

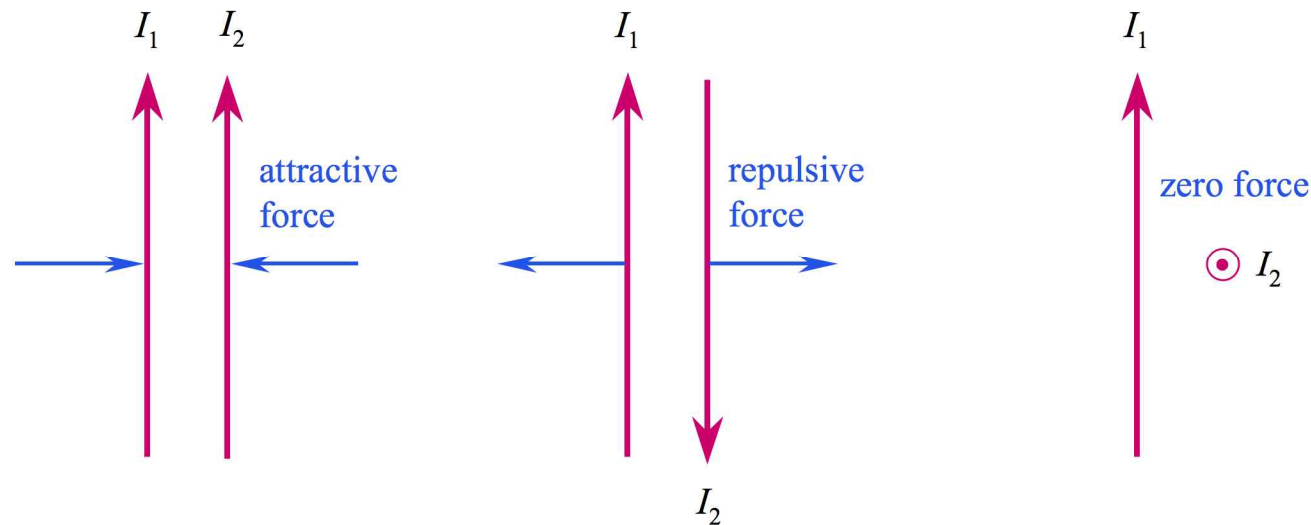
Engineering Electromagnetics

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Chapter 7: The Steady Magnetic Field

Motivating the Magnetic Field Concept: Forces Between Currents

- Magnetic forces due to charge motioning (or current)



- How can we describe a force field around wire 1 that can be used to determine the force on wire 2?

7.1 Biot-Savart Law

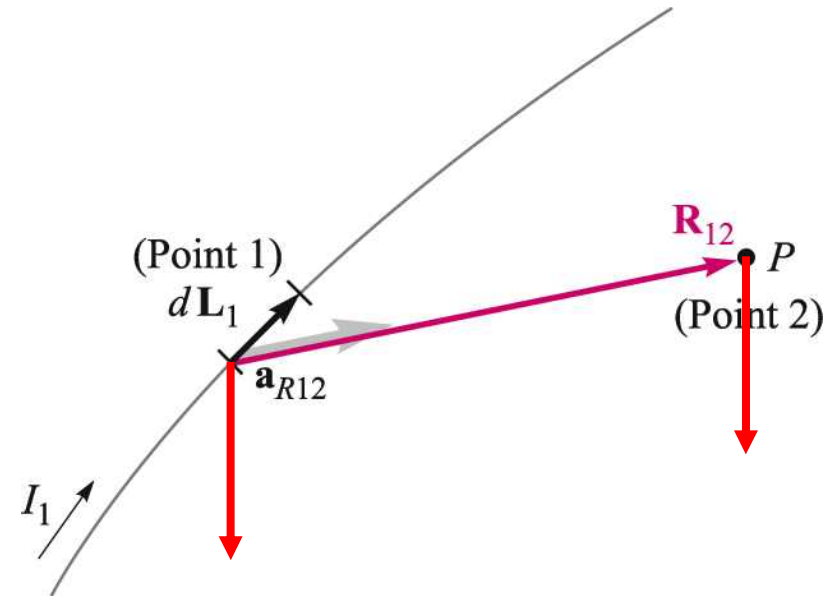
- Source of the steady magnetic field:
 - 1) Permanent magnet
 - 2) An electric field changing linearly with time
 - 3) **Direct current (DC)**
- Differential current element: (**vanishingly**) very small current section of current-carrying filamentary conductor where cross section radius approaches zero.
- **H**: magnetic field intensity [A/m]

- Biot-Savart Law (or Ampere's law for current element)

$$d\vec{H} = \frac{Id\vec{L} \times \vec{a}_R}{4\pi R^2} \quad \leftarrow \vec{a}_R = \frac{\vec{R}}{|\vec{R}|} = \frac{\vec{R}}{R}$$

$$= \frac{Id\vec{L} \times \vec{R}}{4\pi R^3}$$

$$d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}$$



(right-hand rule)

where P_1 : current element location

P_2 : (magnetic) field measurement point location

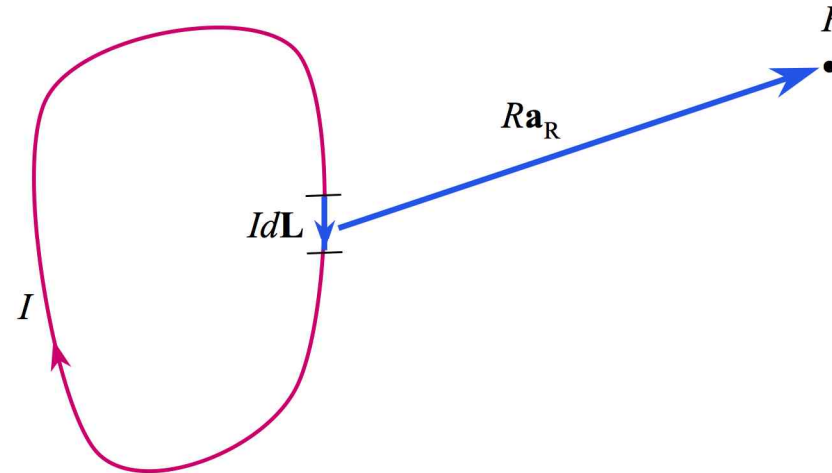
cf.) Coulomb's Law

A point charge of magnitude dQ_1 at point 1 generates electric

field at point 2 as like

$$d\mathbf{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi \epsilon_0 R_{12}^2}$$

- Biot-Savart Law can't be checked experimentally because the differential current element cannot be isolated.



- Since the magnetic field at point P , associated with the differential current element $Id\mathbf{L}$

$$d\mathbf{H} = \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2},$$

the total field arising from closed circuit path is

$$\mathbf{H} = \oint \frac{Id\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

- Expressions of Biot-Savart Law on two- and three-dimensional distributed current source ($\vec{J}$: current density, $\vec{K}$: surface current density)
- Current may be expressed in terms of current density (J):

$$I = JS = J(bt) = \text{constant}$$

- If $t \rightarrow 0$, (**sheet current**)

$$I = \lim_{t \rightarrow 0} J \cdot (bt) = \text{constant}, \therefore J \rightarrow \infty$$

➔ Meaningless where J : [A/m²]

- So current may be expressed in terms of **surface current density** (K).

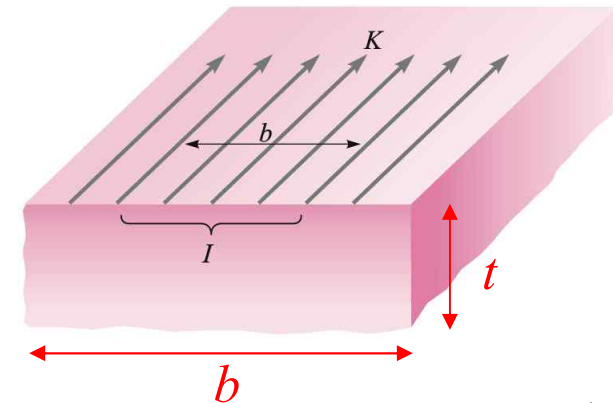
$$\text{Current: } I = Kb,$$

where b : width measured perpendicularly to the direction which the current is flowing.

- For non-uniform surface current density,

$$I = \int K dN$$

where dN : differential element of path across flowing current



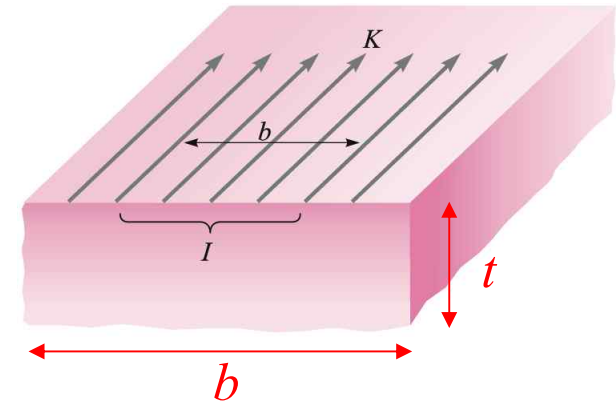
- Current expressions

$$I d\vec{L} = \vec{K} dS = \vec{J} dv$$

$$b \times t \times dL = dv$$

$$b \times dL = dS$$

미소 전류소 (differential current element)



where $\mathbf{K}$ [A/m] : uniform surface current density

- Alternate forms of the Biot-Savart Law

$$\vec{H} = \int_S \frac{\vec{K} dS \times \vec{a}_R}{4\pi R^2} = \int_S \frac{\vec{K} \times \vec{a}_R dS}{4\pi R^2}$$

$$\text{or } \int_{vol} \frac{\vec{J} dv \times \vec{a}_R}{4\pi R^2} = \int_{vol} \frac{\vec{J} \times \vec{a}_R dv}{4\pi R^2}$$

Example of the Biot-Savart Law

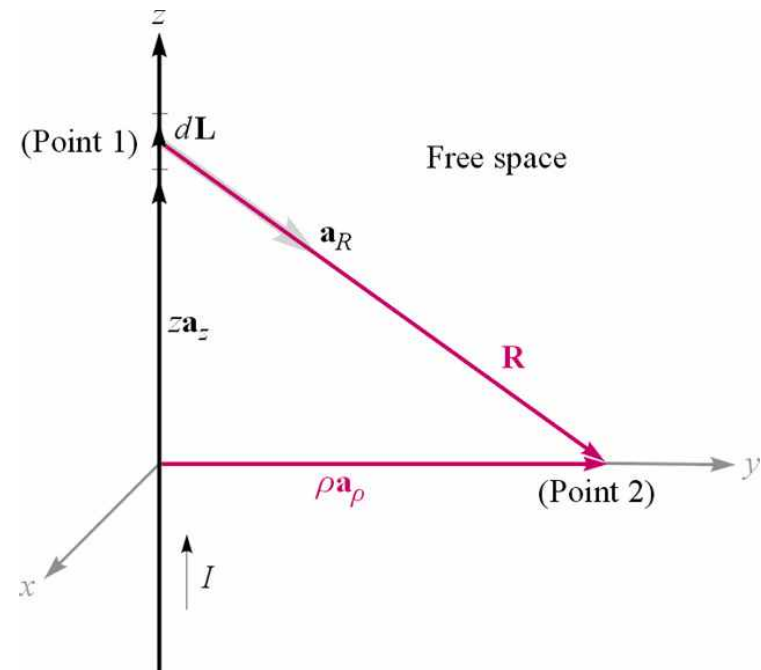
- Magnetic field intensity on y axis (equivalently in xy plane) arising from a filament current element of infinite length on z axis

- Source location: $\vec{r}' = z'\vec{a}_z$
Measurement field point: $\vec{r} = \rho\vec{a}_\rho$

$$\therefore \vec{R}_{12} = \vec{r} - \vec{r}' = \rho\vec{a}_\rho - z'\vec{a}_z$$

$$\vec{a}_{R12} = \frac{\rho\vec{a}_\rho - z'\vec{a}_z}{\sqrt{\rho^2 + z'^2}}$$

- Since $d\vec{L} = dz'\vec{a}_z$,
$$d\vec{H}_2 = \frac{Idz'\vec{a}_z \times (\rho\vec{a}_\rho - z'\vec{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

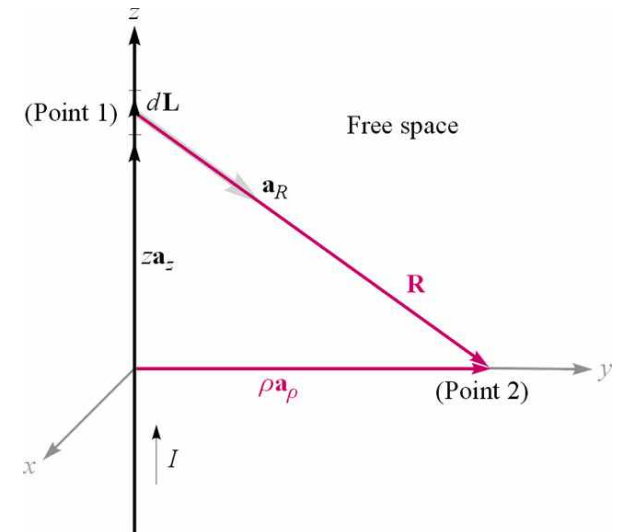


$$\vec{H}_2 = \int_{-\infty}^{\infty} \frac{Idz' \vec{a}_z \times (\rho \vec{a}_\rho - z' \vec{a}_z)}{4\pi(\rho^2 + z'^2)^{3/2}}$$

$$= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{\rho dz' \vec{a}_\phi}{(\rho^2 + z'^2)^{3/2}} = \frac{I\rho \vec{a}_\phi}{4\pi} \int_{-\infty}^{\infty} \frac{dz'}{(\rho^2 + z'^2)^{3/2}}$$

$$z' = \rho \tan \theta \quad \Rightarrow \quad dz' = \rho \sec^2 \theta d\theta$$

$$z': -\infty \leftrightarrow \infty \quad \theta: -\frac{\pi}{2} \leftrightarrow \frac{\pi}{2}$$



$$\vec{H}_2 = \frac{I\rho \vec{a}_\phi}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho \sec^2 \theta d\theta}{(\rho^2 + \rho^2 \tan^2 \theta)^{3/2}} = \frac{I\rho \vec{a}_\phi}{4\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\rho \sec^2 \theta}{\rho^3 \sec^3 \theta} d\theta$$

