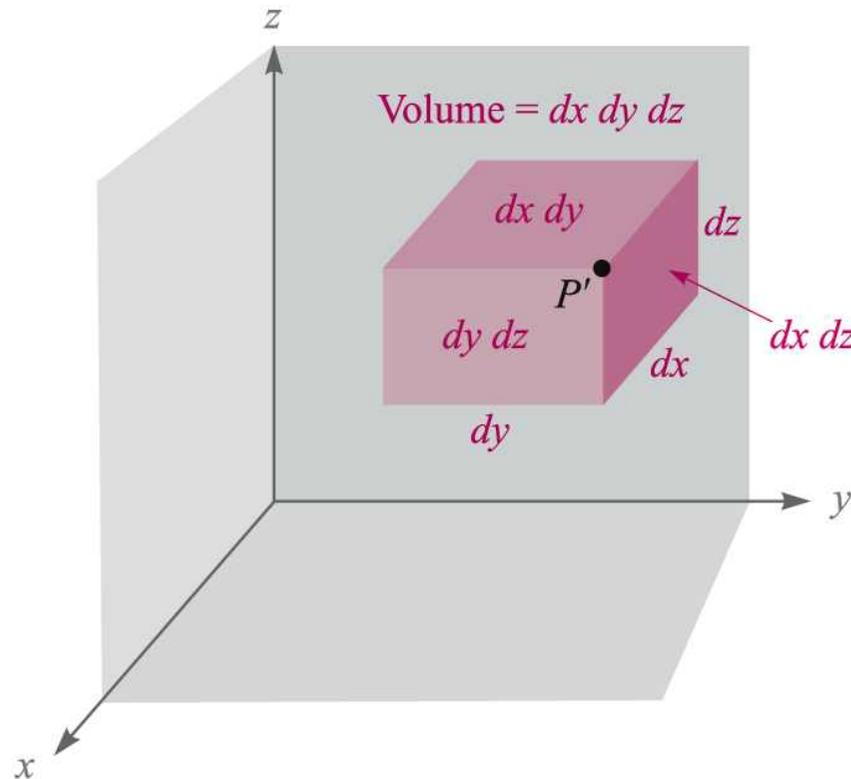
Differential Elements

- Differential lengths: dx, dy, dz

$$dl_{pp'} = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

- Differential areas: $dS = dxdy$, $dydz$, $dzdx$

- Differential volume:
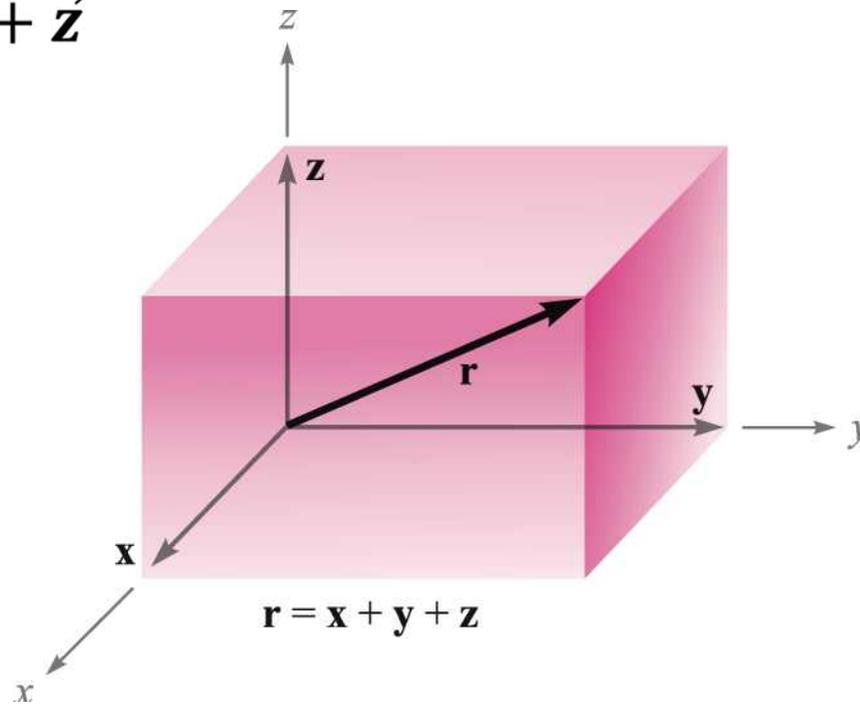
  $dv = dxdydz$

Volume = $dx\,dy\,dz$

1.4 Vector Components and Unit Vectors

- Orthogonal vector components

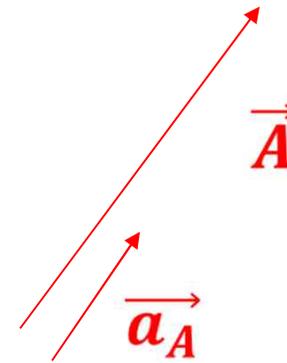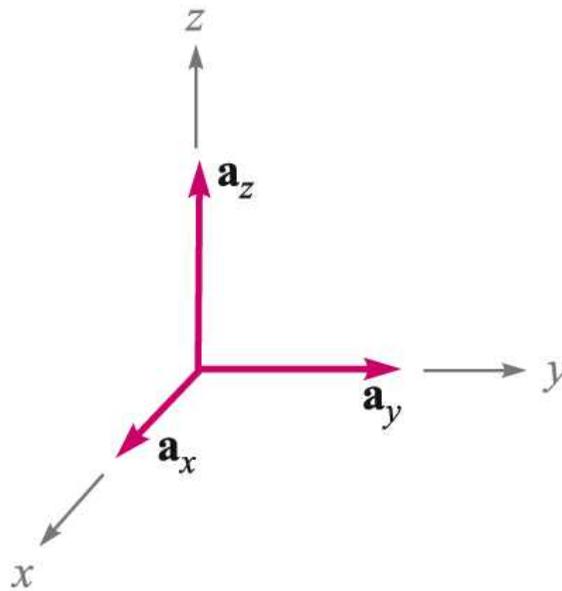$$\vec{x} \perp \vec{y}, \qquad \vec{y} \perp \vec{z}, \qquad \vec{z} \perp \vec{x}$$

