

Chapter 11. The Uniform Plane Wave

11.1 Wave Propagation in (source) Free Space

- Consider wave motion in free space first. ($\rho = 0, J = 0$)

$$\left\{ \begin{array}{ll} \nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} & (1) : \vec{E} \text{ 가 한 점에서 시간적인 변화가 있으면 그 주위에 } \vec{H} \text{ curl 발생.} \\ & \rightarrow \text{closed loop linking} \rightarrow \vec{H} \text{ 도 시간적인 변화를 유도} \\ \nabla \times \vec{E} = -\mu_0 \frac{\partial \vec{H}}{\partial t} & (2) : \vec{H} \text{ 가 한 점에서 시간적인 변화가 있으면 그 주위에 } \vec{E} \text{ curl 발생.} \\ \nabla \cdot \vec{E} = 0 & (3) \rightarrow \text{closed loop linking} \rightarrow \vec{E} \text{ 도 시간적인 변화를 유도} \\ \nabla \cdot \vec{H} = 0 & (4) \Rightarrow \text{원래 source point로부터 약간 이격된 곳에 새로운 field를 유기시킴.} \end{array} \right.$$

- Given the electric vector field:

$$\vec{E} = E_x \vec{a}_x \quad : \text{진폭 방향은 } \vec{a}_x \text{이지만 진행방향은 } \vec{a}_z \rightarrow \vec{E} = E_x(z)e^{\pm jz} \vec{a}_x$$

Using Eq. (2),

$$\nabla \times \vec{E} = \frac{\partial E_x}{\partial z} \vec{a}_y = -\mu_0 \frac{\partial \vec{H}}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t} \vec{a}_y \quad (5)$$

$\Rightarrow \vec{E}$ and $\nabla \times \vec{E}$ are mutually orthogonal.

$$\nabla \times \vec{H} = -\frac{\partial H_y}{\partial z} \vec{a}_x = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \frac{\partial E_x}{\partial t} \vec{a}_x \quad (6) \rightarrow \vec{H} = H_y(z)e^{\pm jz} \vec{a}_y$$

With Eqs. (5) and (6),

$$\frac{\partial E_x}{\partial z} = -\mu_0 \frac{\partial H_y}{\partial t} \quad (7)$$

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t} \quad (8)$$

$\leftrightarrow$ Telegraphist's equations for lossless line (Eq. (20) and (21) in Ch.10)

Differentiate Eq. (7) with respect to z and Eq. (8) with respect to t :

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \frac{\partial^2 H_y}{\partial t \partial z} \quad (9)$$

$$\frac{\partial^2 H_y}{\partial z \partial t} = -\epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \quad (10)$$

Eq. (9) $\leftarrow$ Eq. (10):

$$\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_x}{\partial t^2} \quad (11)$$

: **Wave equation** for x -polarized TEM electric field

$\leftrightarrow$ Analogy to Eq. (13) in Ch. 10

Propagation velocity:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/sec} = c \quad (12)$$

where c : velocity of light in free space

- Differentiate Eq. (7) with respect to t and Eq. (8) with respect to z :

$$\begin{aligned} \frac{\partial^2 E_x}{\partial z \partial t} &= -\mu_0 \frac{\partial^2 H_y}{\partial t^2} \\ \frac{\partial^2 H_y}{\partial z^2} &= -\epsilon_0 \frac{\partial^2 E_x}{\partial t \partial z} = \mu_0 \epsilon_0 \frac{\partial^2 H_y}{\partial t^2} \end{aligned} \quad (13)$$

: **Wave equation** for y -polarized TEM magnetic field

- General electric field solution of Eq. (11):

= forward-propagating waves + backward-propagating waves

$$E_x(z, t) = f_1(t - z/v) + f_2(t + z/v) \quad (14)$$

Since the waves are sinusoidal, electric field can be expressed with a phase velocity (v_p) as below.

$$\begin{aligned} E_x(z, t) &= E_x(z, t) + E'_x(z, t) \\ &= |E_{x0}| \cos[\omega(t - z/v_p) + \phi_1] + |E'_{x0}| \cos[\omega(t + z/v_p) + \phi_2] \\ &= \underbrace{|E_{x0}| \cos[\omega t - k_0 z + \phi_1]}_{\text{forward z travel}} + \underbrace{|E'_{x0}| \cos[\omega t + k_0 z + \phi_2]}_{\text{backward z travel}} \end{aligned} \quad (15)$$

$$\text{where } k_0 \equiv \frac{\omega}{c} \text{ [rad/m]} \quad (16)$$

: wavenumber in free space

ω : radian time frequency (or angular frequency) [rad/s]

k_0 : spatial frequency [rad/m] $\leftrightarrow$ phase constant

Wavelength: distance in free space over which the special phase shifts by 2π

$$k_0 z = k_0 \lambda = 2\pi \quad \rightarrow \quad \lambda = \frac{2\pi}{k_0} \text{ (free space)} \quad (17)$$

- Condition for considering the m^{th} crest of the wave:

$$k_0 z = 2m\pi$$

For periodic cosine waveform, the entire cosine argument be the same multiple of 2π for all time.

$$\omega t - k_0 z = \omega t - (\omega / c) z = \omega(t - z / c) = 2m\pi \quad (18)$$

If $t \uparrow$, then $z \uparrow$. $\rightarrow$ moving positive z direction with phase velocity c (in free space)

$$\omega t + k_0 z = \omega(t + z / c) = 2m\pi$$

If $t \uparrow$, then $z \downarrow$. $\rightarrow$ moving negative z direction with phase velocity c (in free space)

• Real instantaneous fields in terms of their phasor forms:

$$E_x(z, t) = \frac{1}{2} \underbrace{|E_{x0}| e^{j\phi}}_{E_{x0}} e^{-jk_0 z} e^{j\omega t} + c.c. = \frac{1}{2} E_{xs} e^{j\omega t} + c.c. = \text{Re} [E_{xs} e^{j\omega t}] \quad (19)$$

where $E_{xs} = E_{x0} e^{-jk_0 z}$: phasor electric field

$E_{x0} = |E_{x0}| e^{j\phi}$: complex amplitude

$c.c.$: complex conjugate

$$[\text{Ex. 11.1}] \quad E_y(z, t) = 100 \cos(10^8 t - 0.5z + 30^\circ) \quad [\text{V/m}]$$

$$\xrightarrow{\text{phasor}} E_{ys}(z) = 100 e^{-j(0.5z - 30^\circ)}$$

$$[\text{Ex. 11.2}] \quad \text{Magnitude } \vec{E}_s = 100 \vec{a}_x + 20 \angle 30^\circ \vec{a}_y \quad [\text{V/m}]$$

$$= 100 e^{j0^\circ} \vec{a}_x + 20 e^{j30^\circ} \vec{a}_y, \quad f = 10^7 \text{ Hz}$$

$$k_0 = \omega / c = (2\pi \times 10^7) / (3 \times 10^8) = 0.21 \text{ rad/m}$$

$$\begin{aligned}
\vec{E}(z,t) &= \text{Re}[100e^{-j0.21z}e^{j2\pi \times 10^7 t}\vec{a}_x + 20e^{j30^\circ}e^{-j0.21z}e^{j2\pi \times 10^7 t}\vec{a}_y] \\
&= \text{Re}[100e^{j(2\pi \times 10^7 t - 0.21z)}\vec{a}_x + 20e^{j(2\pi \times 10^7 t - 0.21z + 30^\circ)}\vec{a}_y] \\
&= 100\cos(2\pi \times 10^7 t - 0.21z)\vec{a}_x \\
&\quad + 20\cos(2\pi \times 10^7 t - 0.21z + 30^\circ)\vec{a}_y
\end{aligned}$$