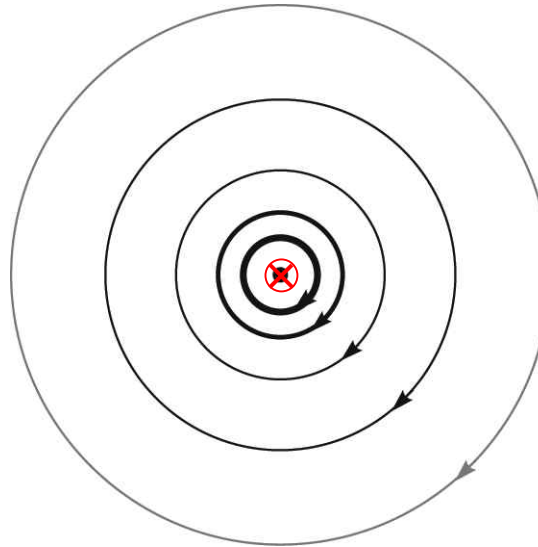
$$= \frac{I \vec{a}_\phi}{4\pi\rho} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos \theta d\theta = \frac{I \vec{a}_\phi}{4\pi\rho} [\sin \theta]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{I}{2\pi\rho} \vec{a}_\phi$$

$$(\therefore |\vec{H}| = f(\rho) \neq f(\phi, z))$$

- Current **go into** the page.
- Magnetic field **streamlines are concentric circles**, whose magnitudes decrease as the **inverse distance** from the z axis

$$\vec{H} \propto \frac{1}{\rho} \vec{a}_\phi$$

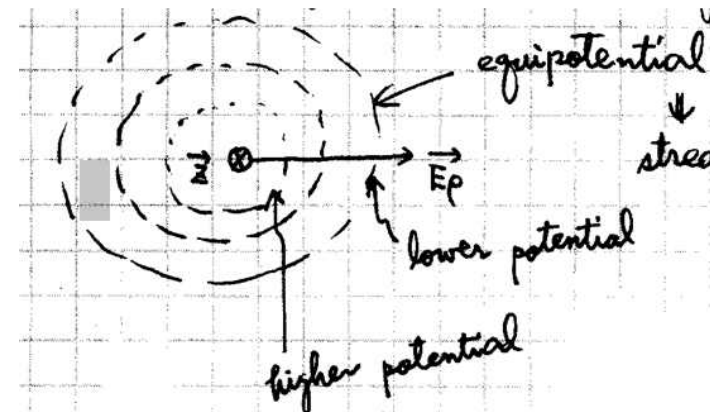
$$\vec{a}_H = \vec{a}_\phi$$



cf.) Electric field intensity for line charge

$$\vec{E} = \frac{\rho_L}{2\pi\epsilon_0\rho} \vec{a}_\rho$$

$$\vec{E} \propto \frac{1}{\rho} \vec{a}_\rho \quad \text{and} \quad \vec{a}_E = \vec{a}_\rho$$



[Ex.] (Magnetic) Field arising from a finite current segment

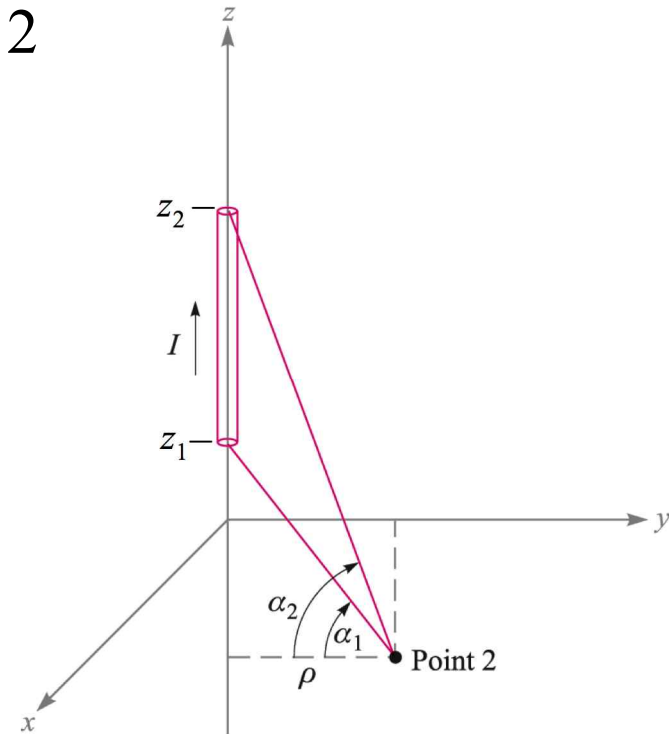
- Field to be found in the xy plane at point 2

$$\mathbf{H} = \int_{z_1}^{z_2} \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}$$

$$\vec{H}_2 = \frac{I \vec{a}_\phi}{4\pi\rho} [\sin \theta]_{\alpha_1}^{\alpha_2} = \frac{I}{4\pi\rho} [\sin \alpha_2 - \sin \alpha_1] \vec{a}_\phi,$$

where

$$\left(\begin{array}{l} z' = \rho \tan \theta \quad \tan \theta = \frac{z'}{\rho} \\ \theta = \tan^{-1} \frac{z'}{\rho} \\ \alpha_2 = \tan^{-1} \frac{z_2}{\rho} \\ \alpha_1 = \tan^{-1} \frac{z_1}{\rho} \end{array} \right)$$



[Ex. 7.1]

$$\alpha_{1x} = -90^\circ$$

$$\alpha_{2x} = \tan^{-1}\left(\frac{0.4}{0.3}\right) = 53.1^\circ$$

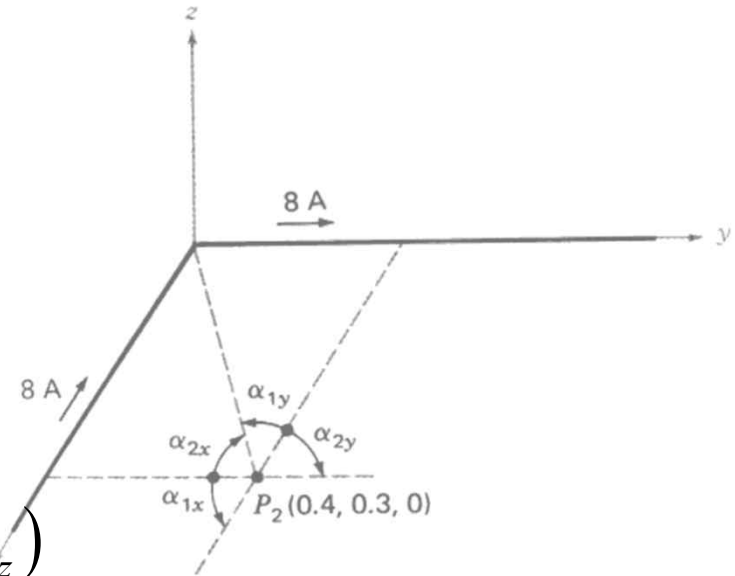
$$\vec{H}_{2(x)} = \frac{8}{4\pi \times (0.3)} [\sin 53.1^\circ - \sin(-90^\circ)](-\vec{a}_z)$$

$$= -\frac{2}{0.3\pi}(1.8)\vec{a}_z = -\frac{12}{\pi}\vec{a}_z \leftarrow Id\vec{L} \times \vec{a}_R = -Idx\vec{a}_x \times \vec{a}_y = -Idx\vec{a}_z$$

$$\alpha_{1y} = -\tan^{-1}\left(\frac{0.3}{0.4}\right) = -36.9^\circ \quad \alpha_{2y} = 90^\circ$$

$$\vec{H}_{2(y)} = \frac{8}{4\pi \times (0.4)} [\sin 90^\circ - \sin(-36.9^\circ)](-\vec{a}_z) = -\frac{8}{\pi}\vec{a}_z \text{ [A/m]}$$

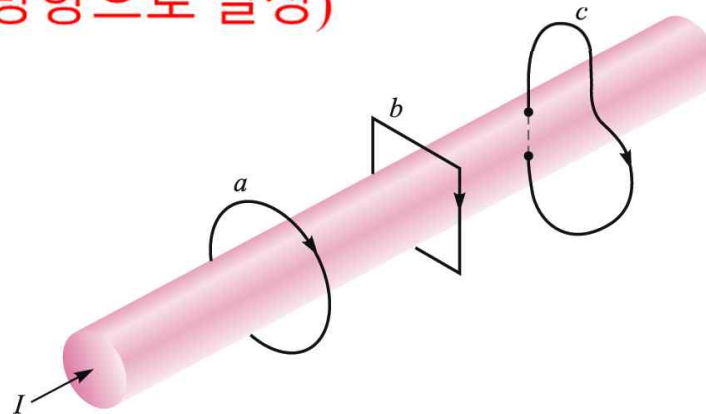
$$\therefore \vec{H}_2 = \vec{H}_{2(x)} + \vec{H}_{2(y)} = \left(-\frac{12}{\pi} - \frac{8}{\pi}\right)\vec{a}_z = -\frac{20}{\pi}\vec{a}_z = -6.37\vec{a}_z \text{ [A/m]}$$



7.2 Ampere's Circuital Law

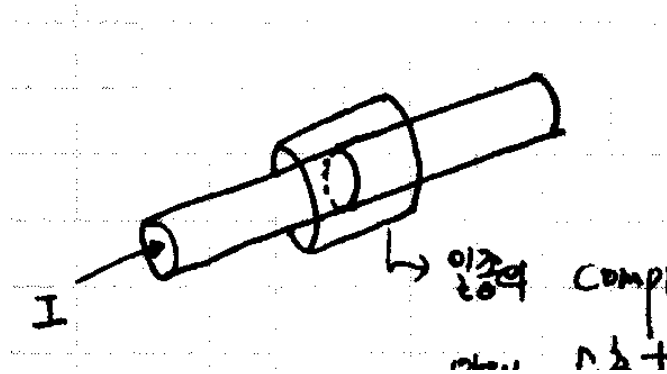
- The line integral of $\mathbf{H}$ about *any closed path* is exactly equal to the direct current enclosed by that path. (Proof: 7-7)
- Define positive current as flowing in the direction of advance of a right-handed screw turned in the direction in which the closed path is traversed. ($\vec{H}$ -field를 오른나사 방향으로 회전하는 방향으로 선적분할 때 나사진행 방향을 (+) 전류 진행방향으로 설정)

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

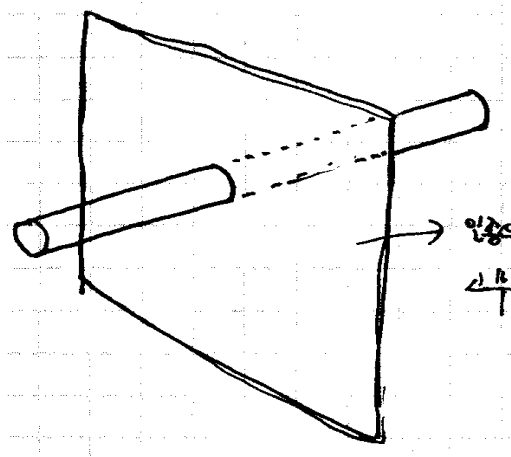


1. Since the closed paths a and b include current path, $\oint \vec{H} \cdot d\vec{L} = I$
2. Since the integral over path c include a portion of current path, $\oint \vec{H} \cdot d\vec{L} \neq I$
3. The direction of current is decided by right-handed screw direction path.
4. Paths a and b are different integration paths, the current are same.

- Compression tube connecting two current wires



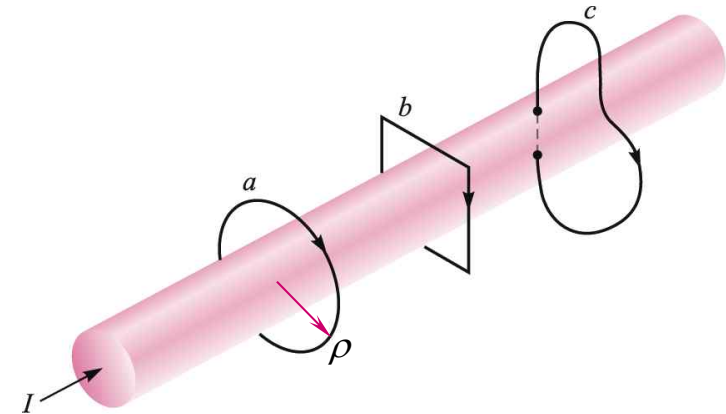
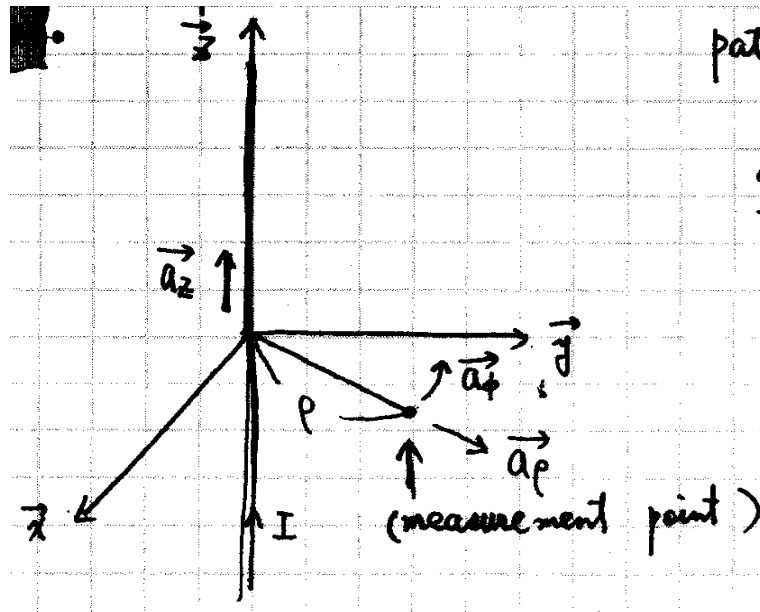
- Rubber plate passing through current wire



Total currents passing through compression tube and rubber plate are same.

cf.) Gauss' law: 폐곡면 내부의 전하량 유도
Ampere's: 폐곡선 내부의 전류량 유도

[Ex. 1] Path is a circle of radius (ρ).