$$\vec{r} = \vec{x} + \vec{y} + \vec{z}$$

$$r = x + y + z$$

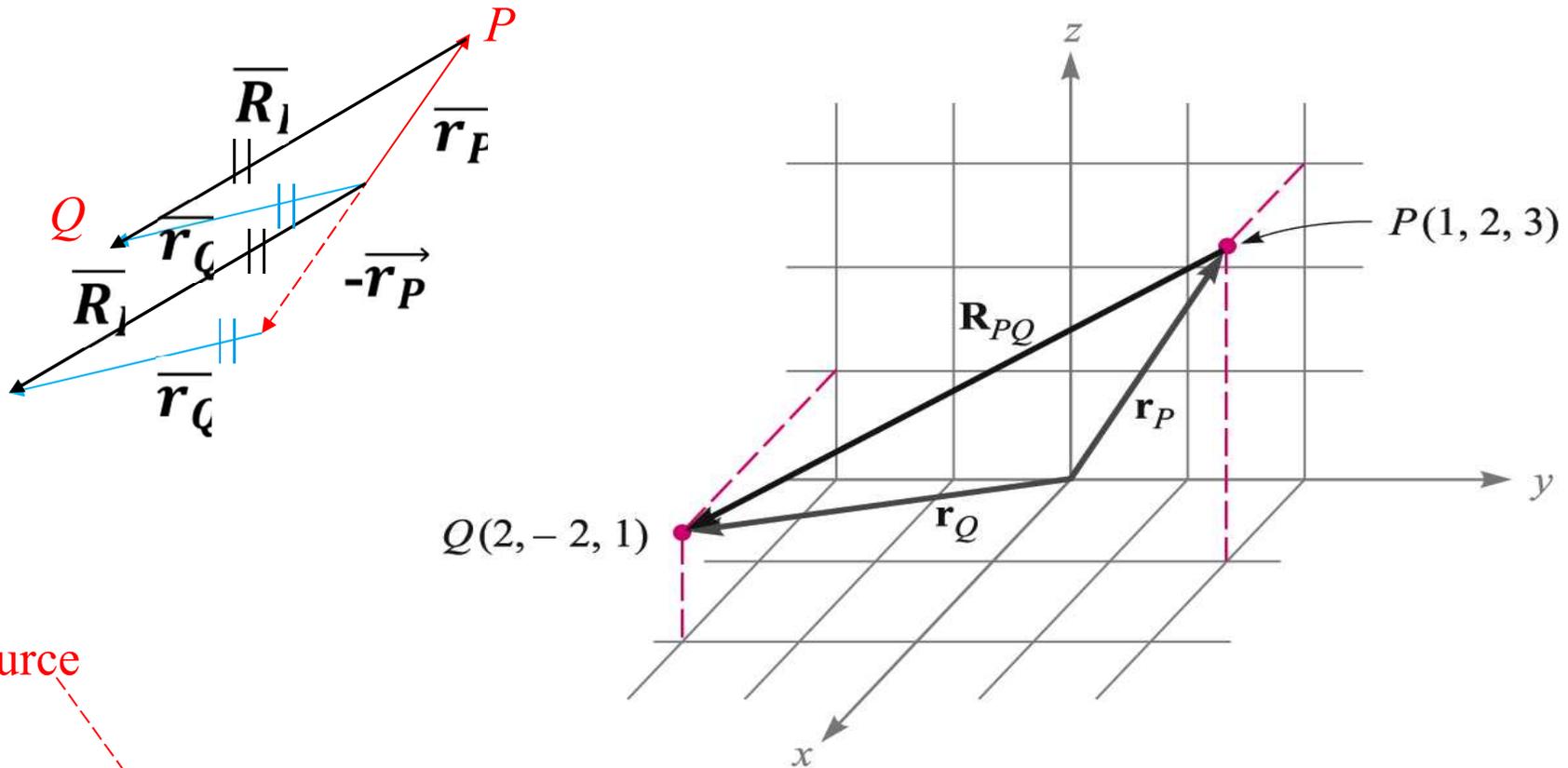- Unit vector: A vector having unit magnitude directed along the direction of the arbitrary vector

Ex.] $\vec{a_x}, \vec{a_y}, \vec{a_z}, \vec{a_A},$ etc,.

· Vector representation in terms of orthogonal rectangular components

$$\mathbf{R}_{PQ} = \mathbf{r}_Q - \mathbf{r}_P = (2-1)\mathbf{a}_x + (-2-2)\mathbf{a}_y + (1-3)\mathbf{a}_z$$

$$= \mathbf{a}_x - 4\mathbf{a}_y - 2\mathbf{a}_z$$

source

target

- Vector expressions in rectangular coordinates

General vector, **B**:
$$\mathbf{B} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z$$

Magnitude of **B**:
$$|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}$$

Unit vector in the direction of **B**:
$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

Example

Specify the unit vector extending from the origin toward the point $G(2, -2, -1)$

$$\mathbf{G} = 2\mathbf{a}_x - 2\mathbf{a}_y - \mathbf{a}_z$$

$$|\mathbf{G}| = \sqrt{(2)^2 + (-2)^2 + (-1)^2} = 3$$

$$\mathbf{a}_G = \frac{\mathbf{G}}{|\mathbf{G}|} = \tfrac{2}{3}\mathbf{a}_x - \tfrac{2}{3}\mathbf{a}_y - \tfrac{1}{3}\mathbf{a}_z = 0.667\mathbf{a}_x - 0.667\mathbf{a}_y - 0.333\mathbf{a}_z$$

1.5 Vector Field

We are accustomed to thinking of a specific vector:

$$\mathbf{v} = v_x \mathbf{a}_x + v_y \mathbf{a}_y + v_z \mathbf{a}_z$$

A vector field is **a *function* defined in space** that has magnitude and direction at all points:

$$\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})\mathbf{a}_x + v_y(\mathbf{r})\mathbf{a}_y + v_z(\mathbf{r})\mathbf{a}_z$$

where $\mathbf{r} = (x,\, y,\, z)$

1.6 The Dot Product (or Scalar Product)

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta_{AB}$$

$$= |\vec{A}||\vec{B}|\cos(-\theta_{BA}) = |\vec{B}||\vec{A}|\cos(\theta_{BA}) = \vec{B} \cdot \vec{A}$$

Commutative Law: $\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$

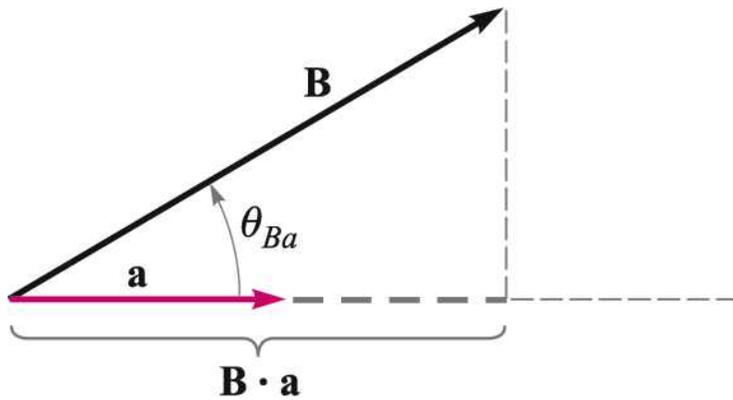$$\vec{A} \cdot \vec{A} = |\vec{A}||\vec{A}|\cos(0) = |\vec{A}|^2$$

$$\vec{a} \cdot \vec{a} = |\vec{a}||\vec{a}|\cos(0) = 1$$

$$\vec{A} \cdot \vec{a} = |\vec{A}||\vec{a}|\cos(\theta_{Aa}) = |\vec{A}|\cos(\theta_{Aa})$$

$$(\vec{A} \cdot \vec{a})\vec{a} = \{|\vec{A}||\vec{a}|\cos(\theta_{Aa})\}\vec{a} = |\vec{A}|\cos(\theta_{Aa})\,\vec{a}$$