- Taking the partial derivative of any field quantity with respect to time is equivalent to multiplying the corresponding phasor by ($j\omega$):

$$\frac{\partial H_y}{\partial z} = -\epsilon_0 \frac{\partial E_x}{\partial t} \quad (20)$$

$$\text{where } E_x(z,t) = \frac{1}{2} E_{xs}(z)e^{j\omega t} + c.c. \text{ and } H_y(z,t) = \frac{1}{2} H_{ys}(z)e^{j\omega t} + c.c. \quad (21)$$

$$\frac{dH_{ys}(z)}{dz} = -j\omega\epsilon_0 E_{xs}(z) \quad (22)$$

- Maxwell's phasor form equations (in free space).

$$\left\{ \begin{array}{l} \nabla \times \vec{H}_s = j\omega\epsilon_0 \vec{E}_s \end{array} \right. \quad (23)$$

$$\left\{ \begin{array}{l} \nabla \times \vec{E}_s = -j\omega\mu_0 \vec{H}_s \end{array} \right. \quad (24)$$

$$\left\{ \begin{array}{l} \nabla \cdot \vec{E}_s = 0 \end{array} \right. \quad (25)$$

$$\left\{ \begin{array}{l} \nabla \cdot \vec{H}_s = 0 \end{array} \right. \quad (26)$$

- Wave equation in free space:

$$\nabla \times \vec{E}_s = -j\omega\mu_0\vec{H}_s$$

$$\nabla \times \nabla \times \vec{E}_s = (-j\omega\mu_0)\nabla \times \vec{H}_s$$

$$\nabla(\nabla \cdot \vec{E}_s) - \nabla^2 \vec{E}_s = (-j\omega\mu_0)(j\omega\epsilon_0)\vec{E}_s \quad (27)$$

$$-\nabla^2 \vec{E}_s = \omega^2 \mu_0 \epsilon_0 \vec{E}_s \quad (\because \nabla \cdot \vec{E}_s = 0)$$

$$\therefore \nabla^2 \vec{E}_s = -\omega^2 \mu_0 \epsilon_0 \vec{E}_s$$

$$\nabla^2 \vec{E}_s = -k_0^2 \vec{E}_s \quad : (\text{vector}) \text{ Helmholtz equation} \quad (28)$$

where $k_0 = \omega\sqrt{\mu_0\epsilon_0}$: free space wavenumber

- One-dimensional form: $\nabla^2 \vec{E}_{xs} = -k_0^2 \vec{E}_{xs} \quad (29)$

($\vec{E}$ - field 가 $\vec{x}$ 방향성분만 가지고 있다고 가정한 것.)

- Expansion of the operator leads to the second-order partial differential equation:

$$\frac{\partial^2 E_{xs}}{\partial x^2} + \frac{\partial^2 E_{xs}}{\partial y^2} + \frac{\partial^2 E_{xs}}{\partial z^2} = -k_0^2 E_{xs} \quad : \vec{E} = E_x(z)e^{\pm jz}\vec{a}_x$$

If E_{xs} does not vary with x or y ,

$$\frac{d^2 E_{xs}}{dz^2} = -k_0^2 E_{xs} \quad (30) \quad (\text{편미분} \rightarrow \text{상미분})$$

$$\Rightarrow E_{xs} = E_{x0}e^{-jk_0z} + E'_{x0}e^{jk_0z} \quad (31)$$

where E_{x0} and E'_{x0} : real

$$\begin{aligned} \nabla \times \vec{E} &= -\mu_0 \frac{\partial \vec{H}}{\partial t} \\ \nabla \times \nabla \times \vec{E} &= -\mu_0 \nabla \times \frac{\partial \vec{H}}{\partial t} \\ \nabla(\nabla \cdot \vec{E}) - \nabla^2 \vec{E} &= -\mu_0 \frac{\partial}{\partial t} (\nabla \times \vec{H}) \\ &= -\mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} \\ \therefore \nabla^2 \vec{E} - \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2} &= 0 \end{aligned}$$

$\vec{E}$ - field 를 공간적 성격과
시간적인 성격으로 분리시킨 식.

- Determine the form of the $\vec{H}$ – field

$$\nabla \times \vec{E}_s = -j\omega\mu_0\vec{H}_s \quad (24)$$

Simplify for a simple E_{xs} component varying only with z

$$\frac{\partial E_{xs}}{\partial z} = -j\omega\mu_0 H_{ys} \quad \leftarrow (31)$$

$$\begin{aligned} \nabla \times \vec{E} &= \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) \vec{a}_x + \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \vec{a}_y \\ &\quad + \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) \vec{a}_z \\ \vec{E} &= E_x(z) \vec{a}_x \\ \nabla \times \vec{E} &= \frac{\partial E_x}{\partial z} \vec{a}_y = \frac{\partial E_x}{\partial z} \vec{a}_y \end{aligned}$$

$$\begin{aligned} H_{ys} &= \frac{1}{-j\omega\mu_0} [(-jk_0)E_{x0}e^{-jk_0z} + (jk_0)E'_{x0}e^{jk_0z}] \leftarrow k_0 = \omega\sqrt{\mu_0\epsilon_0} \\ &= E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{-jk_0z} - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}e^{jk_0z} = H_{y0}e^{-jk_0z} + H'_{y0}e^{jk_0z} \quad (32) \end{aligned}$$

Instantaneous form of magnetic field:

$$H_y(z, t) = E_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t - k_0z) - E'_{x0}\sqrt{\frac{\epsilon_0}{\mu_0}}\cos(\omega t + k_0z) \quad (33)$$

- By comparing the electric and magnetic field amplitudes of the forward-propagating wave in free space,

$$E_{x0} = \sqrt{\frac{\mu_0}{\epsilon_0}}H_{y0} = \eta_0 H_{y0} \quad (34a)$$

Backward propagating wave:

$$E'_{x0} = -\sqrt{\frac{\mu_0}{\epsilon_0}}H'_{y0} = -\eta_0 H'_{y0} \quad (34b)$$

Intrinsic impedance of free space (η [eta]):