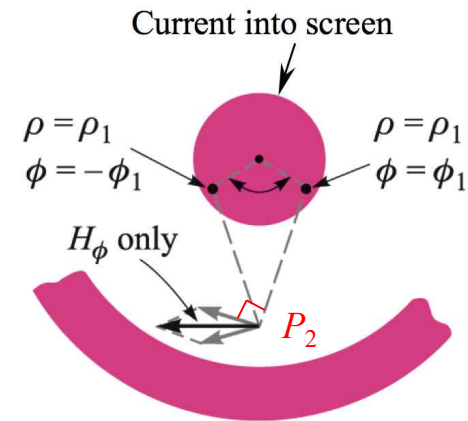
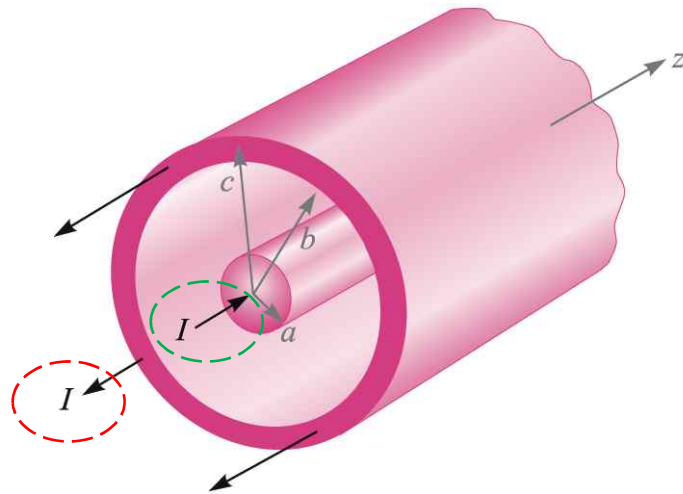


$$\oint \mathbf{H} \cdot d\mathbf{L} = \int_0^{2\pi} H_\phi \rho d\phi = H_\phi \rho \int_0^{2\pi} d\phi = H_\phi 2\pi \rho = I$$

$$\therefore H_\phi = \frac{I}{2\pi\rho} \rightarrow \vec{H} = \frac{I}{2\pi\rho} \vec{a}_\phi$$

[Ex. 2] Infinitely long coaxial transmission line.

- In the coax line, we have two concentric *solid* conductors that carry equal and opposite currents, I .

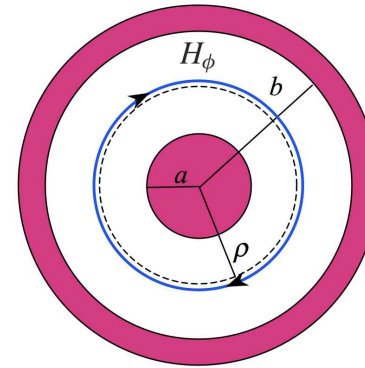


(coaxial cable은 단면적을 가지고 있으므로 하나의 current filament가 아니고, 무한개의 current filament의 합으로 생각할 수 있음.)

(임의의 전류 filament에 의한 P_2 점에서 느끼는 $\vec{H}_2$ 는 단순히 $\vec{a}_\rho$ 및 $\vec{a}_\phi$ 의 함수가 아님. 그러나 전류 filament의 symmetric 특성에 의해 최종적으로 생긴 $\vec{H}_2$ 는 $\vec{a}_\phi$ 의 함수가 됨.)

- Case 1: $a \leq \rho \leq b$

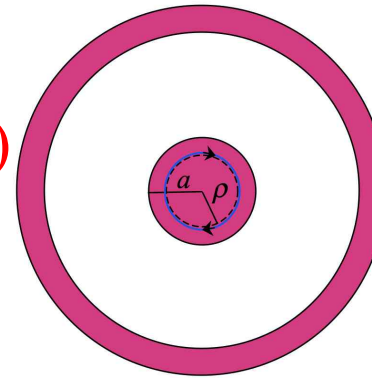
$$H_\phi = \frac{I}{2\pi\rho} \quad (\rho > a \text{ 이므로 전류 } I \text{ 를 전부 고려})$$



- Case 2: $\rho \leq a$

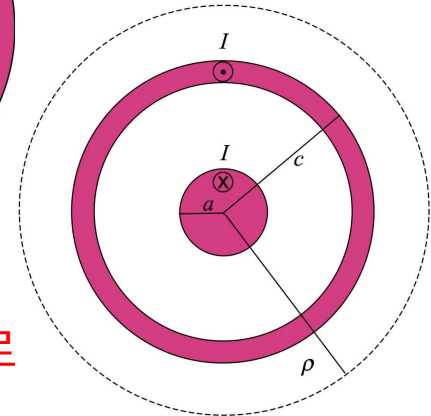
$$I_{\text{encl}} = I \frac{\pi\rho^2}{\pi a^2} = I \frac{\rho^2}{a^2} \quad (\text{closed current})$$

$$\therefore 2\pi\rho H_\phi = I \frac{\rho^2}{a^2} \quad H_\phi = \frac{I\rho}{2\pi a^2}$$



- Case 3: $\rho \geq c$

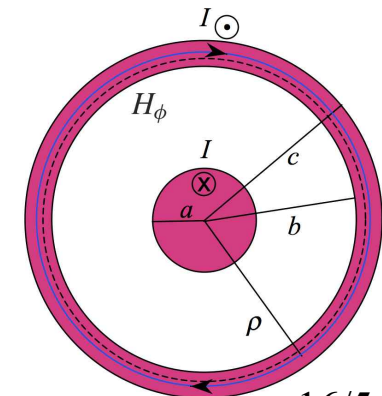
$$I_{\text{encl}} = 0 \rightarrow H_\phi = 0 \quad (\text{내부에 포함된 유효 전류합이 "0"이므로})$$



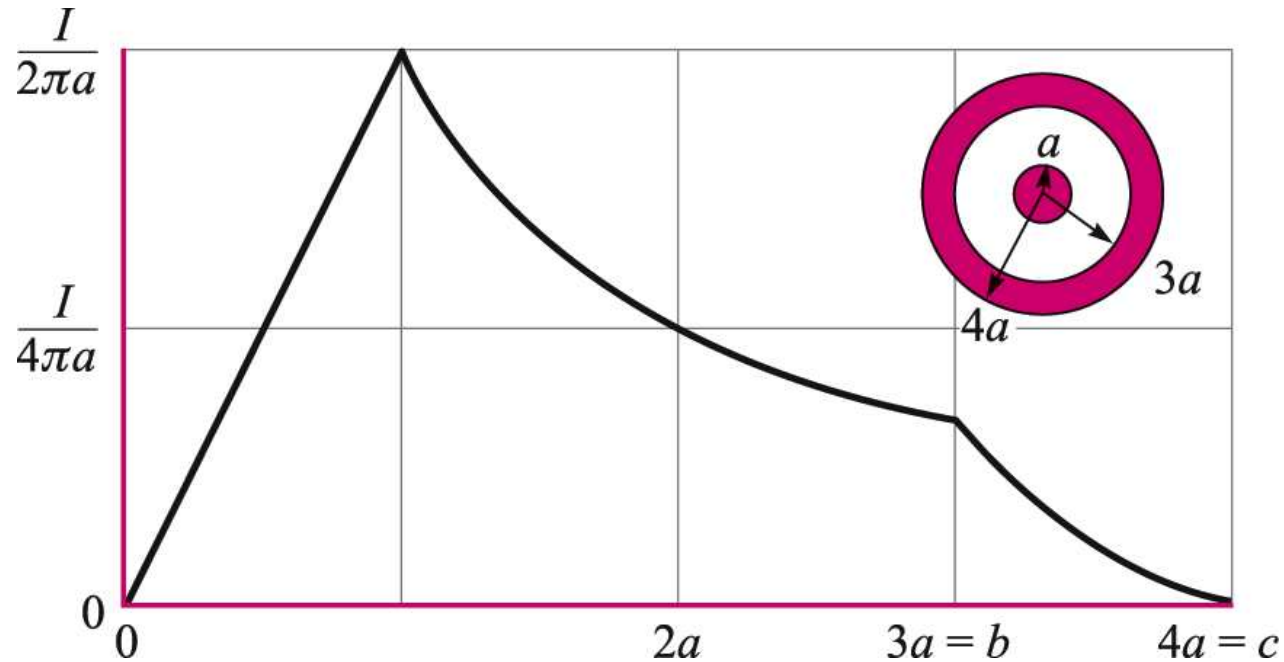
- Case 4: $b \leq \rho \leq c$

$$2\pi\rho H_\phi = I - I \frac{\rho^2\pi - b^2\pi}{c^2\pi - b^2\pi} = I - I \frac{(\rho^2 - b^2)\pi}{(c^2 - b^2)\pi} = I \frac{c^2 - \rho^2}{c^2 - b^2}$$

$$H_\phi = \frac{I}{2\pi\rho} \frac{c^2 - \rho^2}{c^2 - b^2}$$



- $\vec{H}$ -field variation for a coaxial cable with $b = 3a$, $c = 4a$



- 1) H -field is continuous at all the conductor boundaries.
 - 2) The external H -field at outer side of outer conductor is zero.
- ➔ Equal positive and negative currents don't produce any noticeable effect in an adjacent circuit.

7.2.4 Magnetic Field Arising from a Current Sheet

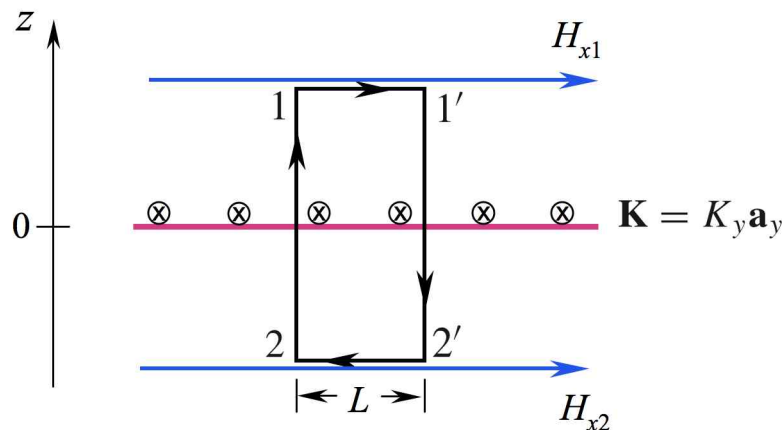
- Current sheet $\vec{K} = K_y \vec{a}_y$ @ $z = 0$
 $\Rightarrow H_y = 0 \leftarrow (d\vec{H} = \frac{\vec{K} dS \times \vec{a}_R}{4\pi R^2}, \quad \vec{H} \perp \vec{K})$

- Current sheet is subdivided into a number of filaments.

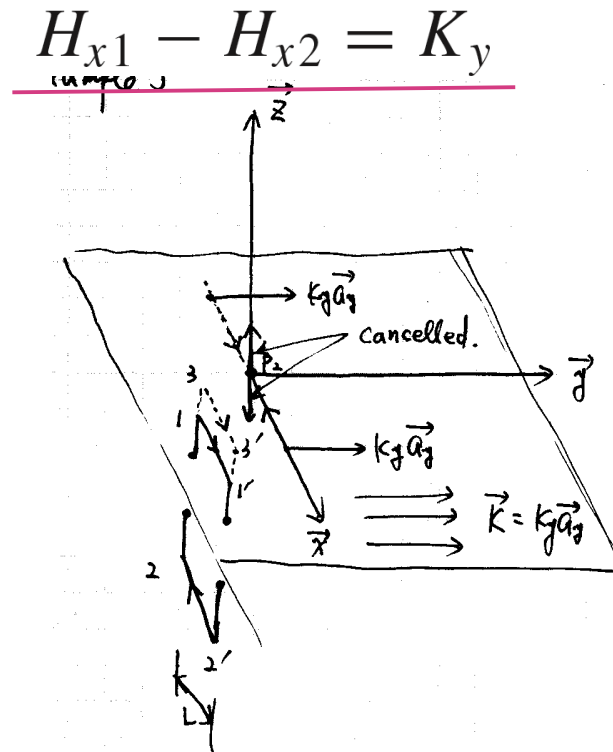
$H_z = 0$ ($\because$ By a symmetrically located pair of filaments) $\Rightarrow H_x \neq 0$

- Choose 1-1'-2'-2-1 path

$$H_{x1}L + H_{x2}(-L) = K_y L \quad \text{or} \quad H_{x1} - H_{x2} = K_y$$



edge view



- Choose 3-3'-2'-2-3 path

$$H_{x3}L + H_{x2}(-L) = K_y L, \quad H_{x3} - H_{x2} = K_y \quad \leftarrow H_{x1} - H_{x2} = K_y$$

$$\Rightarrow H_{x3} = H_{x1}$$

($\therefore H_x$ is the same for all positive z .)