▪ (Vector) **Projection** using the dot product: Finding the magnitude component of a vector in given direction

B • a gives the component of **B** in the horizontal direction

(B • a) a gives the *vector* component of **B** in the horizontal direction

Magnitude of $\vec{B}$ in direction of $\vec{a}$

- Operational Use of the Dot Product

Given $\begin{cases} \mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z \\ \mathbf{B} = B_x \mathbf{a}_x + B_y \mathbf{a}_y + B_z \mathbf{a}_z \end{cases}$

Find

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= \left(A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}\right) \cdot \left(B_x \vec{a_x} + B_y \vec{a_y} + B_z \vec{a_z}\right) \\
&= A_x B_x \vec{a_x} \cdot \vec{a_x} + A_x B_y \vec{a_x} \cdot \vec{a_y} + A_x B_z \vec{a_x} \cdot \vec{a_z} \\
&\quad + A_y B_x \vec{a_y} \cdot \vec{a_x} + A_y B_y \vec{a_y} \cdot \vec{a_y} + A_y B_z \vec{a_y} \cdot \vec{a_z} \\
&\quad + A_z B_x \vec{a_z} \cdot \vec{a_x} + A_z B_y \vec{a_z} \cdot \vec{a_y} + A_z B_z \vec{a_z} \cdot \vec{a_z} \\
&= A_x B_x + A_y B_y + A_z B_z
\end{aligned}$$

where $\begin{cases} \mathbf{a}_x \cdot \mathbf{a}_y = \mathbf{a}_y \cdot \mathbf{a}_z = \mathbf{a}_x \cdot \mathbf{a}_z = 0 \\ \mathbf{a}_x \cdot \mathbf{a}_x = \mathbf{a}_y \cdot \mathbf{a}_y = \mathbf{a}_z \cdot \mathbf{a}_z = 1 \end{cases}$

Note also: $\boxed{\mathbf{A} \cdot \mathbf{A} = A^2 = |\mathbf{A}|^2}$

1.7 Cross Product (or Vector Product)

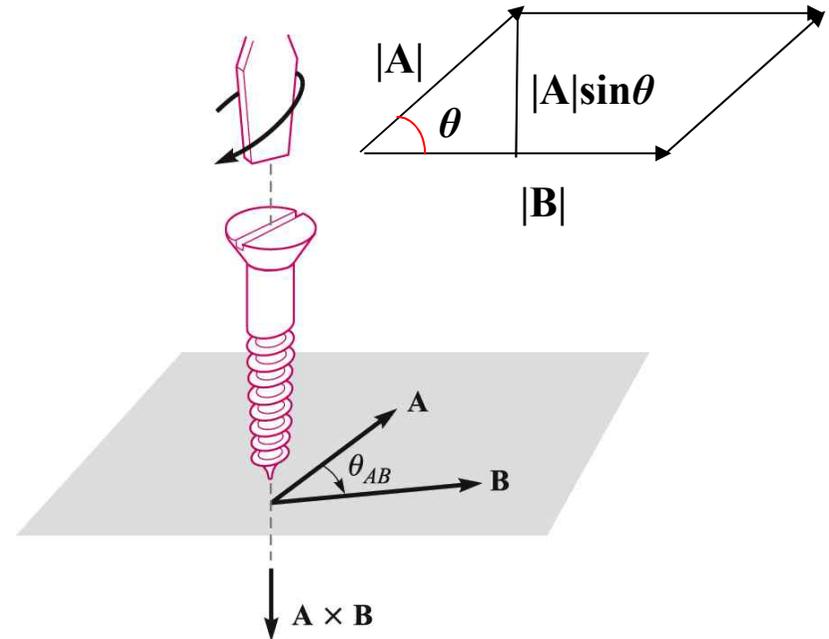$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\,|\mathbf{B}| \sin\theta_{AB}$$

right-handed screw direction

Since $\sin\theta_{AB} = -\sin\theta_{BA}$,

$$\vec{A} \times \vec{B} = -|\vec{A}||\vec{B}| \sin\theta_{BA}\, \overrightarrow{a_N}$$
$$= -|\vec{B}||\vec{A}| \sin\theta_{BA}\, \overrightarrow{a_N}$$
$$= -(\vec{B} \times \vec{A})$$

$$\vec{A} \times \vec{B} = |\vec{A}| sin\theta_{AB} |\vec{B}| \overrightarrow{a_N}$$

← Vector 크기: 평행사변형 면적
방향: 벡터 $\vec{A}$ 와 $\vec{B}$ 의
오른나사 진행방향

1.7.2 Operational Definition of the Cross Product

$$\vec{A} \times \vec{B} = (A_x\vec{a_x} + A_y\vec{a_y} + A_z\vec{a_z}) \times (B_x\vec{a_x} + B_y\vec{a_y} + B_z\vec{a_z})$$

$$= A_xB_x\mathbf{a}_x \times \mathbf{a}_x + A_xB_y\mathbf{a}_x \times \mathbf{a}_y + A_xB_z\mathbf{a}_x \times \mathbf{a}_z$$

$$+ A_yB_x\mathbf{a}_y \times \mathbf{a}_x + A_yB_y\mathbf{a}_y \times \mathbf{a}_y + A_yB_z\mathbf{a}_y \times \mathbf{a}_z$$

$$+ A_zB_x\mathbf{a}_z \times \mathbf{a}_x + A_zB_y\mathbf{a}_z \times \mathbf{a}_y + A_zB_z\mathbf{a}_z \times \mathbf{a}_z$$