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 377 \cong 120\pi \quad [\Omega] \quad (35)$$

$$= E_{xs} / H_{ys}$$

$\Rightarrow$ in-phase in space as well as time

$\Rightarrow$ constant

$\Rightarrow$ analogy to the characteristic impedance, Z_0

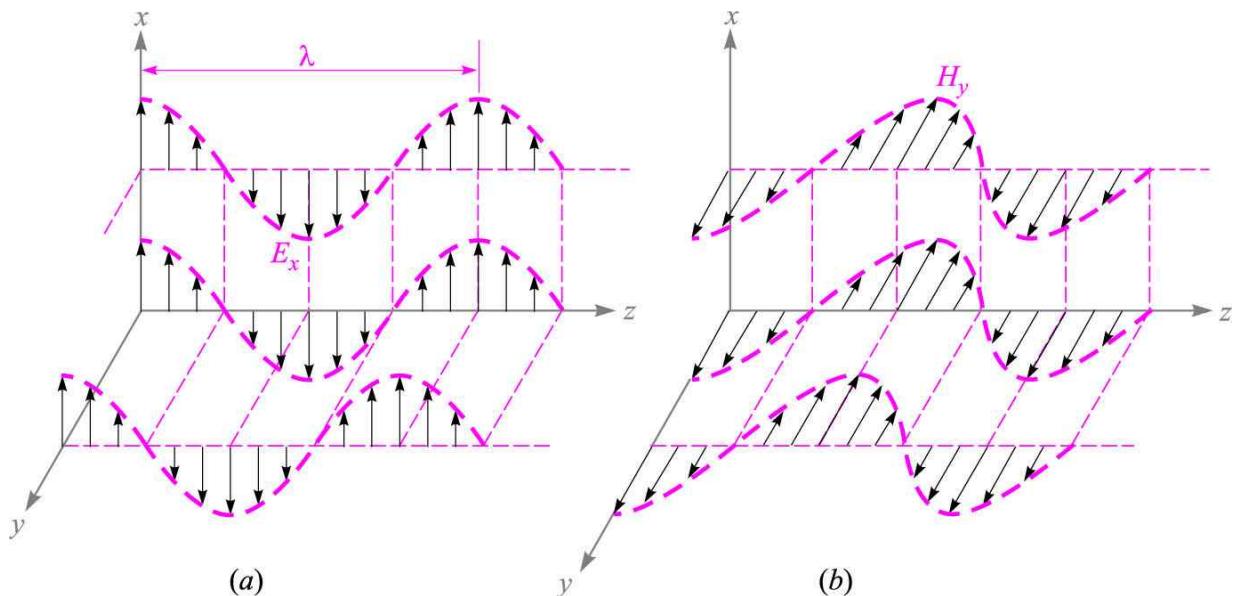
• Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H}$$

• Uniform plane wave: The magnetic field intensity $\vec{H}$ lies in a plane normal to propagation direction ($\vec{z}$), and is perpendicular to $\vec{E}$

$\Rightarrow$ TEM-wave (Transverse Electromagnetic wave)

$$\Rightarrow (\vec{z} \perp \vec{E}, \vec{z} \perp \vec{H}, \vec{E} \perp \vec{H})$$



11.2 Wave Propagation in Dielectrics

- Propagation of uniform plane wave in a perfect (lossless) dielectric of permittivity ϵ and permeability μ which medium is isotropic and homogeneous

- Helmholtz equation:

$$\nabla^2 \vec{E}_s = -k^2 \vec{E}_s \quad (36)$$

$$\begin{aligned} &= -\omega^2 \mu \epsilon \vec{E}_s = -\omega^2 \mu_r \mu_0 \epsilon_r \epsilon_0 \vec{E}_s = -\omega^2 \mu_0 \epsilon_0 \mu_r \epsilon_r \vec{E}_s \\ &= -k_0^2 \mu_r \epsilon_r \vec{E}_s \quad \leftarrow k_0^2 = \omega^2 \mu_0 \epsilon_0 \end{aligned}$$

$$k = \omega \sqrt{\mu \epsilon} = \omega \sqrt{\mu_0 \epsilon_0} \sqrt{\mu_r \epsilon_r} = k_0 \sqrt{\mu_r \epsilon_r} \quad (37)$$

- For E_{xs} ,

$$\frac{d^2 E_{xs}}{dz^2} = -k^2 E_{xs} \quad (38)$$

- General solution:

$$E_{xs} = E_{x0} e^{-jkz} = E_{x0} e^{-\alpha z} e^{-j\beta z} \quad (40) \quad \vec{z} \text{ 방향으로 } \vec{E} - \text{field가 전달되면서 감쇄 성분도 함께 표현.}$$

$$\xrightarrow{\text{instantaneous}} E_x = E_{x0} e^{-\alpha z} \cos(\omega t - \beta z) \quad (41) \quad \text{lossy propagation model}$$

$$\text{where } \begin{cases} \alpha : \text{attenuation constant.} \\ \beta : \text{phase constant} \end{cases} \quad \alpha = 0 \text{ for lossless.}$$

$$\rightarrow jk = \alpha + j\beta$$

$e^{-\alpha z}$:

where α : positive $\rightarrow$ attenuation
negative $\rightarrow$ gain

- Physical processes in material can affect the wave electric field and be described complex permittivity as.

$$\varepsilon = \varepsilon' - j\varepsilon'' = \varepsilon_0(\varepsilon_r' - j\varepsilon_r'') \quad (42)$$

- Losses arising from the response of the medium (e.g. ferromagnetic material or ferrite) to the magnetic field are modeled as

$$\mu = \mu' - j\mu'' = \mu_0(\mu_r' - j\mu_r'')$$

Ex.] Ferrimagnetic material (or Ferrites)

$\rightarrow$ Very weaker than that of dielectric material $\mu \approx \mu_0$.

- Eq. (37) $\leftarrow$ Eq. (42)

$$k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu(\varepsilon' - j\varepsilon'')} = \omega\sqrt{\mu\varepsilon'}\sqrt{1 - j\frac{\varepsilon''}{\varepsilon'}} \quad (43)$$

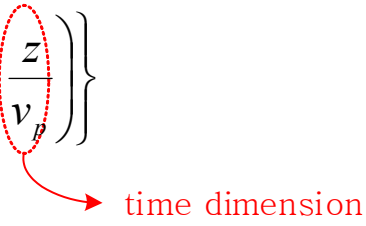
$$\alpha = \text{Re}\{jk\} = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} - 1\right)^{1/2} \quad (44)$$

$$\beta = \text{Im}\{jk\} = \omega\sqrt{\frac{\mu\varepsilon'}{2}}\left(\sqrt{1 + \left(\frac{\varepsilon''}{\varepsilon'}\right)^2} + 1\right)^{1/2} \quad (45)$$

where $\varepsilon''/\varepsilon'$: loss tangent (if $\varepsilon''/\varepsilon' = 0$, then $\alpha = 0$)

- From electric field intensity

$$\begin{aligned}
 E_x &= E_{x0} e^{-\alpha z} \cos(\omega t - \beta z) \\
 &= E_{x0} e^{-\alpha z} \cos \left\{ \omega \left(t - \frac{\beta}{\omega} z \right) \right\} = E_{x0} e^{-\alpha z} \cos \left\{ \omega \left(t - \frac{z}{(\omega / \beta)} \right) \right\} \\
 &= E_{x0} e^{-\alpha z} \cos \left\{ \omega \left(t - \frac{z}{v_p} \right) \right\}
 \end{aligned}$$


 time dimension

$$\therefore v_p = \frac{\omega}{\beta} \quad (46) \quad : \text{phase velocity}$$

- Wavelength: distance required to effect a phase change of 2π

$$\begin{aligned}
 \beta \lambda &= 2\pi \\
 \lambda &= \frac{2\pi}{\beta}
 \end{aligned} \quad (47)$$

- By uniform plane wave property,

$$H_{ys} = \frac{E_{x0}}{\eta} e^{-\alpha z} e^{-j\beta z}$$

$$\text{where } \eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu}{\epsilon' - j\epsilon''}} = \sqrt{\frac{\mu}{\epsilon'}} \frac{1}{\sqrt{1 - j(\epsilon''/\epsilon')}} \quad (48)$$

: intrinsic impedance in dielectric

➔ Electric and magnetic fields are no longer in phase.