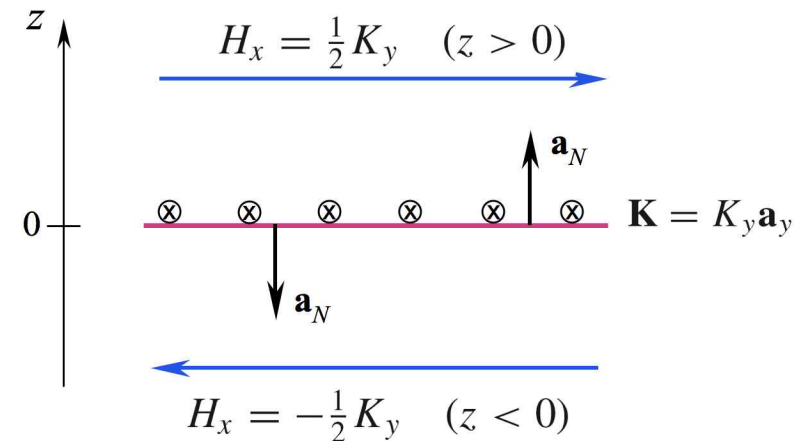
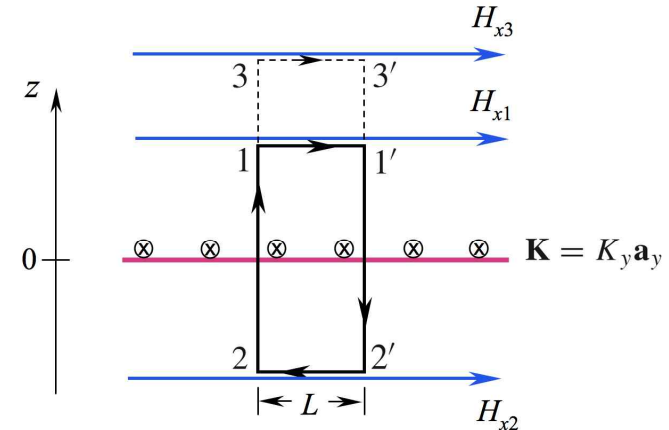
- By symmetric property,

$$H_x = \frac{1}{2} K_y \quad (@ z > 0)$$

(H -field on paths 1-1' and 3-3')

$$H_x = -\frac{1}{2} K_y \quad (@ z < 0)$$

(H -field on path 2-2')



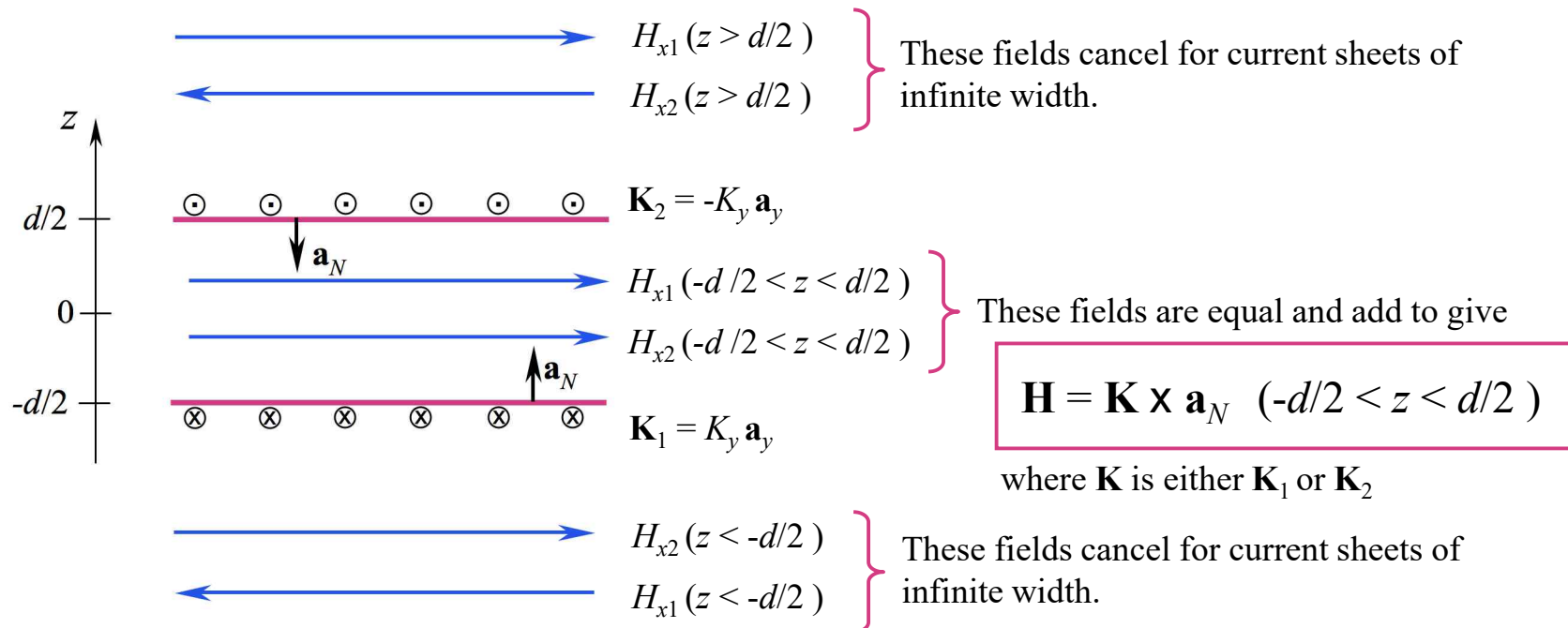
$$\Rightarrow \vec{H} = \frac{1}{2} \vec{K} \times \vec{a}_N \Leftrightarrow \vec{H} \neq f(\text{distance}) \quad \text{cf.} \quad \vec{E} = \frac{\rho_s}{2\epsilon_o} \vec{a}_N$$

where $\vec{a}_N$: unit vector normal to current sheet

- Magnetic field intensity in case two current sheets are located at

$$\vec{K}_1 = K_y \vec{a}_y \text{ (@ } z = -d/2) \text{ and } \vec{K}_2 = -K_y \vec{a}_y \text{ (@ } z = d/2)$$

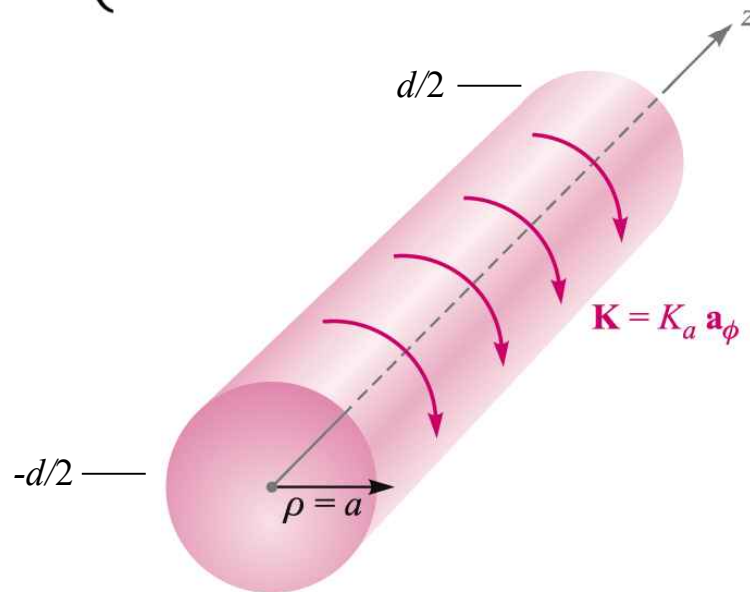
$$\Rightarrow \vec{H} = K_y \vec{a}_x \text{ (@ } -d/2 < z < d/2) \text{ and } \vec{H} = 0 \text{ (@ } z < -d/2, z > d/2)$$



Magnetic Fields within Solenoids and Toroids

- Infinite long solenoid for radius a and uniform current density $K_a \vec{a}_\phi$ [A/m]. (Exercise #7-13)

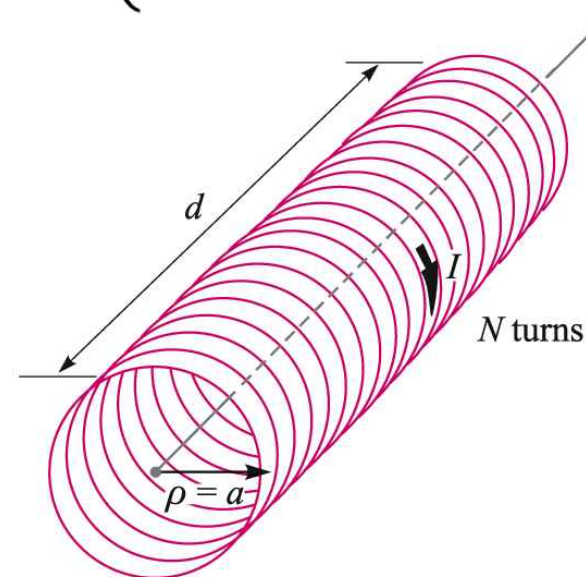
$$\begin{cases} \vec{H} = K_a \vec{a}_z & (@ \rho < a) \\ \vec{H} = 0 & (@ \rho > a) \end{cases}$$



$$\begin{aligned} \mathbf{H} &= K_a \mathbf{a}_z, \rho < a \\ \mathbf{H} &= 0, \rho > a \end{aligned}$$

(a)

$$\begin{cases} \vec{H} = \frac{NI}{d} \vec{a}_z & (@ \rho < a) \\ \vec{H} = 0 & (@ \rho > a) \end{cases}$$



$$\mathbf{H} = \frac{NI}{d} \mathbf{a}_z$$

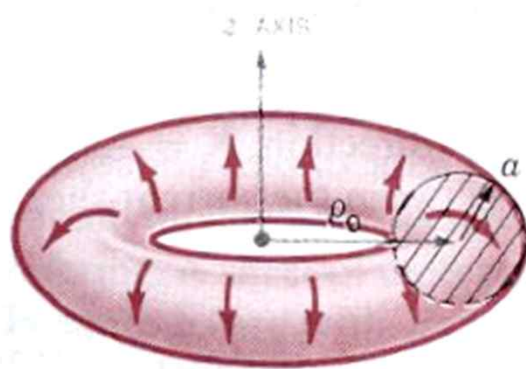
(well inside coil)

(b)

Surface Current Model of a Toroid

$$\vec{K} = K_a \vec{a}_z \text{ @ } \rho = \rho_0 - a, z = 0$$

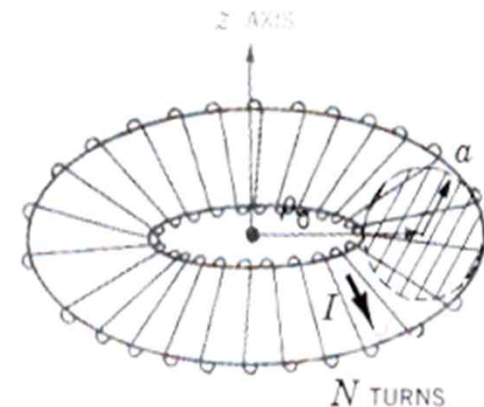
$$\begin{cases} \vec{H} = K_a \frac{\rho_0 - a}{\rho} \vec{a}_\phi & \text{(inside toroid)} \\ \vec{H} = 0 & \text{(outside)} \end{cases} \quad \begin{cases} \vec{H} = \frac{NI}{2\pi\rho} \vec{a}_\phi & \text{(inside toroid)} \\ \vec{H} = 0 & \text{(outside)} \end{cases}$$



$$\mathbf{K} = K_a \mathbf{a}_z \text{ at } \rho = \rho_0 - a, z = 0$$

$$\mathbf{H} = K_a \frac{\rho_0 - a}{\rho} \mathbf{a}_\phi \text{ (INSIDE TOROID)}$$

$$\mathbf{H} = 0 \text{ (OUTSIDE)}$$



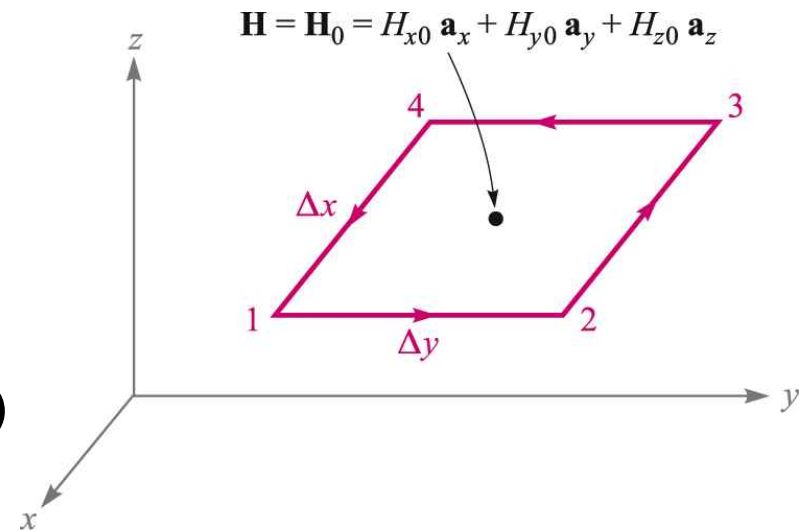
$$\mathbf{H} = \frac{NI}{2\pi\rho} \mathbf{a}_\phi \text{ (WELL INSIDE TOROID)}$$