where $\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z, \ \mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x, \ \mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$
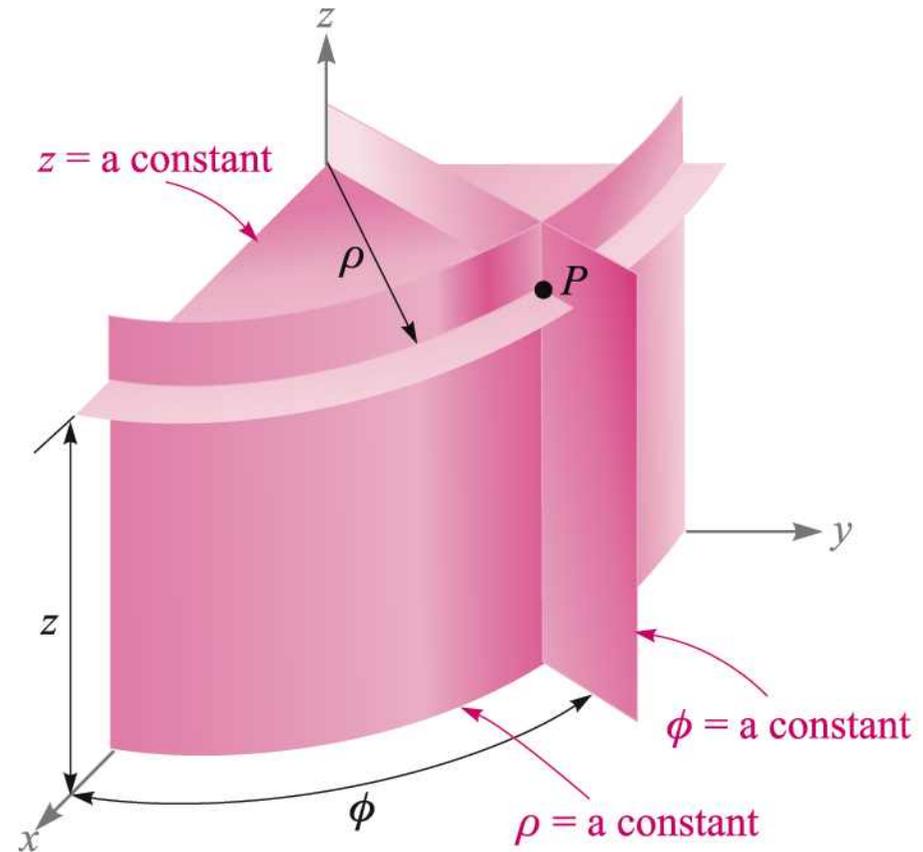
Therefore:

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\mathbf{a}_x + (A_zB_x - A_xB_z)\mathbf{a}_y + (A_xB_y - A_yB_x)\mathbf{a}_z$$

Or…

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$
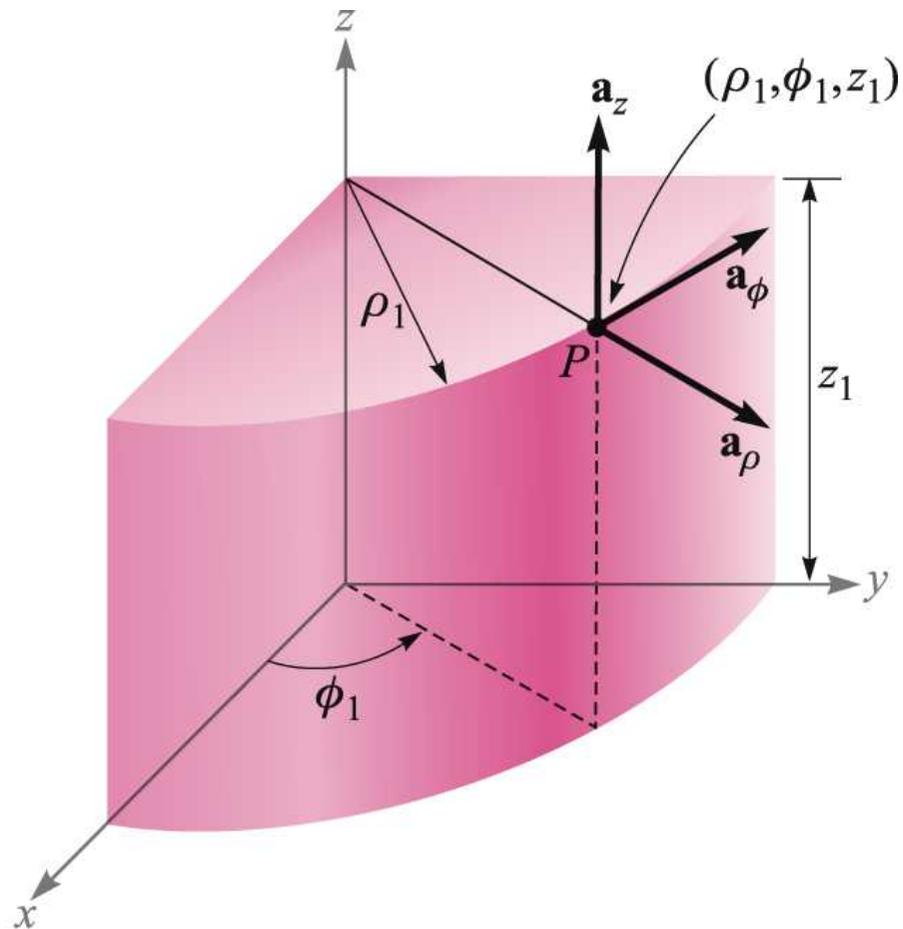
1.8 (Circular) Cylindrical Coordinates

- Point P has coordinately specified by $P(\rho, \phi, z)$.

- Right-handed coordinate

- $\rho \rightarrow \phi \rightarrow z \rightarrow \rho \rightarrow \cdots$

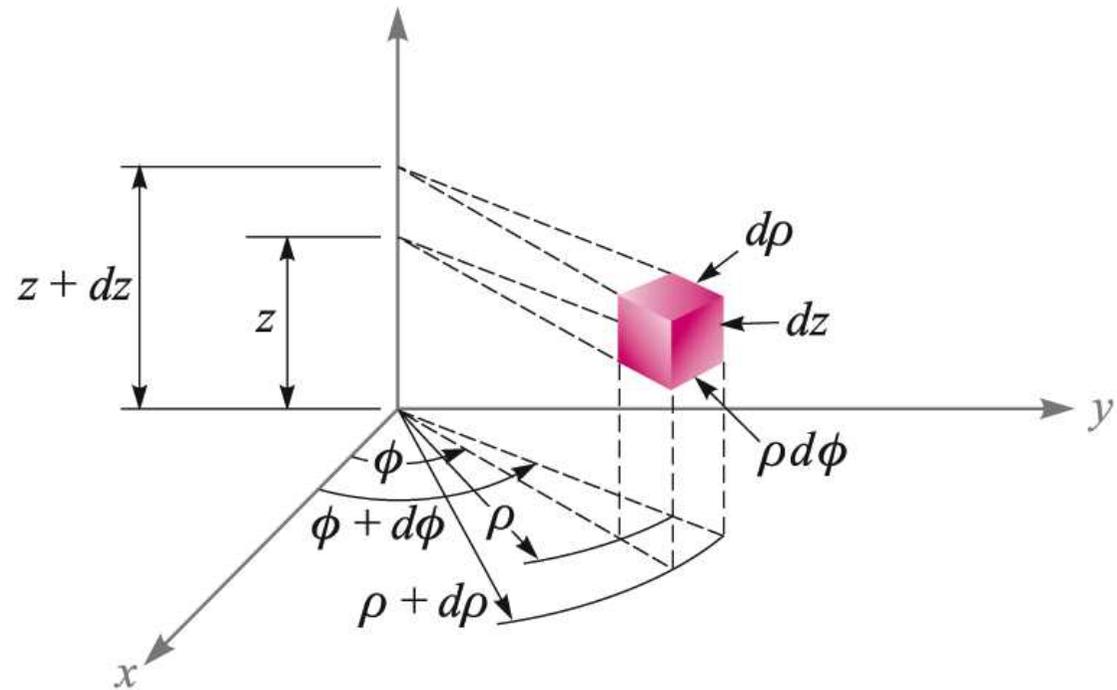- Orthogonal unit vectors in cylindrical coordinates
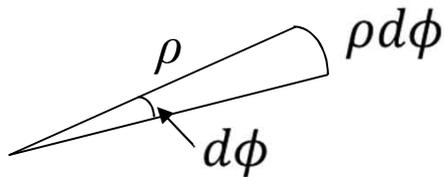
$$\vec{a_\rho} \times \vec{a_\phi} = \vec{a_z}, \quad \vec{a_\phi} \times \vec{a_z} = \vec{a_\rho}, \quad \vec{a_z} \times \vec{a_\rho} = \vec{a_\phi}$$