- Lossless medium (or perfect dielectric): $\varepsilon'' = 0$ and $\varepsilon = \varepsilon'$

➔ From Eq. (44), $\alpha = 0$

$$\text{From Eq. (45), } \beta = \omega\sqrt{\mu\varepsilon'} \quad (\text{lossless medium}) \quad (49)$$

- With $\alpha = 0$,

$$E_x = E_{x0} \cos(\omega t - \beta z)$$

- Phase velocity and wavelength in medium except free space:

$$v_p = \frac{\omega}{\beta} = \frac{\omega}{\omega\sqrt{\mu\varepsilon}} = \frac{1}{\sqrt{\mu_0\varepsilon_0}\sqrt{\mu_r\varepsilon_r}} = \frac{\overset{\text{speed of light}}{c}}{\sqrt{\mu_r\varepsilon_r}} < c$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\omega\sqrt{\mu\varepsilon'}} = \frac{1}{f\sqrt{\mu\varepsilon'}} = \frac{c}{f\sqrt{\mu_r\varepsilon_r}} = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}} < \lambda_0$$

- Magnetic field associated with E_x :

$$H_y = \frac{E_{x0}}{\eta} \cos(\omega t - \beta z)$$

$$\text{Where } \eta = \sqrt{\frac{\mu}{\varepsilon}} \quad (52)$$

[Ex. 11.3] 1 MHz plane wave propagating in fresh water.

Neglect losses in fresh water ($\epsilon'' = 0$)

$$\mu_r = 1, \quad \epsilon_r = \epsilon'_r = 81$$

$$\Rightarrow \beta = \omega \sqrt{\mu \epsilon'} = \omega \sqrt{\mu_0 \epsilon_0} \sqrt{\epsilon_r} = \frac{\omega \sqrt{\epsilon_r}}{c} = \frac{2\pi \times 10^6 \times \sqrt{81}}{3 \times 10^8} = 0.19 \quad [\text{rad/m}]$$

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{0.19} = 33 \text{ [m]} \quad < \quad 300 \text{ [m]} = \lambda_0$$

$$v_p = \frac{\omega}{\beta} = \frac{2\pi \times 10^6}{0.19} = 3.3 \times 10^7 \text{ [m/sec]} \quad < \quad 3 \times 10^8 \text{ [m/sec]} = v_{p0}$$

$$\eta = \sqrt{\frac{\mu}{\epsilon'}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \cdot \frac{1}{\sqrt{\epsilon_r}} = \frac{377}{\sqrt{81}} = 42 \text{ } [\Omega]$$

If $E_{x0} = 0.1 \text{ [V/m]}$,

$$E_x = 0.1 \cos(2\pi 10^6 t - 0.19z) \text{ [V/m]}$$

$$H_y = \frac{E_x}{\eta} = \frac{0.1}{42} \cos(2\pi \times 10^6 t - 0.19z) = 2.4 \times 10^{-3} \cos(2\pi \times 10^6 t - 0.19z) \text{ [A/m]}$$

[Ex. 11.4] In previous example, $f = 2.5 \text{ GHz}$, $\epsilon_r' = 78$, $\epsilon_r'' = 7$

(Dielectric loss can't be ignored.)

$$\begin{aligned} \Rightarrow \alpha &= \omega \sqrt{\frac{\mu \epsilon'}{2}} \left(\sqrt{1 + \left(\frac{\epsilon_r''}{\epsilon_r'} \right)^2} - 1 \right)^{1/2} = \omega \frac{\sqrt{\mu_0 \epsilon_0} \sqrt{\epsilon_r'}}{\sqrt{2}} \left(\sqrt{1 + \left(\frac{\epsilon_r''}{\epsilon_r'} \right)^2} - 1 \right)^{1/2} \\ &= \frac{\omega \sqrt{\epsilon_r'}}{\sqrt{2} c} \left(\sqrt{1 + \left(\frac{\epsilon_r''}{\epsilon_r'} \right)^2} - 1 \right)^{1/2} \end{aligned}$$

$$= \frac{2\pi \times 2.5 \times 10^9 \times \sqrt{78}}{\sqrt{2} \times 3 \times 10^8} \left(\sqrt{1 + \left(\frac{7}{78}\right)^2} - 1 \right)^{1/2} = 21 \quad [\text{Np/m}]$$

- Frequency of microwave oven: 2.5 GHz.

$$e^{-\alpha z} = e^{-1} \quad z = \frac{1}{\alpha} = \frac{1}{21} = 0.048 \text{ m} : \text{penetration depth.}$$

$= 0.3679$

- If $f > 2.5 \text{ GHz}$, $\epsilon'' \uparrow$. But $z = \frac{1}{\alpha} \downarrow \downarrow \Rightarrow$ Too much power is needed.

$$f < 2.5 \text{ GHz}, \quad z = \frac{1}{\alpha} \uparrow \quad \epsilon'' \downarrow \downarrow .$$

$$\beta = \frac{2\pi \times 2.5 \times 10^9 \times \sqrt{78}}{\sqrt{2} \times 3 \times 10^8} \left(\sqrt{1 + \left(\frac{7}{78}\right)^2} + 1 \right)^{1/2} = 464 \quad \text{rad/m}$$

$$\lambda = \frac{2\pi}{\beta} = 0.014 < 0.12 = \lambda_0$$

$$\eta = \sqrt{\frac{\mu}{\epsilon}} = \sqrt{\frac{\mu_0}{\epsilon_0}} \cdot \frac{1}{\sqrt{\epsilon'_r - j\epsilon''_r}} = \frac{377}{\sqrt{78} \sqrt{1 - j\left(\frac{7}{78}\right)}}$$

$$= 43 + j1.9 = 43.04 \angle 2.6^\circ [\Omega]$$

- Real conductor does not have infinite conductivity. ($\sigma \neq \infty$)

$\Rightarrow$ The wave loses power through resistive heating of the material.