7.3 Curl

- Incremental closed magnetic field closed path (1-2-3-4-1) of sides with Δx and Δy
- Magnetic field at center due to some current

$$\vec{H}_0 = H_{x0}\vec{a}_x + H_{y0}\vec{a}_y + H_{z0}\vec{a}_z$$

- Overall magnetic field intensity over specific closed path (1-2-3-4-1)



$$\oint \vec{H} \cdot d\vec{L} = (\vec{H} \cdot \Delta\vec{L})_{1-2} + (\vec{H} \cdot \Delta\vec{L})_{2-3} + (\vec{H} \cdot \Delta\vec{L})_{3-4} + (\vec{H} \cdot \Delta\vec{L})_{4-1},$$

$$\text{where } (\vec{H} \cdot \Delta\vec{L})_{1-2} = (H_y \vec{a}_y \cdot \Delta y \vec{a}_y)_{1-2} = H_{y,1-2} \Delta y$$

$$\approx \left[\underline{H_{y0}} + \underline{\frac{\partial H_y}{\partial x}} \left(\underline{\frac{1}{2} \Delta x} \right) \right] \Delta y \quad (\text{by Taylor series})$$

▪ By the same way,

$$(\vec{H} \cdot \Delta \vec{L})_{2-3} \cong H_{x,2-3}(-\Delta x) \cong -[H_{x0} + \frac{\partial H_x}{\partial y}(\frac{1}{2}\Delta y)]\Delta x$$

$$(\vec{H} \cdot \Delta \vec{L})_{3-4} \cong H_{y,3-4}(-\Delta y) \cong [H_{y0} + \frac{\partial H_y}{\partial x}(-\frac{1}{2}\Delta x)](-\Delta y)$$

$$(\vec{H} \cdot \Delta \vec{L})_{4-1} \cong H_{x,4-1}(\Delta x) \cong [H_{x0} + \frac{\partial H_x}{\partial y}(-\frac{1}{2}\Delta y)](\Delta x)$$

$$\therefore \oint \vec{H} \cdot d\vec{L} \cong \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \Delta x \Delta y$$

$$= I = J_z \Delta x \Delta y \quad : \text{current enclosed by the path}$$

$$\text{or } \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \Delta y} \cong \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \quad \lim_{\Delta x, \Delta y \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z$$

: unit area에 대한 $\vec{H}$ -field의 외곽선 적분과 unit area 면적을 지나는 전류와의 관계

cf.) Gauss's Law

$$\text{div } \vec{D} = \lim_{\Delta v \rightarrow 0} \frac{\oint \vec{D} \cdot d\vec{S}}{\Delta v} = \rho_v$$

- By analogous process,

$$\lim_{\Delta y, \Delta z \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \quad \lim_{\Delta z, \Delta x \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y$$

- Generally $(\text{curl } \vec{H})_N = \lim_{\Delta S_N \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S_N} = J_N$

where ΔS_N : planar area enclosed by closed integral

$$\boxed{\text{curl } \mathbf{H} = \nabla \times \mathbf{H}} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}$$

$$= \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \mathbf{a}_z$$

$$= \vec{J} = J_x \vec{a}_x + J_y \vec{a}_y + J_z \vec{a}_z \quad (\text{Cartesian coordinate})$$

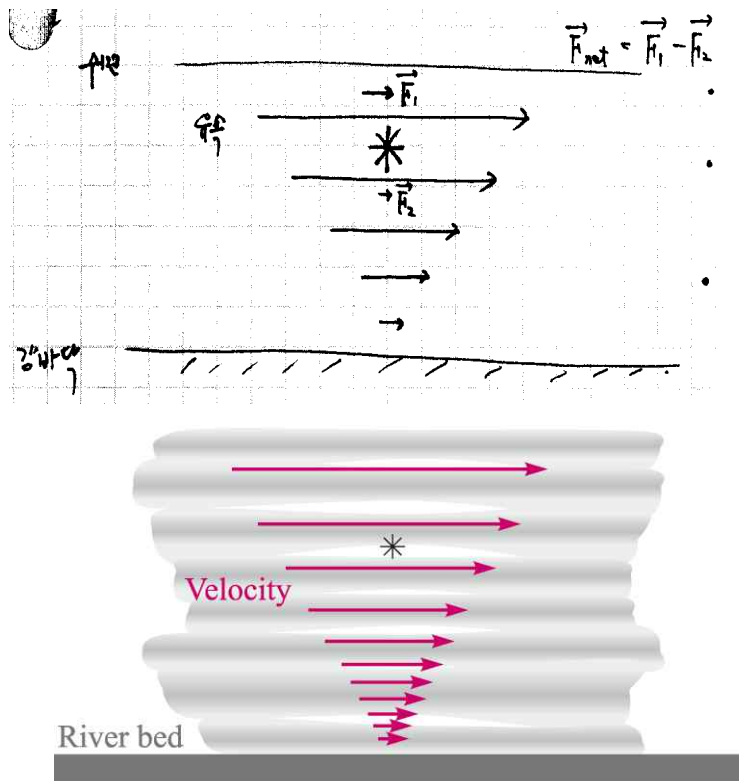
$$\nabla \times \mathbf{H} = \left(\frac{1}{\rho} \frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z} \right) \mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) \mathbf{a}_\phi + \left(\frac{1}{\rho} \frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial H_\rho}{\partial \phi} \right) \mathbf{a}_z \quad (\text{cylindrical})$$

$$\nabla \times \mathbf{H} = \frac{1}{r \sin \theta} \left(\frac{\partial(H_\phi \sin \theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right) \mathbf{a}_r + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial(r H_\phi)}{\partial r} \right) \mathbf{a}_\theta + \frac{1}{r} \left(\frac{\partial(r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right) \mathbf{a}_\phi \quad (\text{spherical})$$

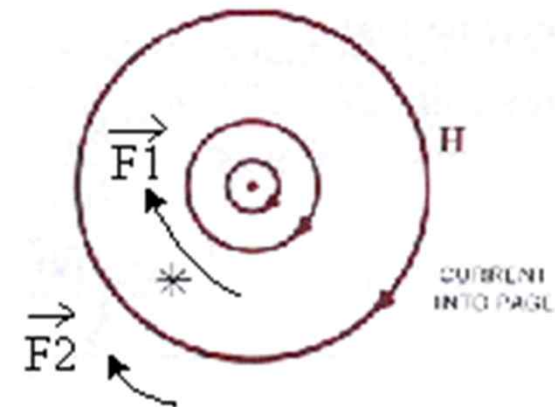
$$\left\{ \begin{array}{l} \oint : \text{circulation} \\ \oint \vec{E} \cdot d\vec{L} = 0 \Rightarrow \text{Work required to carry a charge around a closed path is zero.} \\ \nabla \times \vec{E} = \lim_{\Delta S_N \rightarrow 0} \frac{\oint \vec{E} \cdot d\vec{L}}{\Delta S_N} = 0 \quad (\because \oint \vec{E} \cdot d\vec{L} = 0) \quad (\text{for electrostatic}) \\ \Leftrightarrow \nabla \times \vec{H} = \lim_{\Delta S_N \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{\Delta S_N} = \lim_{\Delta S_N \rightarrow 0} \frac{I_N}{\Delta S_N} \neq 0 \end{array} \right.$$

Visualization of Curl

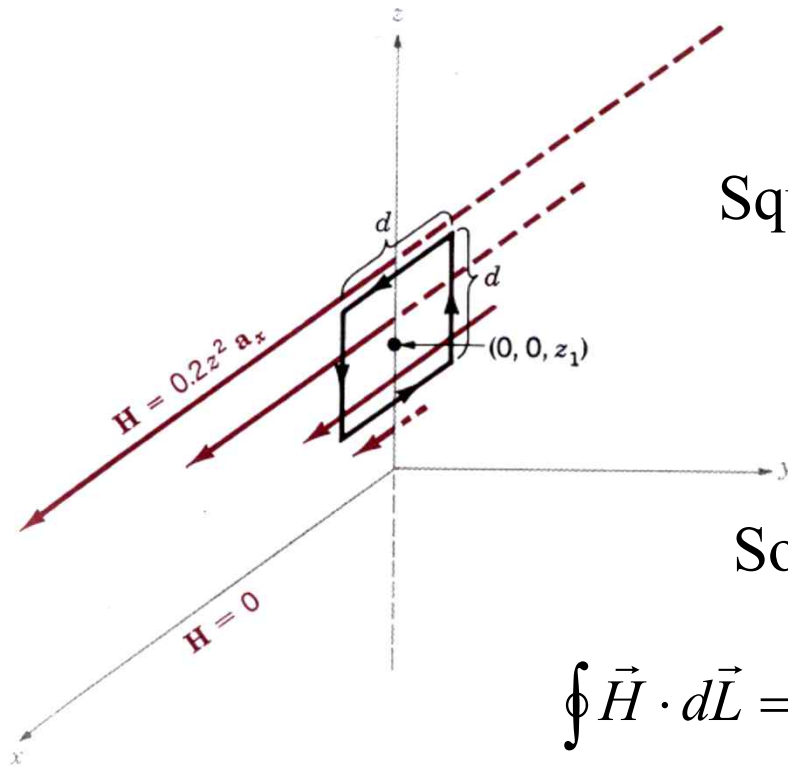
- Curl meter: “paddle wheel” in a flowing stream of water (wheel axis points into the screen.)
- Since $\vec{F}_1 > \vec{F}_2$, paddle wheel rotates clockwise.
- Current go through page.
- Since $\vec{F}_1 > \vec{F}_2$, paddle wheel rotates count-clockwise.



(a)



[Example] $\vec{H} = \begin{cases} 0.2z^2 \vec{a}_x & @z > 0 \\ 0 & @\text{elsewhere.} \end{cases}$



Square path: length = d

center $(0, 0, z_1)$ @ $y = 0$ plane

$$z_1 > d/2$$

Solution 1)

$$\oint \vec{H} \cdot d\vec{L} = 0.2 \left(z_1 + \frac{1}{2}d \right)^2 d + 0 + 0.2 \left(z_1 - \frac{1}{2}d \right)^2 (-d) + 0$$

$$= 0.4z_1 d^2$$

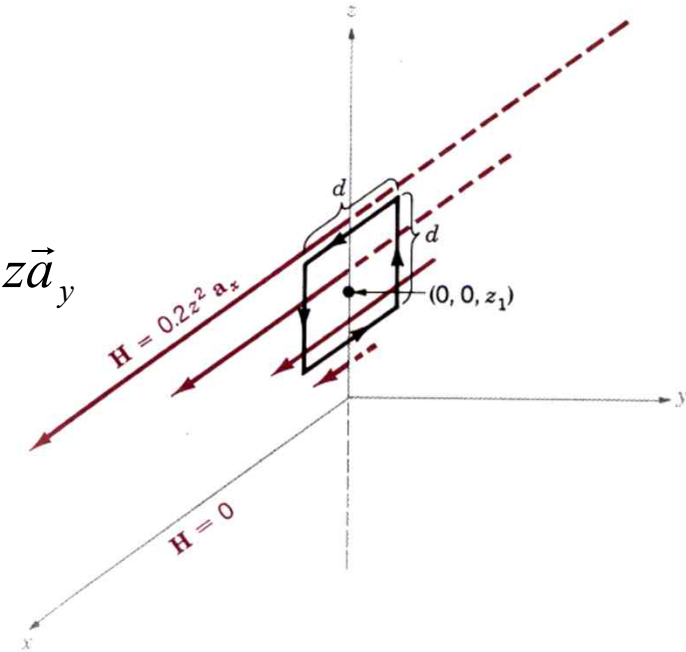
$$(\nabla \times \vec{H})_y = \lim_{d \rightarrow 0} \frac{\oint \vec{H} \cdot d\vec{L}}{d^2} = \lim_{d \rightarrow 0} \frac{0.4z_1 d^2}{d^2} = 0.4z_1$$

: 사각형 변의 길이가 d 인 loop를 관통하여 지나는 전류

$$\therefore \nabla \times \vec{H} = 0.4z_1 \vec{a}_y \quad (\because \text{선적분 면에 수직(normal) 방향은 } \vec{a}_y \text{ 방향})$$

Solution 2)

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{a}_x & \vec{a}_y & \vec{a}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z} (0.2z^2) \vec{a}_y = 0.4z \vec{a}_y$$



@ $z = z_1$, $\nabla \times \vec{H} = 0.4z_1 \vec{a}_y$ ///

▪ $\text{Curl } \vec{H} = \nabla \times \vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) \vec{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \vec{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) \vec{a}_z$
 $= \vec{J}$

: point form of Ampere's circuital law (time-invariant condition)

➔ The second equation of Maxwell's four equations

cf.) $\nabla \times \vec{E} = 0$ ($\because \oint \vec{E} \cdot d\vec{L} = 0$)

➔ The fourth equation of Maxwell's four equations
in case of time-invariant condition

7.4 Stokes' Theorem

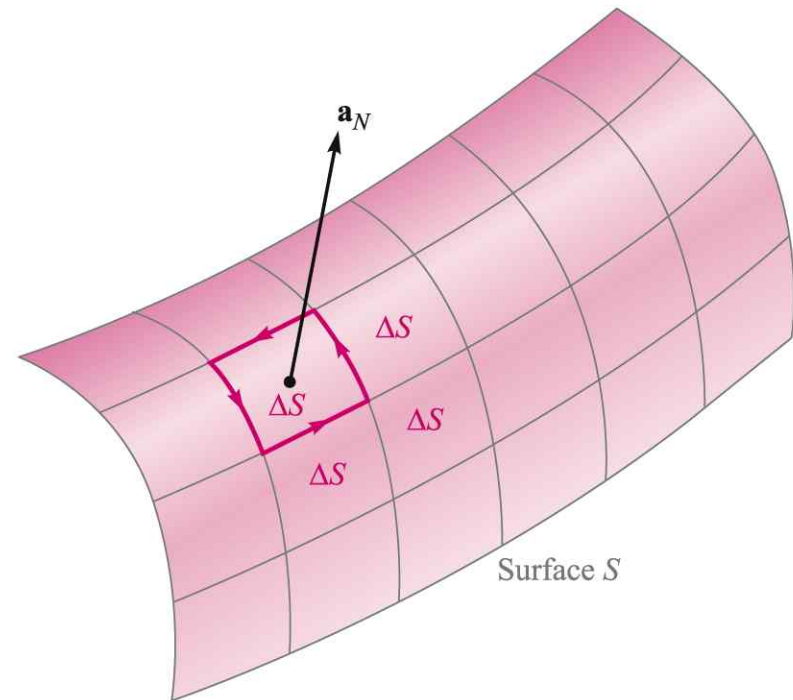
- The surface S is broken up into incremental surfaces of areas ΔS .

$$\frac{\oint \vec{H} \cdot d\vec{L}_{\Delta S}}{\Delta S} \cong (\nabla \times \vec{H})_N$$

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

where N : right-handed direction
normal to surface

$d\vec{L}_{\Delta S}$: closed path vector of
perimeter of ΔS



$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S} \quad \leftarrow d\vec{S} = dS \vec{a}_N$$

where $\vec{a}_N$: normal unit vector in right-handed direction normal
to ΔS .