Differential elements in Cylindrical Coordinates

- Differential lengths: $d\rho$, $\rho d\phi$, dz

- Differential areas: $dS = \rho d\rho d\phi$, $\rho d\phi dz$, $dz d\rho$

- Differential volume:

$$dv = \rho d\rho d\phi dz$$

Relation between Cartesian and cylindrical Coordinates

$$\rho = \sqrt{x^2 + y^2} \quad (\rho \geq 0)$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$z = z$$

$$\frac{\rho \sin\phi}{\rho \cos\phi} = \tan\phi = \frac{y}{x}$$
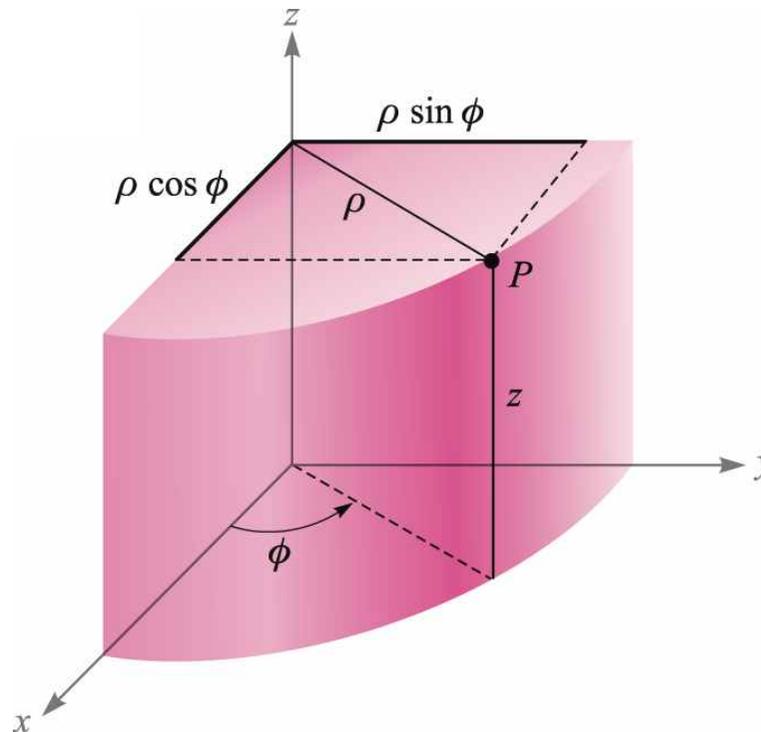
$$x = \rho \cos\phi$$

$$y = \rho \sin\phi$$

$$z = z$$

Ex.] $x = -3$, $y = 4$ $\rightarrow$ 2nd quadrature plane $\rho = 5$, $\phi = 180° - 53.1° = 126.9°$

$x = 3$, $y = -4$ $\rightarrow$ 4th quadrature plane $\rho = 5$, $\phi = -53.1°$

Relation between Cartesian and cylindrical Coordinates

$$\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z} = A_\rho \vec{a_\rho} + A_\phi \vec{a_\phi} + A_z \vec{a_z}$$

where

$$A_\rho = \vec{A} \cdot \vec{a_\rho} = (A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}) \cdot \vec{a_\rho}$$
$$= A_x \vec{a_x} \cdot \vec{a_\rho} + A_y \vec{a_y} \cdot \vec{a_\rho}$$

$$A_\phi = \vec{A} \cdot \vec{a_\phi} = A_x \vec{a_x} \cdot \vec{a_\phi} + A_y \vec{a_y} \cdot \vec{a_\phi}$$

$$A_z = \vec{A} \cdot \vec{a_z} = (A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z}) \cdot \vec{a_z} = A_z$$

$$\overrightarrow{a_\rho} \cdot \overrightarrow{a_x} = \cos \boldsymbol{\phi}$$

$$\overrightarrow{a_\rho} \cdot \overrightarrow{a_y} = \cos(90^\circ - \phi) = \sin \phi$$

$$\overrightarrow{a_\phi} \cdot \overrightarrow{a_x} = -\cos(90^\circ - \phi) = -\sin \phi$$