These phenomena can be explained with the complex permittivity.

$$\nabla \times \vec{H}_s = j\omega(\epsilon' - j\epsilon'')\vec{E}_s = \omega\epsilon''\vec{E}_s + j\omega\epsilon'\vec{E}_s \quad (53)$$

$$\nabla \times \vec{H}_s = \vec{J}_s + j\omega\epsilon'\vec{E}_s = \sigma\vec{E}_s + j\omega\epsilon'\vec{E}_s \quad \leftarrow \epsilon' \gg \epsilon''$$

$$= (\sigma + j\omega\epsilon')\vec{E}_s = \underbrace{\vec{J}_{\sigma s}}_{\text{conduction current}} + \underbrace{\vec{J}_{ds}}_{\text{displacement current}} \quad (55)$$

displacement current와
비교되는 측면에서 " σ "를 붙임.

$$\vec{J}_{\sigma s} = \omega\epsilon''\vec{E}_s = \sigma\vec{E}_s$$

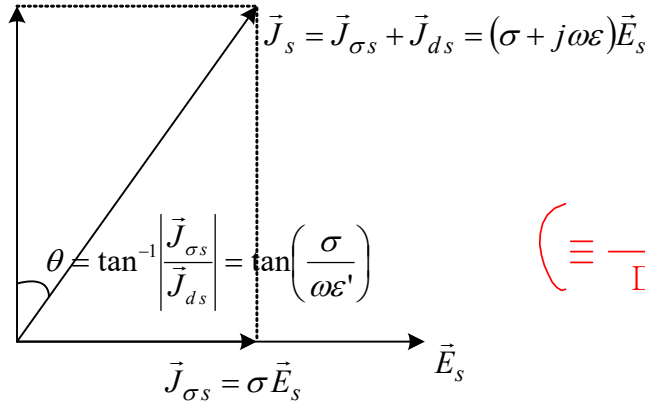
$$\therefore \epsilon'' = \frac{\sigma}{\omega} \quad (56)$$

- Loss tangent (ϵ''/ϵ') is a barometer to judge whether the loss is small or not.

$$\therefore \alpha = \text{Re}\{jk\} = \omega \sqrt{\frac{\mu\epsilon'}{2}} \left(\sqrt{1 + \left(\frac{\epsilon''}{\epsilon'} \right)^2} - 1 \right)^{1/2} \quad (44)$$

$$\frac{\vec{J}_{\sigma s}}{\vec{J}_{ds}} = \frac{\omega\epsilon''\vec{E}_s}{j\omega\epsilon'\vec{E}_s} = \frac{\epsilon''}{j\epsilon'} = \frac{\sigma/\omega}{j\epsilon'} = \frac{\sigma}{j\omega\epsilon'} \quad (57)$$

$$\vec{J}_{ds} = j\omega\epsilon'\vec{E}_s = j\omega\epsilon\vec{E}_s$$



$$\therefore \tan \theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon'}$$

$$\left(\equiv \frac{\text{Conduction current density}}{\text{Displacement current density}} \right)$$

- If the loss tangent is small,

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\epsilon''}{\epsilon'}}$$

$$= j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}} \quad (59)$$

$$\leftarrow (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

where $|x| \ll 1$

$$= j\omega\sqrt{\mu\epsilon'} \left[1 - j\frac{\sigma}{2\omega\epsilon'} + \frac{1}{8} \left(\frac{\sigma}{\omega\epsilon'} \right)^2 - \dots \right]$$

$$= \alpha + j\beta$$

$$\therefore \alpha = \text{Re}(jk) = j\omega\sqrt{\mu\epsilon'} \left(-j\frac{\sigma}{2\omega\epsilon'} \right) = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon'}} \quad (60a) \leftarrow (43) \sim (45)$$

$$\beta = \text{Im}(jk) = \omega\sqrt{\mu\epsilon'} \left[1 + \frac{1}{8} \left(\frac{\sigma}{\omega\epsilon'} \right)^2 \right]$$

$$(60b) \leftarrow |\sigma| \ll |\omega\epsilon'|$$

$$\approx \omega\sqrt{\mu\epsilon'} \quad (61)$$

$$\leftarrow (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots$$

$$\text{where } |x| \ll 1$$

$$\eta = \sqrt{\frac{\mu}{\varepsilon'}} \frac{1}{\sqrt{1 - j\left(\frac{\varepsilon''}{\varepsilon'}\right)}} \quad \leftarrow \quad x = -j\frac{\sigma}{\omega\varepsilon'}, \quad n = -\frac{1}{2}$$

$$= \sqrt{\frac{\mu}{\varepsilon'}} \left[1 + j\frac{\sigma}{2\omega\varepsilon'} + \frac{(-1/2)(-3/2)}{2} \left(-j\frac{\sigma}{\omega\varepsilon'}\right)^2 + \dots \right]$$

$$\approx \sqrt{\frac{\mu}{\varepsilon'}} \left[1 - \frac{3}{8} \left(\frac{\sigma}{\omega\varepsilon'}\right)^2 + j\frac{\sigma}{2\omega\varepsilon'} \right] \quad (62a)$$

$$\approx \sqrt{\frac{\mu}{\varepsilon'}} \left(1 + j\frac{\sigma}{2\omega\varepsilon'} \right) \quad (62b)$$

[Ex. 11.5] Repeat the computation of [Ex. 11.4].

$$\frac{\varepsilon''}{\varepsilon'} = \frac{7}{78} = 0.09 \quad \varepsilon'' = \frac{\sigma}{\omega} \quad \sigma = \varepsilon''\omega$$

$$\begin{aligned} \alpha &= \frac{\sigma}{2} \sqrt{\frac{\mu}{\varepsilon'}} = \frac{\varepsilon''\omega}{2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \frac{1}{\sqrt{78}} \\ &= \frac{7 \times 8.854 \times 10^{-12} \times 2\pi \times 2.5 \times 10^9}{2} \frac{377}{\sqrt{78}} = 20.78 (\approx 21) \quad [\text{Np/m}] \end{aligned}$$

$$\beta = \omega\sqrt{\mu\varepsilon'} = \omega\sqrt{\mu_0\varepsilon_0}\sqrt{78} = \frac{2\pi \times 2.5 \times 10^9 \times \sqrt{78}}{3 \times 10^8} = 462.4 (\approx 464) \quad [\text{rad/m}]$$

$$\begin{aligned} \eta &= \sqrt{\frac{\mu}{\varepsilon'}} \left(1 + j\frac{\sigma}{2\omega\varepsilon'} \right) = \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \frac{1}{\sqrt{78}} \left(1 + j\frac{\varepsilon''}{2\varepsilon'} \right) \\ &= \frac{377}{\sqrt{78}} \left(1 + j\frac{0.09}{2} \right) = 42.7 + j1.92 (\approx 43 + j1.9) \, \Omega \Rightarrow \text{very similar results.} \end{aligned}$$

11.3 Poynting's Theorem and Wave Power

- Consider Maxwell's equation

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad (63)$$

$$\underline{\vec{E} \cdot \nabla \times \vec{H}} = \vec{J} \cdot \vec{E} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad (64)$$

Using vector identity,

$$\nabla \cdot (\vec{E} \times \vec{H}) = \underline{-\vec{E} \cdot \nabla \times \vec{H}} + \vec{H} \cdot \nabla \times \vec{E} \quad (65)$$

$$\vec{H} \cdot \underline{\nabla \times \vec{E}} - \nabla \cdot (\vec{E} \times \vec{H}) = \vec{E} \cdot \nabla \times \vec{H} = \vec{J} \cdot \vec{E} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \quad (66)$$

$$\underline{-\vec{H} \cdot \frac{\partial \vec{B}}{\partial t}} - \nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}$$

$$-\nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \quad (67)$$

Since $\epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} = \frac{\epsilon}{2} \frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\epsilon E^2}{2} \right) = \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{D} \cdot \vec{E} \right) \leftarrow \frac{\partial}{\partial t} F(x) = F'(x) \cdot \frac{\partial x}{\partial t}$

and $\mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \frac{\mu}{2} \frac{\partial H^2}{\partial t} = \frac{\partial}{\partial t} \left(\frac{\mu H^2}{2} \right) = \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{B} \cdot \vec{H} \right),$

$$-\nabla \cdot (\vec{E} \times \vec{H}) = \vec{J} \cdot \vec{E} + \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{D} \cdot \vec{E} \right) + \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{B} \cdot \vec{H} \right) \quad (69)$$