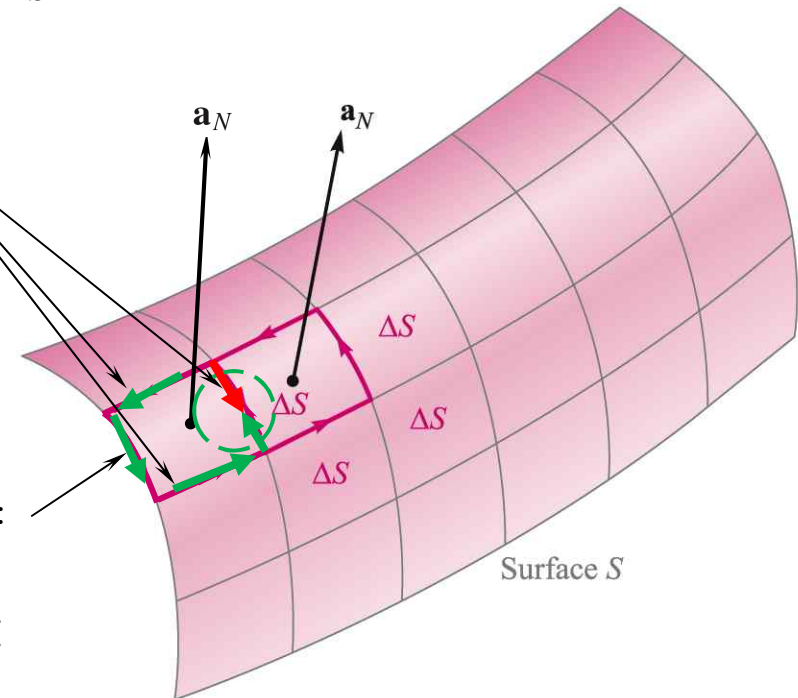
- Let us comprise S for every ΔS .

$$\oint \vec{H} \cdot d\vec{L}_{\Delta S} \doteq \oint \vec{H} \cdot d\vec{L} = \int (\nabla \times \vec{H}) \cdot \Delta \vec{S} = \int_S \vec{J}_N \cdot \Delta \vec{S} = I$$

: Stokes' theorem
(holding for any vector field)
where $d\vec{L}$: closed path vector
of perimeter S

Cancellation here:

No cancellation here:



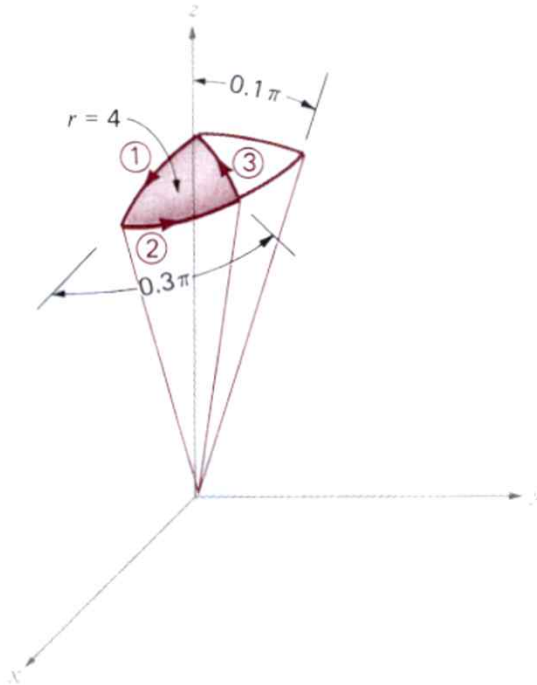
cf.) Divergence theorem (Gauss's law):

$$\oint \vec{D}_s \cdot d\vec{S} = \int \rho_v dv = \int (\nabla \cdot \vec{D}) dv \quad (\text{closed 면적분} \leftrightarrow \text{체적 적분})$$

- Stokes' theorem: $\oint \vec{H} \cdot d\vec{L} = I = \int \vec{J} \cdot d\vec{S} = \int (\nabla \times \vec{H}) \cdot d\vec{S}$

(선적분 $\leftrightarrow$ open 면적분)

[Ex. 7.3] Portion of sphere on $r = 4$, $0 \leq \theta \leq 0.1\pi$, $0 \leq \phi \leq 0.3\pi$



Path segment 1)

$$r = 4, 0 \leq \theta \leq 0.1\pi, \phi = 0$$

Path segment 2)

$$r = 4, \theta = 0.1\pi, 0 \leq \phi \leq 0.3\pi$$

Path segment 3)

$$r = 4, 0 \leq \theta \leq 0.1\pi, \phi = 0.3\pi$$

$$\vec{H} = 6r \sin \phi \vec{a}_r + 18r \sin \theta \cos \phi \vec{a}_\phi = H_r \vec{a}_r + H_\phi \vec{a}_\phi$$

▪ Solution 1) $d\vec{L} = \textcolor{red}{dr} \vec{a}_r + r d\theta \vec{a}_\theta + r \sin \theta d\phi \vec{a}_\phi = r d\theta \vec{a}_\theta + r \sin \theta d\phi \vec{a}_\phi$

: in spherical coordinate ($\because \textcolor{red}{r = 4: \text{constant}}$)

$$\therefore \oint \vec{H} \cdot d\vec{L} = \int_1 H_\theta r d\theta + \int_2 H_\phi r \sin \theta d\phi + \int_3 H_\theta r d\theta \leftarrow \textcolor{red}{H_\theta = 0}$$

$r, \phi : \text{constant}$

$$\rightarrow dr = 0 = d\phi$$

$r, \theta : \text{constant}$

$$\rightarrow dr = 0 = d\theta$$

$r, \phi : \text{constant}$

$$\rightarrow dr = 0 = d\phi$$

$$\begin{aligned}
\therefore \oint \vec{H} \cdot d\vec{L} &= \int_2 H_\phi r \sin \theta d\phi \quad (\because H_\theta = 0) \leftarrow H_\phi = 18r \sin \theta \cos \phi \\
&= \int_0^{0.3\pi} [18 \cdot (4) \cdot \sin(0.1\pi) \cdot \cos \phi] \cdot (4) \sin(0.1\pi) d\phi \\
&= 288 \sin^2(0.1\pi) \sin(0.3\pi) = 22.2 \text{ [A]}
\end{aligned}$$

▪ Solution 2)

$$\begin{aligned}
\nabla \times \vec{H} &= \frac{1}{r \sin \theta} \left[\frac{\partial(H_\phi \sin \theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi} \right] \vec{a}_r + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial H_r}{\partial \phi} - \frac{\partial(r H_\phi)}{\partial r} \right] \vec{a}_\theta + \frac{1}{r} \left[\frac{\partial(r H_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta} \right] \vec{a}_\phi \\
&= \frac{1}{r \sin \theta} [36r \sin \theta \cos \theta \cos \phi] \vec{a}_r + \frac{1}{r} \left[\frac{6r \cos \phi}{\sin \theta} - 36r \sin \theta \cos \phi \right] \vec{a}_\theta + \frac{1}{r} [0] \vec{a}_\phi \\
d\vec{S} &= r^2 \sin \theta d\theta d\phi \vec{a}_r
\end{aligned}$$

$$\begin{aligned}
\therefore \int_s (\nabla \times \vec{H}) \cdot d\vec{S} &= \int_0^{0.3\pi} \int_0^{0.1\pi} (36 \cos \theta \cos \phi)(16 \sin \theta) d\theta d\phi \leftarrow \vec{a}_r \text{ 성분} \\
&= \int_0^{0.3\pi} 576 \left[\frac{1}{2} \sin^2 \theta \right]_0^{0.1\pi} \cos \phi d\phi = 288 (\sin^2 0.1\pi) \times (\sin 0.3\pi) \\
&= 22.2 \text{ [A]} \quad : \text{ Stokes' theorem is satisfied. } ///
\end{aligned}$$

- Let us obtain Ampere's circuital law from $\nabla \times \vec{H} = \vec{J}$ (curl)

$$\int_S (\nabla \times \vec{H}) \cdot d\vec{S} = \int_S \vec{J} \cdot d\vec{S} = I$$

$$= \oint \vec{H} \cdot d\vec{L} \quad (\text{By Stoke's theorem})$$

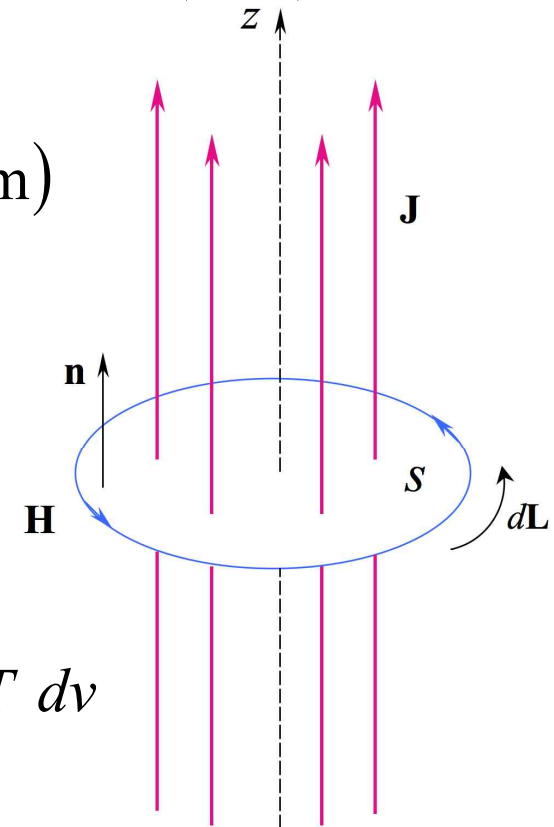
$$\therefore \oint \vec{H} \cdot d\vec{L} = I$$

- Vector identity

$$\nabla \cdot \underbrace{\nabla \times \vec{A}}_{\substack{\text{vector} \\ \text{scalar}}} = T \quad \vec{A}: \text{arbitrary vector}$$

$$\int_{vol} \nabla \cdot (\nabla \times \vec{A}) dv = \int_{vol} T dv \quad \oint_S (\nabla \times \vec{A}) \cdot d\vec{S} = \int_{vol} T dv$$

(s) → closed surface



- (- Gauss 법칙은 체적(volume) v를 감싸고 있는 폐곡면(closed surface)에 대한 면적분임.
- Stokes' theorem은 임의의 폐경로에 의한 개곡면(open surface)에 대한 면적분임.
- 폐곡면의 경우 개곡면적 = 0, 개곡면의 경우 폐곡면적 = 0
- Ex.) 펼친 보자기(wrapping cloth): 개곡면, 묶은 보자기: 폐곡면)

$$\int_{vol} T dv = 0 \rightarrow T dv = 0 \rightarrow T = 0 \quad (\because v \neq 0) \rightarrow \nabla \cdot \underbrace{\nabla \times \vec{A}}_{\substack{\text{open surface} \\ \text{closed surface}}} = 0$$

7.5 Magnetic Flux and Flux Density

- Magnetic flux density: $\vec{B}$

$$\vec{B} = \mu_0 \vec{H}, \quad ([\text{Wb/m}^2] \text{ or } [\text{T: telsa}] \text{ or } [\text{G: gauss}])$$

where $\mu_0 = 4\pi \times 10^{-7} \text{ [H/m]}$: free space *permeability*

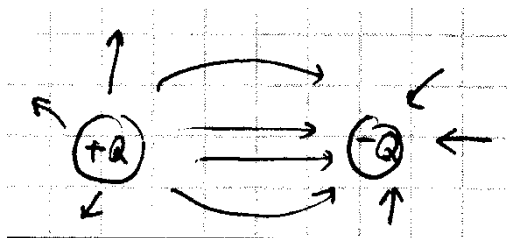
cf.) $\vec{D} = \epsilon_0 \vec{E}$ (isotropic material)