$$\overrightarrow{a_\phi} \cdot \overrightarrow{a_y} = \sin(90^\circ - \phi) = \cos\phi$$
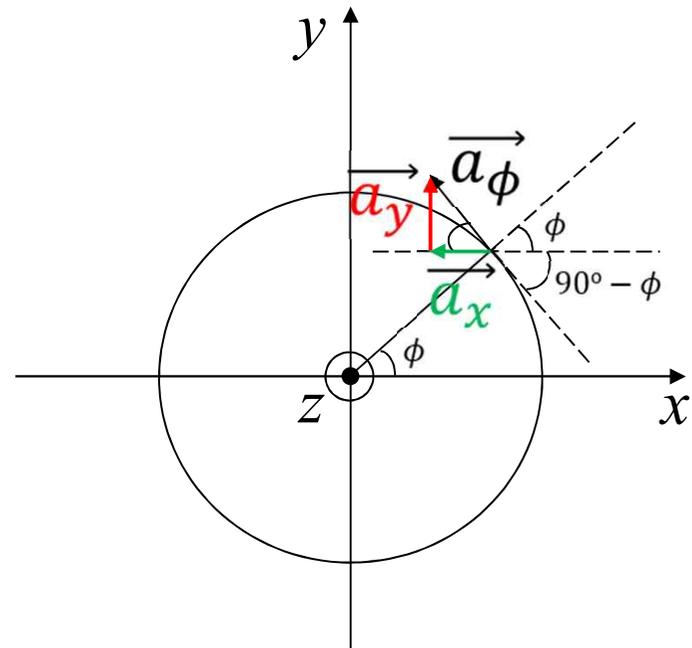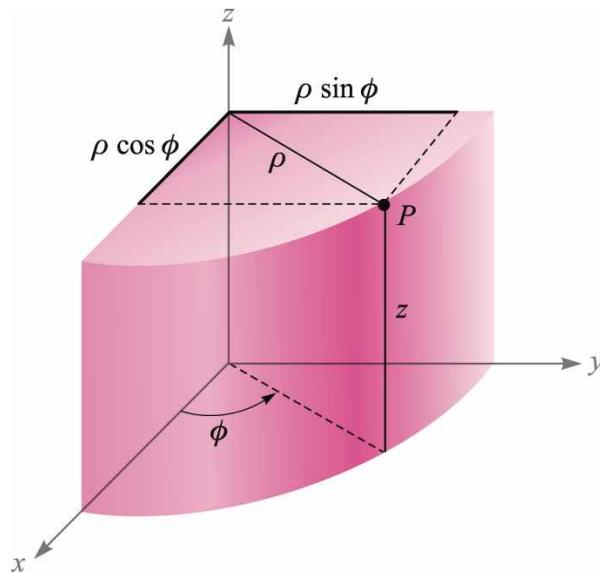
Table 1.1 Dot Products of Unit Vectors in Cylindrical and Rectangular Coordinate Systems

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	**1**

Example

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Start with:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$

into cylindrical coordinates:

Then:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y\cos\phi - x\sin\phi = \rho\sin\phi\cos\phi - \rho\cos\phi\sin\phi = 0$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

$$= -y\sin\phi - x\cos\phi = -\rho\sin^2\phi - \rho\cos^2\phi = -\rho$$

Transform the vector,

$$\mathbf{B} = y\mathbf{a}_x - x\mathbf{a}_y + z\mathbf{a}_z$$
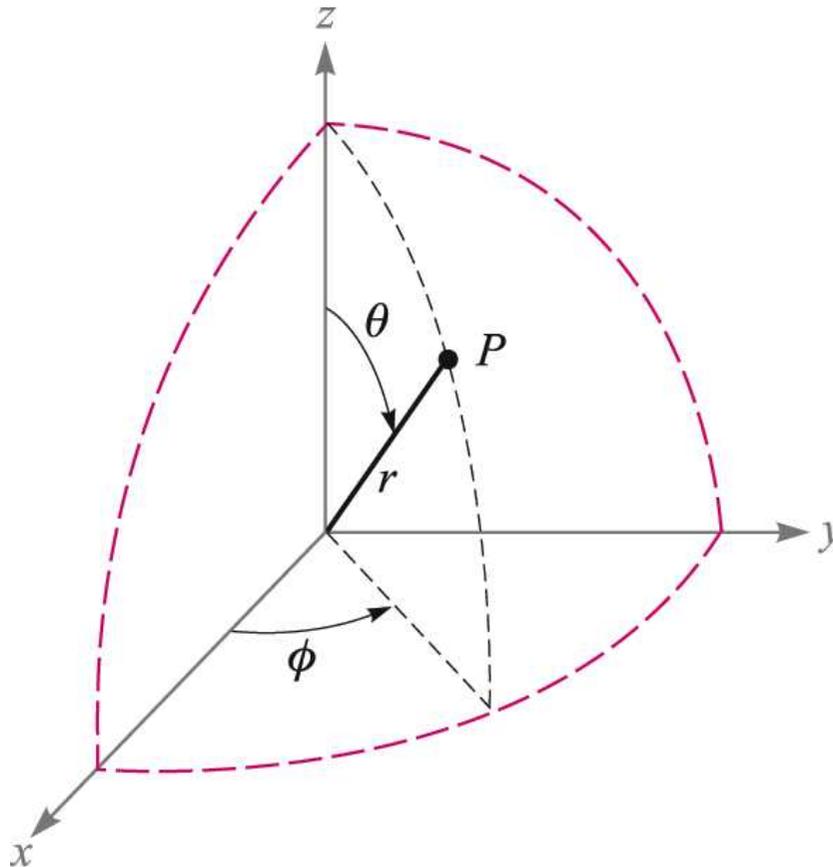
into cylindrical coordinates:

Finally:

$$B_\rho = \mathbf{B} \cdot \mathbf{a}_\rho = y(\mathbf{a}_x \cdot \mathbf{a}_\rho) - x(\mathbf{a}_y \cdot \mathbf{a}_\rho)$$

$$= y \cos \phi - x \sin \phi = \rho \sin \phi \cos \phi - \rho \cos \phi \sin \phi = 0$$

$$B_\phi = \mathbf{B} \cdot \mathbf{a}_\phi = y(\mathbf{a}_x \cdot \mathbf{a}_\phi) - x(\mathbf{a}_y \cdot \mathbf{a}_\phi)$$

$$= -y \sin \phi - x \cos \phi = -\rho \sin^2 \phi - \rho \cos^2 \phi = -\rho$$
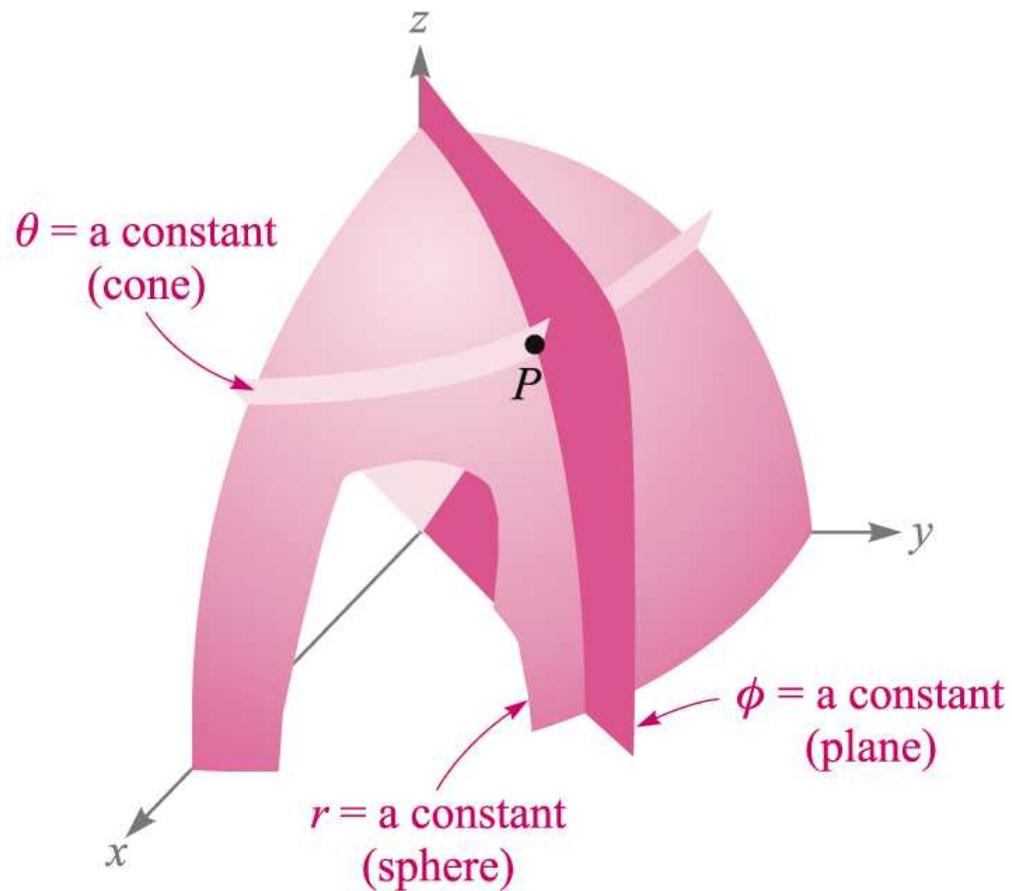
$$\boxed{\mathbf{B} = -\rho\mathbf{a}_\phi + z\mathbf{a}_z}$$