Integrate throughout a volume,

$$-\int_{vol} \nabla \cdot (\vec{E} \times \vec{H}) dv = \int_{vol} \vec{J} \cdot \vec{E} dv + \int_{vol} \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{D} \cdot \vec{E} \right) dv + \int_{vol} \frac{\partial}{\partial t} \left(\frac{1}{2} \vec{B} \cdot \vec{H} \right) dv$$

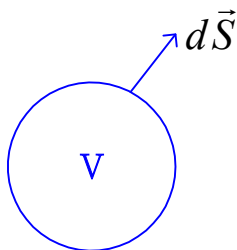
Apply divergence theorem

$$-\oint_S (\vec{E} \times \vec{H}) \cdot d\vec{S} = \int_{vol} \vec{J} \cdot \vec{E} dv + \frac{d}{dt} \int_{vol} \frac{1}{2} \vec{D} \cdot \vec{E} dv + \frac{d}{dt} \int_{vol} \frac{1}{2} \vec{B} \cdot \vec{H} dv \quad (70)$$

(-)부호이므로 구(체적)
안으로 들어오는 total
transmitting power

volume 안에서
소모되는
(ohmic) power

체적 안에서 electric 또는
magnetic field 형태로 저장되는
total energy. (in Ch. 8)



$$\vec{J} \cdot \vec{E} = \frac{|\vec{J}|^2}{\sigma} = R |\vec{J}|^2$$

: Poynting Theorem

• Poynting vector:

$$\vec{S} = \vec{E} \times \vec{H} \quad [\text{W/m}^2] \quad (72): \quad \text{Instantaneous power density, measured in watts per square meter [W/m}^2]$$

$$\Rightarrow (\vec{S} \perp \vec{E}) \quad \text{and} \quad (\vec{S} \perp \vec{H}) \quad \text{and} \quad (\vec{E} \perp \vec{H})$$

- In the uniform plane wave,

$$E_x \vec{a}_x \times H_y \vec{a}_y = S_z \vec{a}_z$$

In a perfect dielectric

$$E_x = E_{x0} \cos(\omega t - \beta z)$$

$$H_y = \frac{E_{x0}}{\eta} \cos(\omega t - \beta z)$$

$$\Rightarrow S_z = \frac{E_{x0}^2}{\eta} \cos^2(\omega t - \beta z) \quad (73)$$

To find the time-average power density,

$$\begin{aligned} \langle S_z \rangle &= \frac{1}{T} \int_0^T \frac{E_{x0}^2}{\eta} \cos^2(\omega t - \beta z) dt \\ &= \frac{1}{2T} \frac{E_{x0}^2}{\eta} \int_0^T [1 + \cos(2\omega t - 2\beta z)] dt \\ &= \frac{1}{2T} \frac{E_{x0}^2}{\eta} \left[t + \frac{1}{2\omega} \sin(2\omega t - 2\beta z) \right]_0^T \\ &= \frac{E_{x0}^2}{2\eta} \\ \therefore \langle S_z \rangle &= \frac{E_{x0}^2}{2\eta} = \frac{(E_{x0}/\sqrt{2})^2}{\eta} = \frac{E_{x0.rms}^2}{\eta} \left(\begin{array}{ll} E_{x0} & : \text{ peak amplitude} \\ E_{x0.rms} & : \text{ rms amplitude} \end{array} \right. \end{aligned}$$

Average power flowing through any area S normal to the propagation direction:

$$S_{z.av} = \frac{1}{2} \frac{E_{x0}^2}{\eta} S \quad [\text{W}]$$

- In the lossy dielectric, E_x and H_y are not in-phased.

$$E_x = E_{x0} e^{-\alpha z} \cos(\omega t - \beta z)$$

$$\text{Let } \eta = |\eta| \angle \theta_\eta$$

$$\Rightarrow H_y = \frac{E_{x0}}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta)$$

$$S_z = E_x H_y = \frac{E_{x0}^2}{|\eta|} e^{-2\alpha z} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) \quad (74)$$

$$= \frac{E_{x0}^2}{2|\eta|} e^{-2\alpha z} [\cos(2\omega t - 2\beta z - \theta_\eta) + \cos \theta_\eta]$$

$$\leftarrow \cos A \cos B = [\cos(A+B) + \cos(A-B)]/2$$

Time-average power density:

$$\langle S_z \rangle = \frac{1}{T} \int_0^T \frac{E_{x0}^2}{2|\eta|} e^{-2\alpha z} [\cos(2\omega t - 2\beta z - \theta_\eta) + \cos \theta_\eta] dt \quad (75)$$

$$= \frac{E_{x0}^2}{2|\eta|} e^{-2\alpha z} \cos \theta_\eta \quad (76)$$

$$\therefore \langle S_z \rangle = \frac{1}{2} \text{Re} [\vec{E}_s \times \vec{H}_s^*] \quad \text{W/m}^2 \quad (77)$$

 **phsor form** $\left(\begin{array}{l} E_x, H_y \text{ 는 } e^{-\alpha z} \text{ 로 감쇠하나} \\ P_{z.av} \text{ 는 } e^{-2\alpha z} \text{ 로 감쇠. 더 빠른 감쇠} \end{array} \right)$

$$\begin{aligned} \Rightarrow \langle \vec{S} \rangle &= \frac{1}{2} \text{Re} [E_{x0} e^{-\alpha z} e^{-j\beta z} \vec{a}_x \times \frac{E_{x0}}{|\eta|} e^{-\alpha z} e^{j(\beta z + \theta_\eta)} \vec{a}_y] = \frac{1}{2} \text{Re} [\frac{E_{x0}^2}{|\eta|} e^{-2\alpha z} e^{j\theta_\eta} \vec{a}_z] \\ &= \frac{1}{2} \frac{E_{x0}^2}{|\eta|} e^{-2\alpha z} \cos \theta_\eta \vec{a}_z \end{aligned}$$

11.4 Propagation in Good Conductors: Skin Effect

- Investigate the behavior of a good conductor

For example, consider nichrome (Ni) $\sigma \cong 10^6$, $f = 100$ MHz, $\epsilon'_r \cong 90$

$$\frac{\sigma}{\omega\epsilon'} = \frac{10^6}{2\pi \times 100 \times 10^6 \times 8.854 \times 10^{-12} \times 90} = 199.7 \times 10^4 \gg 1$$

$$\Rightarrow \vec{J}_{\sigma S} \gg \vec{J}_{dS}$$

- Propagation constant:

$$jk = j\omega\sqrt{\mu\epsilon'}\sqrt{1 - j\frac{\sigma}{\omega\epsilon'}} \approx j\omega\sqrt{\mu\epsilon'}\sqrt{-j\frac{\sigma}{\omega\epsilon'}} \leftarrow (59)$$

$$= j\sqrt{-j\omega\mu\sigma} = j\sqrt{-2j}\sqrt{\pi f\mu\sigma}$$

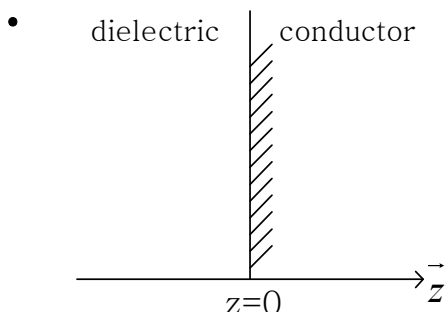
$$= 1\angle 90^\circ \times \sqrt{2}\angle -45^\circ \sqrt{\pi f\mu\sigma} = \sqrt{2}\angle 45^\circ \sqrt{\pi f\mu\sigma}$$

$$= (1+j)\sqrt{\pi f\mu\sigma} = \alpha + j\beta \quad (78)$$

$$\therefore \alpha = \beta = \sqrt{\pi f\mu\sigma} \quad (79)$$

- E_x -component traveling in the (+z)-direction,

$$E_x = E_{x0}e^{-\sqrt{\pi f\mu\sigma}z} \cos(\omega t - \sqrt{\pi f\mu\sigma}z) \quad (80)$$