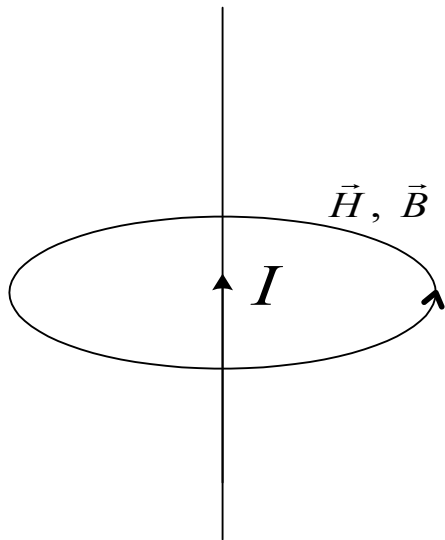
- Magnetic flux: Φ [phi]

$$\Phi = \int_s \vec{B} \cdot d\vec{S} \quad [\text{wb}]$$

cf.) Electric flus: Ψ [psi] (Gauss's law: **The total flux passing through any closed surface is equal to the charge enclosed.**)

$$\Psi = \oint_s \vec{D} \cdot d\vec{S} = Q$$





$\vec{B} = \mu_0 \vec{H}$: relation between magnetic flux
and magnetic field intensity

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B} dv = 0$$

(Since the magnetic flux line forms the open surface, the magnetic flux line is closed and do not terminate on a magnetic charge(s). ➔ No magnetic flux source

- Differential form: $\nabla \cdot \vec{B} = 0$
- (Differential) Maxwell's equations for static electric field and steady magnetic field:

$$\nabla \cdot \mathbf{D} = \rho_v$$

Gauss' Law for the electric field

$$\nabla \times \mathbf{E} = 0$$

Conservative property of the static electric field

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Ampere's Circuital Law

$$\nabla \cdot \mathbf{B} = 0$$

Gauss' Law for the Magnetic Field

where

$$\mathbf{D} = \epsilon_0 \mathbf{E}$$

$$\mathbf{B} = \mu_0 \mathbf{H}$$

in free space

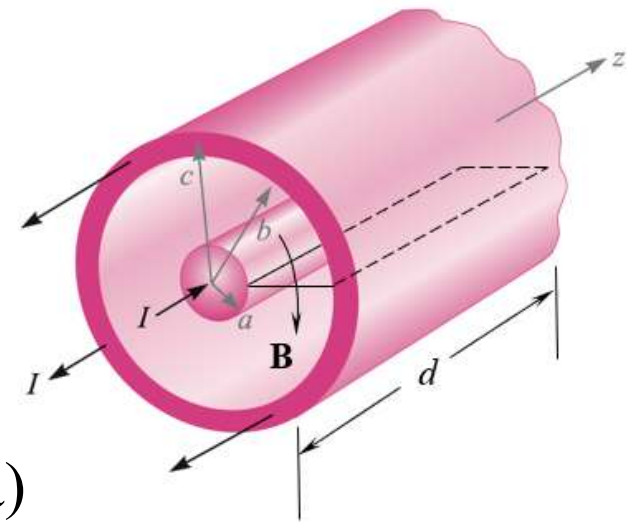
- Maxwell's equations in large scale (or integral) form using divergence theorem and Stokes' theorem

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv$$

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$



[Ex.] $H_\phi = \frac{I}{2\pi\rho} \quad (a < \rho < b)$ from Fig. 7.8(a)

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi$$

Magnetic flux crossing any radial plane on $a \leq \rho \leq b$ and $0 \leq z \leq d$:

$$\begin{aligned} \Phi &= \int_S \vec{B} \cdot d\vec{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi \cdot (d\rho \, dz \, \vec{a}_\phi) \\ &= \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a} \end{aligned}$$

7.6 The Scalar and Vector Magnetic Potentials

- For electric field, Q or charge distribution $\longrightarrow \vec{E} \longrightarrow V \longrightarrow C$
 $\searrow$
 $V \longrightarrow \vec{E} \longrightarrow C$
- Let us introduce scalar magnetic potential (V_m) with similarity of electric potential

$$\mathbf{H} = -\nabla V_{(m)} \quad \Leftrightarrow \quad \mathbf{E} = -\nabla V$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \nabla \times (-\nabla(V_m))$$

: (1) Curl 적용

“zero”
“zero”

Scalar (not slope): current is constant

$$\therefore \mathbf{H} = -\nabla V_m \quad (\mathbf{J} = 0)$$

(Ex. Only region $a < \rho < b$ in coaxial cable)

- In free space,

$$\nabla \cdot \vec{B} = \mu_0 \nabla \cdot \vec{H} = 0 \quad : (2) \text{ Divergence 적용}$$

$$\mu_0 \nabla \cdot (-\nabla V_m) = 0$$

$$\therefore \nabla^2 V_m = 0 \quad (\text{for } \vec{J} = 0) : (\text{Magnetic}) \text{ Laplace's eq.}$$

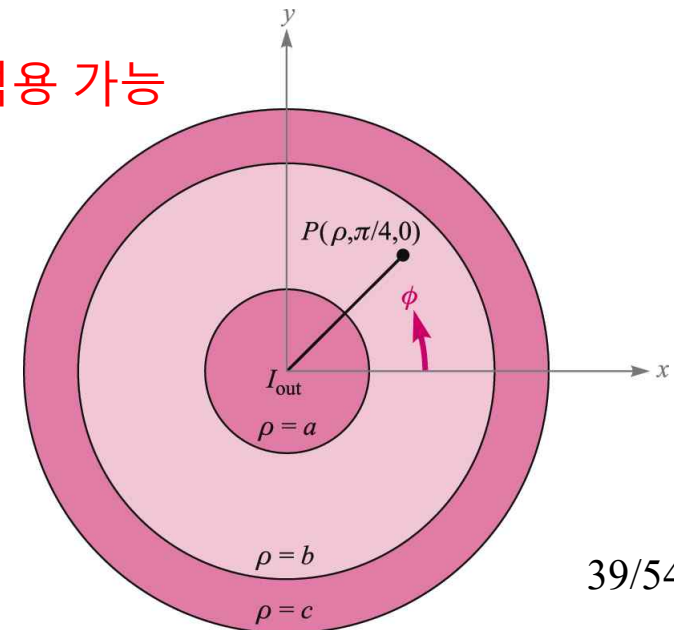
(특정면에 전류가 흐르고, 전류가 흐르지 않는 다른 특정면에서 (magnetic) Laplace's eq. 을 적용 가능)

- Cross section of coaxial line

At $a < \rho < b$, $\vec{J} = 0 \rightarrow \text{magnetic potential 적용 가능}$

$$\Rightarrow \vec{H} = \frac{I}{2\pi \rho} \vec{a}_\phi$$

where I : total current flowing in
inner conductor
directional to $\vec{a}_z$



$$\vec{H} = \frac{I}{2\pi\rho} \vec{a}_\phi = -\nabla V_m|_\phi = -\frac{1}{\rho} \frac{\partial V_m}{\partial \phi} \vec{a}_\phi$$

$$\therefore \frac{\partial V_m}{\partial \phi} = -\frac{I}{2\pi}$$

$$V_m = -\frac{I}{2\pi} \phi (+k) = \frac{I}{2\pi} (-\phi + k')$$

Integral constant = 0 (in this text)

- If $V_m = 0$ at $\phi = 0$ and is proceed countclockwise around the circle, magnetic potential at point P (@ $\phi = \pi/4$) is

$$\begin{aligned} V_{mp} &= \frac{I}{2\pi} \left(2n - \frac{1}{4}\right) \pi & (n = 0, \pm 1, \pm 2, \dots) \\ &= I \left(n - \frac{1}{8}\right) & (\quad " \quad) \end{aligned}$$

➔ Magnetic scalar potential has a **multivaluedness** property.

cf.) Electrostatic case

$$\nabla \times \vec{E} = 0 \quad (\text{point form})$$

$$\oint \vec{E} \cdot d\vec{L} = 0 \quad (\text{integral form})$$

$$V_{ab} = - \int_b^a \vec{E} \cdot d\vec{L} \quad \underline{\underline{(\text{independent of the path})}}$$

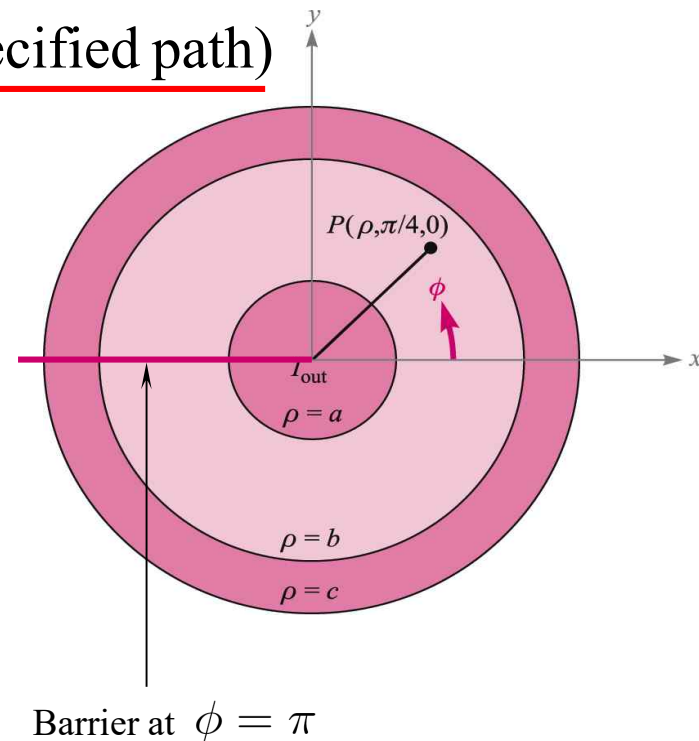
: conservative (or singular) //

$$V_{m,ab} = - \int_a^b \vec{H} \cdot d\vec{L} \quad \underline{\underline{(\text{depend on the specified path})}}$$

- In the above example, assume that ϕ variation is restricted as $-\pi \sim \pi$.

$$V_m = -\frac{I}{2\pi} \phi,$$

then $V_{mp} = -\frac{I}{8} \quad @ \quad \phi = \frac{\pi}{4}$



7.6.2 Vector (Magnetic) Potential

- Time varying case
- Useful in studying an electromagnetic wave radiation from antennas

$$\nabla \cdot \mathbf{B} = 0 \quad (\text{Maxwell equation})$$

- Let $\mathbf{B} = \nabla \times \mathbf{A} \quad \leftarrow \vec{A} : \text{vector magnetic potential [wb/m]}$
(since $\nabla \cdot \nabla \times \vec{A} = 0$ is proven in the previous section, $\nabla \cdot \vec{B} = 0$ is satisfied.)

cf.) $\vec{H} = -\nabla V_m$: scalar magnetic potential

$$\nabla \cdot \vec{B} = \nabla \cdot (\mu_0 \vec{H}) = \nabla \cdot (\nabla \times \vec{A})$$

$$\vec{H} = \frac{1}{\mu_0} \nabla \times \vec{A}$$

$$\Rightarrow \vec{A} = \oint \frac{\mu_0 I d\vec{L}}{4\pi R}$$

$$\nabla \times \vec{H} = \vec{J} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}$$

(This eq. will be proven in the next section)
(If we know $\vec{A}$, then $\vec{B}$ can be known naturally.)

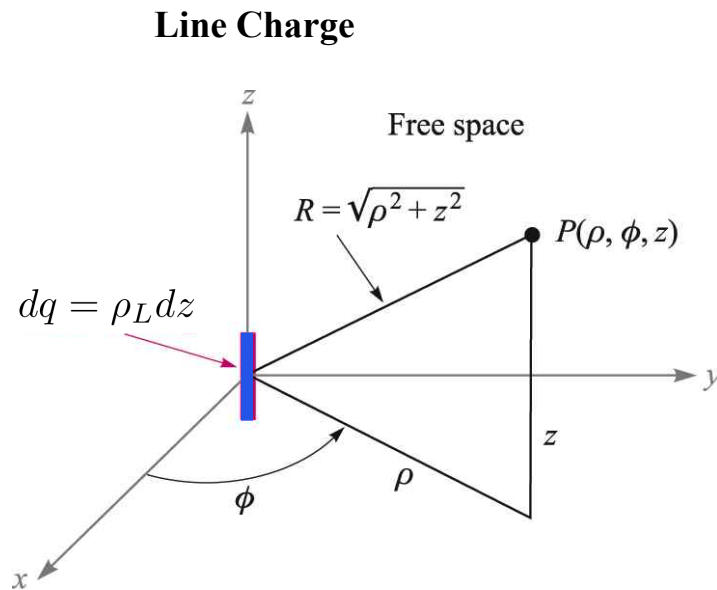
where I : DC current along a filamentary conductor

R : distance from differential current length $d\vec{L}$ to a point where $\vec{A}$ is to be found.