1.9 Spherical Coordinate System

- Point P has coordinately specified by $P(r, \theta, \phi)$.

- Right-handed coordinate $\quad r \rightarrow \theta \rightarrow \phi \rightarrow r \rightarrow \cdots$
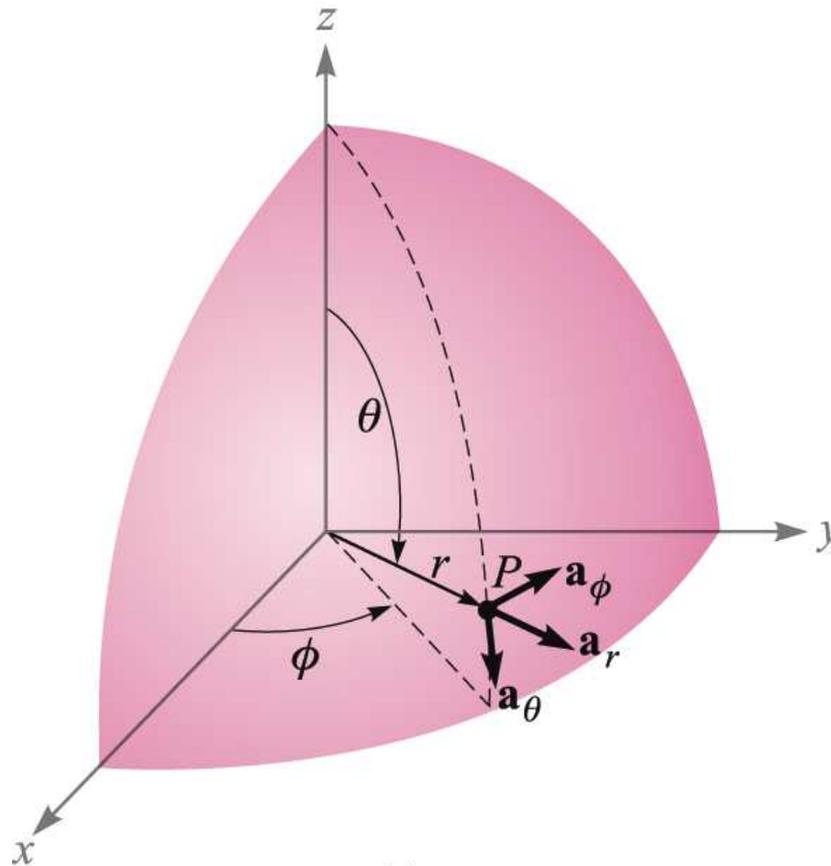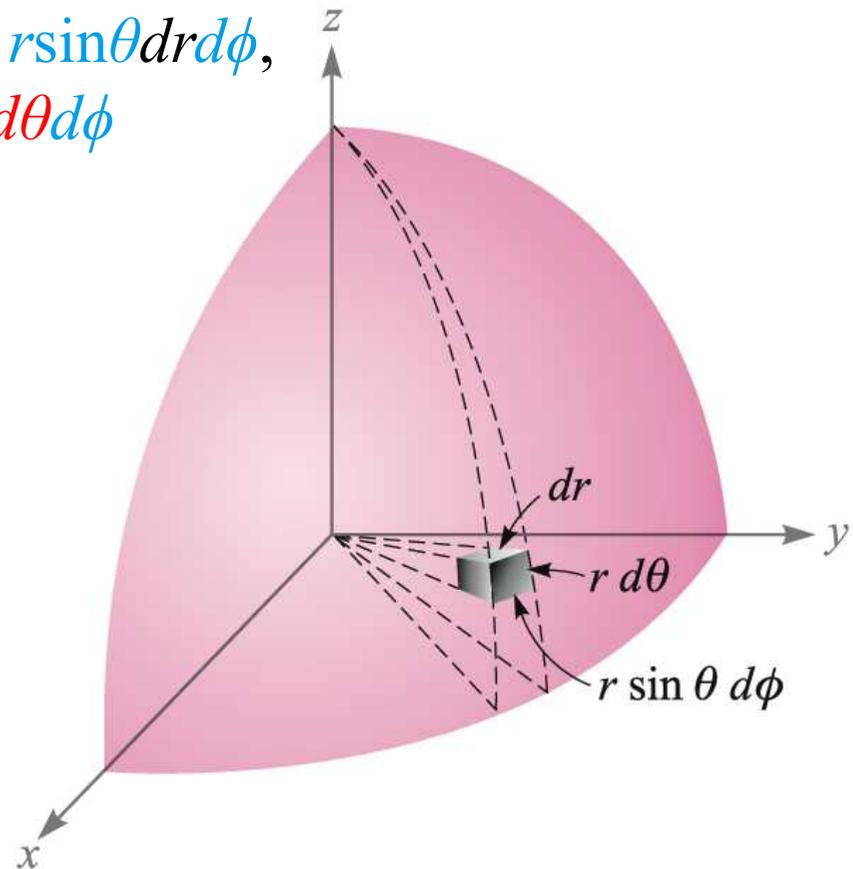
Constant Coordinate Surfaces in Spherical Coordinates
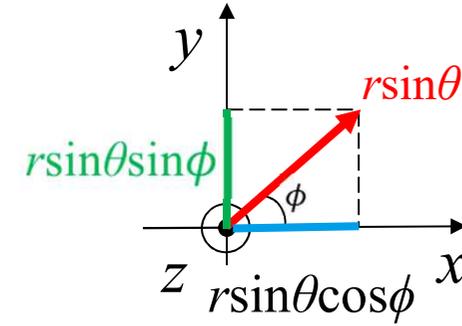
Unit Vector Components in Spherical Coordinates