At $z = 0$, $E_x = E_{x0} \cos \omega t$.

Since the displacement current is negligible in the conductor,

$$\vec{J} = \sigma \vec{E}$$

$$J_x = \sigma E_x = \sigma E_{x0} e^{-\sqrt{\pi f\mu\sigma}z} \cos(\omega t - \sqrt{\pi f\mu\sigma}z)$$

⇒ Conduction current decreases exponentially in the conductor.

- Skin depth (or the depth of penetration):

$$z = \delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} = \frac{1}{\beta} \quad (82) \quad \left(\begin{array}{l} J_x = \sigma E_{x0} e^{-\sqrt{\pi f \mu \sigma} z} \cos(\omega t - \sqrt{\pi f \mu \sigma} z) \text{에서} \\ e^{-\sqrt{\pi f \mu \sigma} z} = e^{-1} \quad \therefore z = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} \end{array} \right)$$

[Ex.] Copper $\sigma = 5.8 \times 10^7$ [S/m]

$$\delta_{\text{Cu}} = \frac{1}{\sqrt{\pi \mu \sigma} \sqrt{f}} = \frac{1}{\sqrt{\pi \times 4\pi \times 10^{-7} \times 5.8 \times 10^7} \sqrt{f}} = \frac{0.0661}{\sqrt{f}}$$

At $f = 60$ [Hz], $\delta_{\text{Cu}} = 8.532$ [mm] $= \frac{1}{3}$ [inch] (→ 대전력선은 2~4" 정도로 하는 것이 좋음)

$$f = 10$$
 [GHz], $\delta_{\text{Cu}} = \frac{0.0661}{\sqrt{10^{10}}} = 6.61 \times 10^{-7}$ [m]

⇒ All field intensities in a good conductor are zero at distance greater than a few skin depths from the surfaces.

⇒ Electromagnetic energy is not transmitted in interior of a conductor and travels surrounding conductor.

- Velocity and wavelength within a good conductor:

$$\alpha = \beta = \frac{1}{\delta} = \sqrt{\pi f \mu \sigma} = \frac{2\pi}{\lambda}$$

$$\therefore \lambda = 2\pi\delta \quad (83)$$

$$v = \frac{\omega}{\beta} = \omega\delta \quad (84)$$

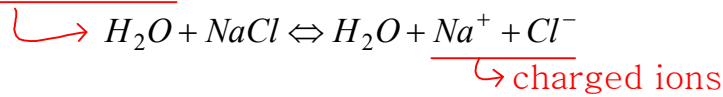
$$\leftarrow \beta = \omega\sqrt{\mu\epsilon} = \frac{\omega}{v_p}, \quad v_p = \frac{\omega}{\beta} = \omega\delta$$

[Ex.] For copper at 60[Hz]

$$\lambda = 2\pi\delta = 2\pi \sqrt{\frac{1}{\pi\mu\sigma f}} = 2\pi \times 8.532 \times 10^{-3} = 0.0536 \text{ [m]} \quad \ll 0.5 \times 10^7 \text{ [m]} = \lambda_0$$

$$v = \omega\delta = 2\pi \times 60 \times 8.532 \times 10^{-3} = 3.216 \text{ [m/s]} \quad < 3 \times 10^8 \text{ [m/s]} = v_0$$

[Ex. 11.6] Sea water $f = 1 \text{ MHz}$, $\sigma = 4 \text{ S/m}$, $\epsilon'_r = 81$



$$\left(\frac{\epsilon''}{\epsilon'}\right) = \frac{\sigma}{\omega\epsilon'} = \frac{4}{2\pi \times 10^6 \times 81 \times 8.854 \times 10^{-12}} = 887.7 \gg 1$$

$\rightarrow$ Good conductor

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 10^6 \times 4\pi \times 10^{-7} \times 4}} = 0.25 \text{ m} \quad \rightarrow \text{RF communication in seawater is quite impractical.}$$

$$\lambda = 2\pi\delta = 1.58 \text{ m} \ll 300 \text{ m} = \lambda_0$$

$$v_p = \omega\delta = 2\pi f \delta = 1.58 \times 10^6 \text{ m/s} \ll 3 \times 10^8 \text{ m/s} = v_0 \quad \left(\because \delta \propto \frac{1}{\sqrt{f}} \right)$$

$\rightarrow f = 10 \text{ Hz}$, $\delta = 79.6 \text{ m}$, $\lambda = 500 \text{ m} \ll \lambda_0 = 3 \times 10^7 \text{ m}$, $v_p = 5000 \text{ m/s}$

- A suspended wire antenna of length shorter than 500m is sufficient to receive signal.
- Drawback: slow signal data rate. ($\because v_p = 5 \text{ km/s}$)

- To find H_y associated with E_x in good conductor,

$$\begin{aligned} \eta &= \sqrt{\frac{\mu}{\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon'(1-j\frac{\varepsilon''}{\varepsilon'})}} = \sqrt{\frac{\mu}{\varepsilon'(1-j\frac{\sigma}{\omega\varepsilon'})}} = \sqrt{\frac{\mu}{\varepsilon'-j\frac{\sigma}{\omega}}} = \sqrt{\frac{j\omega\mu}{\sigma+j\omega\varepsilon'}} \\ &\cong \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{j2\pi f\mu\sigma}{\sigma^2}} = \frac{\sqrt{j2}}{\sigma\delta} = \frac{\sqrt{2}\angle 45^\circ}{\sigma\delta} = \frac{1}{\sigma\delta} + j\frac{1}{\sigma\delta} \quad (85) \end{aligned}$$

(intrinsic impedance)