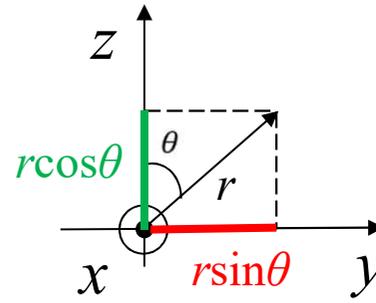$$\vec{a_r} \times \vec{a_\theta} = \vec{a_\phi}, \quad \vec{a_\theta} \times \vec{a_\phi} = \vec{a_r}, \quad \vec{a_\phi} \times \vec{a_r} = \vec{a_\theta}$$

Differential Elements in Spherical Coordinates

- Differential lengths: dr, $r\sin\theta d\phi$
  $$\leftarrow (r + dr)d\theta \approx rd\theta, \; (r+dr)\sin\theta d\phi \approx r\sin\theta d\phi$$

- Differential areas: $dS = rdrd\theta$, $r\sin\theta drd\phi$,
  $$r^2\sin\theta d\theta d\phi$$

- Differential volume:
  $$dv = r^2\sin\theta drd\theta d\phi$$
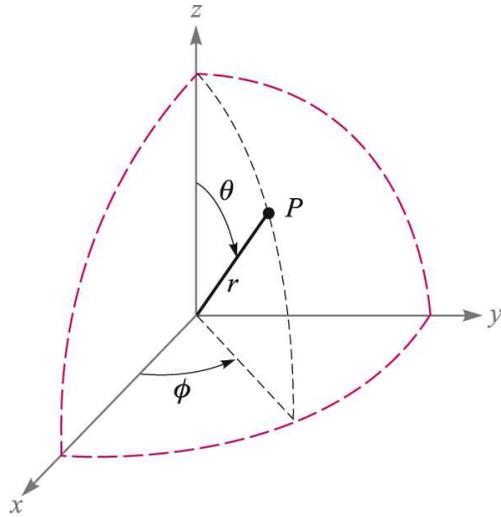
$$x = r \sin \theta \cos \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$

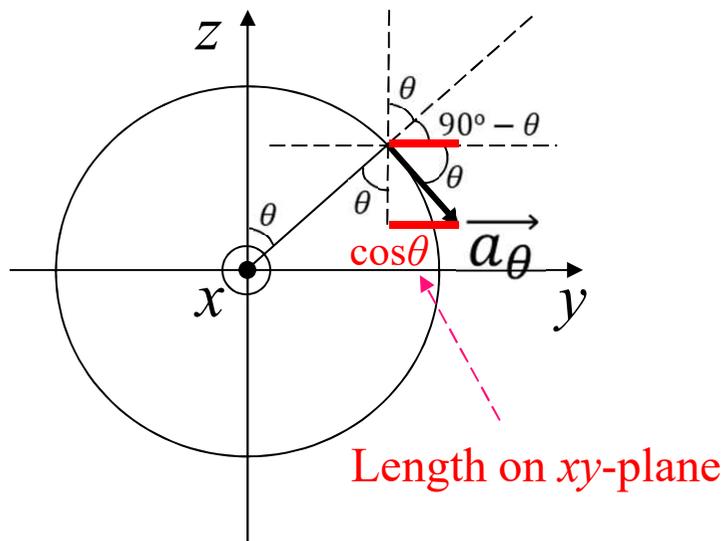$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0° \leq \theta \leq 180°)$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$\vec{A} = A_x \vec{a_x} + A_y \vec{a_y} + A_z \vec{a_z} = A_r \vec{a_r} + A_\theta \vec{a_\theta} + A_\phi \vec{a_\phi}$$

$$\vec{a_r} \cdot \vec{a_x} = \sin\theta \left(\cos\phi \, \vec{a_x} + \sin\phi \, \vec{a_y} \right) \cdot \vec{a_x} = \sin\theta \cos\phi$$

$$\vec{a_r} \cdot \vec{a_y} = \sin\theta \left(\cos\phi \, \vec{a_x} + \sin\phi \, \vec{a_y} \right) \cdot \vec{a_y} = \sin\theta \sin\phi$$

$$\vec{a_r} \cdot \vec{a_z} = \cos\theta \, \vec{a_z} \cdot \vec{a_z} = \cos\theta$$

Length on *xy*-plane

$$\vec{a_\theta} \cdot \vec{a_x} = \cos\theta \cos\phi$$

$$\vec{a_\theta} \cdot \vec{a_y} = \cos\theta \sin\phi$$

$$\vec{a_\theta} \cdot \vec{a_z} = -\cos(90^o - \theta) = \text{-}\sin\theta$$

$$\vec{a_\phi} \cdot \vec{a_x}, \vec{a_\phi} \cdot \vec{a_y}, \vec{a_\phi} \cdot \vec{a_z}: \text{Table 1.1}$$

Table 1.2 Dot Products of Unit Vectors in the Spherical and Rectangular Coordinate Systems

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta \cos\phi$	$\cos\theta \cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta \sin\phi$	$\cos\theta \sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

Example: Vector Component Transformation

Transform the field, $\mathbf{G} = (xz/y)\mathbf{a}_x$, into spherical coordinates and components

$$G_r = \mathbf{G} \cdot \mathbf{a}_r = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_r = \frac{xz}{y}\sin\theta\cos\phi = \frac{r\sin\theta\cos\phi \times r\cos\theta}{r\sin\theta\sin\phi} \times \sin\theta\cos\phi$$

$$= r\sin\theta\cos\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G_\theta = \mathbf{G} \cdot \mathbf{a}_\theta = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\theta = \frac{xz}{y}\cos\theta\cos\phi = \frac{r\sin\theta\cos\phi \times r\cos\theta}{r\sin\theta\sin\phi} \times \cos\theta\cos\phi$$

$$= r\cos^2\theta\frac{\cos^2\phi}{\sin\phi}$$

$$G\phi = \mathbf{G} \cdot \mathbf{a}_\phi = \frac{xz}{y}\mathbf{a}_x \cdot \mathbf{a}_\phi = \frac{xz}{y}(-\sin\phi) = \frac{r\sin\theta\cos\phi \times r\cos\theta}{r\sin\theta\sin\phi} \times (-\sin\phi)$$

$$= -r\cos\theta\cos\phi$$

$$\mathbf{G} = r\cos\theta\cos\phi\,(\sin\theta\cot\phi\,\mathbf{a}_r + \cos\theta\cot\phi\,\mathbf{a}_\theta - \mathbf{a}_\phi)$$