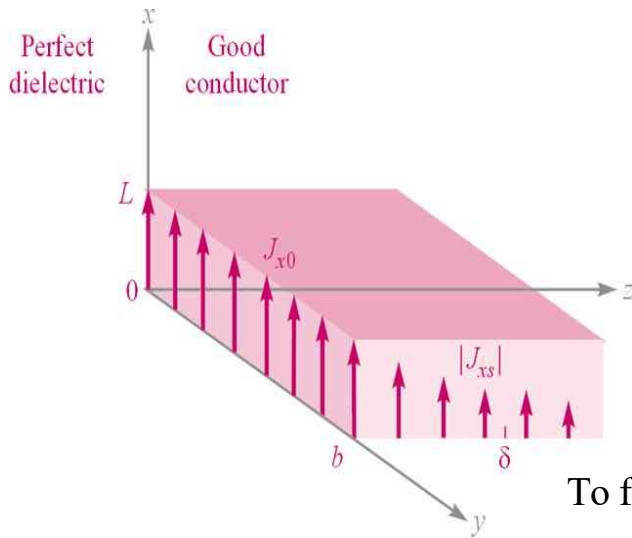
$$\begin{aligned} E_x &= E_{x0} e^{-\sqrt{\pi f \mu \sigma} z} \cos\left(\omega t - \sqrt{\pi f \mu \sigma} z\right) \\ &= E_{x0} e^{-\frac{z}{\delta}} \cos\left(\omega t - \frac{z}{\delta}\right) \quad (86) \end{aligned}$$

$$\begin{aligned} H_{ys} &= \frac{E_{xs}}{\eta} = \frac{E_{x0} e^{-\frac{z}{\delta}} e^{j(-\frac{z}{\delta})}}{\frac{\sqrt{2}}{\sigma\delta} e^{j\pi/4}} \\ &= \frac{\sigma\delta}{\sqrt{2}} E_{x0} e^{-\frac{z}{\delta}} e^{j(-\frac{z}{\delta}-\pi/4)} \end{aligned}$$

$$H_y = \frac{E_x}{\eta} = \frac{\sigma\delta}{\sqrt{2}} E_{x0} e^{-\frac{z}{\delta}} \cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right) \quad (87)$$

$$\begin{aligned} \langle S_z \rangle &= \frac{1}{T} \int_0^T \frac{\sigma\delta}{\sqrt{2}} E_{x0}^2 e^{-\frac{2z}{\delta}} \cos\left(\omega t - \frac{z}{\delta}\right) \cos\left(\omega t - \frac{z}{\delta} - \frac{\pi}{4}\right) dt \\ &= \frac{\sigma\delta}{2\sqrt{2}} E_{x0}^2 e^{-\frac{2z}{\delta}} \cos\frac{\pi}{4} \\ &= \frac{1}{4} \sigma\delta E_{x0}^2 e^{-\frac{2z}{\delta}} \end{aligned}$$

- Total average power loss in $0 < y < b$, $0 < x < L$:



$$P_{L,av} = \int_S \langle S_z \rangle dS = \int_0^b \int_0^L \frac{1}{4} \sigma \delta E_{x0}^2 e^{-2z/\delta} \bigg|_{z=0} dx dy$$

$$= \frac{1}{4} \sigma \delta b L E_{x0}^2$$

$$= \frac{1}{4\sigma} \delta b L J_{x0}^2$$

To find total current

$$I = \int_0^\infty \int_0^b J_x dy dz$$

$$\text{where } J_x = J_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)$$

$$(88) J_x = \sigma E_x \quad (\because \text{good conductor})$$

$$= \sigma E_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)$$

$$= J_{x0} e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right)$$

$$J_{xs} = J_{x0} e^{-z/\delta} e^{-jz/\delta} = J_{x0} e^{-(1+j)z/\delta}$$

$$\therefore I_s = \int_0^\infty \int_0^b J_{xs} dy dz = \int_0^\infty \int_0^b J_{x0} e^{-(1+j)z/\delta} dy dz$$

$$= J_{x0} b \frac{(-\delta)}{1+j} e^{-(1+j)z/\delta} \bigg|_0^\infty = \frac{J_{x0} b \delta}{1+j} = \frac{J_{x0} b \delta}{\sqrt{2}} e^{-j\pi/4}$$

$$\therefore I = \frac{J_{x0} b \delta}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

- If the above current is distributed with a uniformly throughout

$$(0 < y < b, \quad 0 < z < \delta)$$

$$(\vec{z} \text{ 축 방향으로는 } \alpha = \frac{1}{\sqrt{\pi f \mu \sigma}} \text{ 에}$$

의해 감소하지만, total 전류를 구해서
 $0 < y < b$, $0 < z < \delta$ 의 체적에
 고르게 분포되었다고 가정)

$$I = J' S = J' b \delta = \frac{J_{x0} b \delta}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

$$J' = \frac{J_{x0}}{\sqrt{2}} \cos\left(\omega t - \frac{\pi}{4}\right)$$

Ohmic power loss per unit volume: $\vec{J} \cdot \vec{E}$ ← poynting vector 에서 ohmic loss로 이미 언급됨.

Total instantaneous power dissipated in the volume:

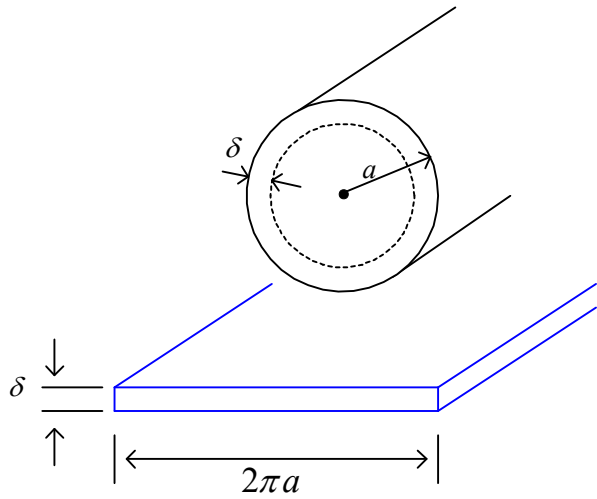
$$\begin{aligned} P_{Li}(t) &= (J' \cdot E) b L \delta = J' \cdot \frac{J'}{\sigma} b L \delta = \frac{(J')^2}{\sigma} b L \delta \\ &= \frac{J_{x0}^2}{2\sigma} b L \delta \cos^2\left(\omega t - \frac{\pi}{4}\right) \end{aligned}$$

$$\underline{\underline{P_{L,av} = \frac{J_{x0}^2}{4\sigma} b L \delta \quad (\text{same result with Eq.(88)})}}$$

⇒ The average power loss in a conductor with skin effect may be calculated by assuming that the total current is distributed uniformly in one skin depth.

(저항적인 측면에서는 폭이 b 이고 길이가 L 인 무한두께의 slab이 skin effect를 가질때의 저항은 skin effect가 없이 폭이 b 이고 길이가 L 이면서 두께가 δ 인 slab의 저항과 같다.)

[Ex.] Circular cross section copper wire.



$$R = \frac{L}{\sigma S} \quad f = 1 \text{ [MHz]}$$

$$a = 1 \text{ [mm]}, \quad L = 1 \text{ [km]}, \quad \sigma = 5.8 \times 10^7$$

$$R_{dc} = \frac{10^3}{5.8 \times 10^7 \times \pi \times 10^{-6}} = 5.488 \text{ } [\Omega]$$

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\sqrt{\pi \times 10^6 \times 4\pi \times 10^{-7} \times 5.8 \times 10^7}} = 66.09 \times 10^{-6}$$

$$\therefore R_{RF} = \frac{L}{\sigma 2\pi a \delta} = \frac{10^3}{5.8 \times 10^7 \times 2\pi \times 10^{-3} \times 66.09 \times 10^{-6}} = 41.52 \text{ } [\Omega]$$