- Electrostatic potential

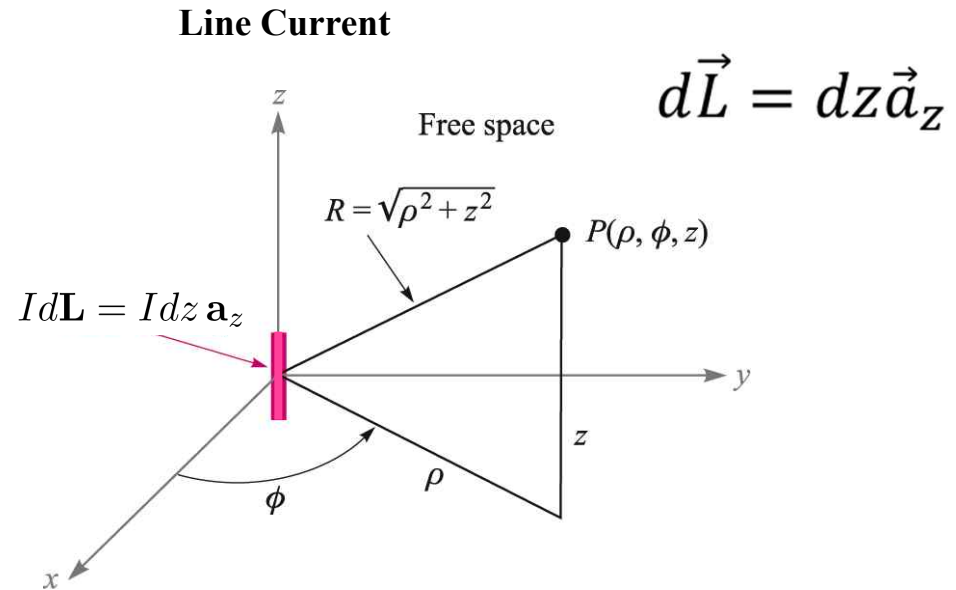
$$V = \int \frac{\rho_L dl}{4\pi\epsilon_0 R} \quad \Leftarrow V = \frac{Q}{4\pi\epsilon_0 R} \quad (4.15)$$

- When compare with the electrostatic potential, differential vector magnetic ($\vec{A}$) is $d\vec{A} = \frac{\mu_0 I d\vec{L}}{4\pi R} \rightarrow \vec{A} \parallel d\vec{L}$ and $|\vec{A}| \propto \frac{1}{R}$



Scalar Electrostatic Potential

$$dV = \frac{dq}{4\pi\epsilon_0 R} = \frac{\rho_L dL}{4\pi\epsilon_0 R}$$



Vector Magnetic Potential

$$d\mathbf{A} = \frac{\mu_0 I d\mathbf{L}}{4\pi R} = \frac{\mu_0 I dz \mathbf{a}_z}{4\pi R} = \frac{\mu_0 I dz \vec{a}_z}{4\pi\sqrt{\rho^2 + z^2}}$$

- In cylindrical coordinate at $P(\rho, \phi, z)$

$$dA_z = \frac{\mu_0 Idz}{4\pi\sqrt{\rho^2 + z^2}} = f(\rho, z)$$

$$dA_\phi = 0$$

$$dA_\rho = 0$$

- Since $\nabla \times \vec{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \vec{a}_\rho + \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \vec{a}_\phi + \left(\frac{1}{\rho} \frac{\partial(\rho A_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right) \vec{a}_z,$

$$d\vec{H} = \frac{1}{\mu_0} \nabla \times d\vec{A} = \frac{1}{\mu_0} \left(-\frac{\partial dA_z}{\partial \rho} \right) \vec{a}_\phi \quad \leftarrow A_z = f(\rho, z) \neq f(\phi)$$

$$= -\frac{Idz}{4\pi} \left(-\frac{1}{2} \right) \frac{2\rho}{(\rho^2 + z^2)^{3/2}} \vec{a}_\phi = \frac{Idz \vec{a}_z}{4\pi(\rho^2 + z^2)} \times \frac{\rho \vec{a}_\rho + z \vec{a}_z}{\sqrt{\rho^2 + z^2}}$$

$$d\vec{H} = \frac{Id\vec{L} \times \vec{a}_r}{4\pi R^2} = \frac{Id\vec{L}}{4\pi R^2} \times \vec{a}_r \text{ 과 같은 형태}$$

- For a current sheet $\vec{K}$,

$$Id\vec{L} = \vec{K}dS$$

For a current throughout a volume with a density $\vec{J}$,

$$Id\vec{L} = \vec{J}dv$$

- Alternative expressions for $\vec{A}$,

$$\vec{A} = \int_s \frac{\mu_0 \vec{K}dS}{4\pi R}$$

$$\vec{A} = \int_{vol} \frac{\mu_0 \vec{J}dv}{4\pi R}$$

(magnetic potential = 0 (@ $R = \infty$) as like electrostatic potential = 0 (@ $R = \infty$)

($\because |\vec{A}| \propto \frac{1}{R}$ ← current location (@ $r = 0$) & measurement point (@ $r = \infty$))

7.7 Derivation of Steady-Magnetic-Field Law

- Relationships among the magnetic field quantities

$$\vec{H} = \oint \frac{Id\vec{L} \times \vec{a}_R}{4\pi R^2}$$

$$\vec{B} = \mu_0 \vec{H}$$

$$\vec{B} = \nabla \times \vec{A}$$

* Proof of Biot-Savart Law

- Let $\vec{A} = \int_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R}$

where current element location: (x_1, y_1, z_1)

$\vec{A}$ measurement location: (x_2, y_2, z_2)

differential volume element location: $dv_1 (= dx_1 dy_1 dz_1)$

- So $\vec{A}_2 = \int_{vol} \frac{\mu_0 \vec{J}_1 dv_1}{4\pi R_{12}}$
- Since $\vec{H} = \frac{\vec{B}}{\mu_0} = \frac{\nabla \times \vec{A}}{\mu_0}$, $\vec{H}_2 = \frac{\nabla_2 \times \vec{A}_2}{\mu_0} = \frac{1}{\mu_0} \nabla_2 \times \int_{vol} \frac{\mu_0 \vec{J}_1 dv_1}{4\pi R_{12}}$

$$= \frac{1}{4\pi} \int_{vol} \nabla_2 \times \frac{\vec{J}_1 dv_1}{R_{12}}$$

$$= \frac{1}{4\pi} \int_{vol} \left(\nabla_2 \times \frac{1}{R_{12}} \vec{J}_1 \right) dv_1$$
- By using vector identity, $\nabla \times (S\vec{V}) = (\nabla S) \times \vec{V} + S(\nabla \times \vec{V})$

$$\vec{H}_2 = \frac{1}{4\pi} \int_{vol} \left[\left(\nabla_2 \frac{1}{R_{12}} \right) \times \vec{J}_1 + \frac{1}{R_{12}} (\nabla_2 \times \vec{J}_1) \right] dv_1$$

= 0 $\because (\vec{J}_1, (x_1, y_1, z_1))$ 에 대한 함수를 $\nabla_2, (x_2, y_2, z_2)$ 에 관하여 미분하므로.

- Because of $R_{12} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ and

$$\nabla_2 \frac{1}{R_{12}} = -\frac{\vec{R}_{12}}{R_{12}^3} = -\frac{\vec{a}_{R_{12}}}{R_{12}^2} \quad (\leftarrow \#7-42)$$

$$\begin{aligned} \vec{H}_2 &= -\frac{1}{4\pi} \int_{vol} \frac{\vec{a}_{R_{12}} \times \vec{J}_1}{R_{12}^2} dv_1 \\ &= \int_{vol} \frac{\vec{J}_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2} dv_1 = \int_{vol} \frac{\vec{J}_1 dv_1 \times \vec{a}_{R_{12}}}{4\pi R_{12}^2} \quad (\text{proven}) \end{aligned}$$

- Replacing $\vec{J} dv_1 = \vec{I}_1 d\vec{L}_1$,

$$\begin{aligned} \vec{H}_2 &= \int \frac{I_1 d\vec{L}_1 \times \vec{a}_{R_{12}}}{4\pi R} \\ \Rightarrow \vec{A} &= \int_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R} \quad \text{is correct.} \end{aligned}$$

* Proof of Ampere's Circuital Law

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}$$

이하 ○ 이 '0'임을 증명하는 과정

- By using vector identities,

$$\nabla \times \nabla \times \vec{A} = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$$

$$\text{where } \nabla^2 \vec{A} \equiv \nabla^2 A_x \vec{a}_x + \nabla^2 A_y \vec{a}_y + \nabla^2 A_z \vec{a}_z$$

$$\nabla \times \vec{H} = \frac{1}{\mu_0} [\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}]$$

- Since $\nabla \cdot (V\vec{D}) \equiv V(\nabla \cdot \vec{D}) + \vec{D} \cdot (\nabla V)$,

$$\nabla_2 \cdot \vec{A}_2 = \nabla_2 \cdot \int_{vol} \frac{\mu_o \vec{J}_1 dv_1}{4\pi R_{12}} = \frac{\mu_o}{4\pi} \int_{vol} \nabla_2 \cdot \frac{1}{R_{12}} \vec{J}_1 dv_1$$

= 0

$$= \frac{\mu_o}{4\pi} \int_{vol} \left[\frac{1}{R_{12}} (\nabla_2 \cdot \vec{J}_1) + \vec{J}_1 \cdot \left(\nabla_2 \frac{1}{R_{12}} \right) \right] dv_1 \quad (62)$$

▪ Because of $\nabla_2 \frac{1}{R_{12}} = -\frac{\vec{R}_{12}}{R_{12}^3}, \quad \nabla_1 \frac{1}{R_{12}} = \frac{\vec{R}_{12}}{R_{12}^3}$

$$\nabla_1 \frac{1}{R_{12}} = -\nabla_2 \frac{1}{R_{12}}$$

▪ So, $\nabla_2 \cdot \vec{A}_2 = \frac{\mu_0}{4\pi} \int_{vol} \left[-\vec{J}_1 \cdot \left(\nabla_1 \frac{1}{R_{12}} \right) \right] dv_1 \leftarrow -\vec{D} \cdot (\nabla V) = V(\nabla \cdot \vec{D}) - \nabla \cdot (V\vec{D})$

$$= \frac{\mu_0}{4\pi} \int_{vol} \left[\frac{1}{R_{12}} (\nabla_1 \cdot \vec{J}_1) - \nabla_1 \cdot \left(\frac{\vec{J}_1}{R_{12}} \right) \right] dv_1$$

$$= -\frac{\mu_0}{4\pi} \oint \frac{\vec{J}_1}{R_{12}} \cdot d\vec{S}_1$$

$$= 0$$

=0 (왜냐하면 steady magnetic fields)

$$\vec{J}_1 = \text{constant} \Rightarrow \frac{\partial J_{1x}}{\partial x} = 0, \frac{\partial J_{1y}}{\partial y} = 0, \frac{\partial J_{1z}}{\partial z} = 0$$

(S_1 은 모든 체적을 둘러싼 폐곡면. S_1 을 모든 전류가 포함되도록 체적을 포함한 폐곡면으로 설정하면, 체적 표면의 밖으로 나가는 전류밀도

$$\vec{J}_1 = 0 \Rightarrow \oint \frac{\vec{J}_1}{R_{12}} \cdot d\vec{S}_1 = 0)$$

- Comparison of magnetic vector potential with electric potential

$$A_x = \int_{vol} \frac{\mu_0 J_x dv}{4\pi R} \quad \leftrightarrow \quad V = \int_{vol} \frac{\rho dv}{4\pi \epsilon_0 R}$$

$$\rightarrow \left(J_x \leftrightarrow \rho, \quad \mu_0 \leftrightarrow \frac{1}{\epsilon_0}, \quad A_x \leftrightarrow V \right)$$

$$\nabla^2 V = -\frac{\rho}{\epsilon_0} : \text{Poisson's equation}$$

$$\Leftrightarrow \begin{cases} \nabla^2 A_x = -\mu_0 \vec{J}_x \\ \nabla^2 A_y = -\mu_0 \vec{J}_y \\ \nabla^2 A_z = -\mu_0 \vec{J}_z \end{cases} \Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}$$

$$\therefore \nabla \times \vec{H} = \nabla \times \frac{\vec{B}}{\mu_0} = \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A}$$

$$= \frac{1}{\mu_0} [\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}] = -\frac{1}{\mu_0} \nabla^2 \vec{A} \quad (\because \nabla(\nabla \cdot \vec{A}) = 0)$$

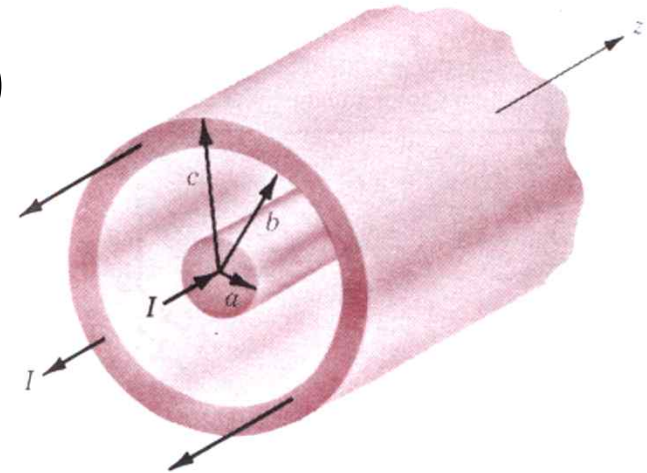
$$= \vec{J} \quad (\text{proven})$$

[Ex.] Coaxial cable

At $a < \rho < b$, $\vec{J} = 0$ (in dielectric)

$$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J} = 0 \quad (\text{Laplace equation})$$

$$\begin{aligned} \text{In general, } \nabla^2 \vec{A} &= \nabla^2 A_x \vec{a}_x + \nabla^2 A_y \vec{a}_y + \nabla^2 A_z \vec{a}_z \\ &\neq \nabla^2 A_\rho \vec{a}_\rho + \nabla^2 A_\phi \vec{a}_\phi + \nabla^2 A_z \vec{a}_z \end{aligned}$$



(But $\vec{a}_z$ component is same in both coordinates.)

$$\nabla^2 \vec{A} \Big|_z = \nabla^2 A_z \quad \therefore \nabla^2 A_z = \mu_0 J_z = 0 \quad \left(\leftarrow \nabla^2 A_z = \nabla \cdot \nabla A_z \right)$$

(Since the coaxial cable is directed to z-axis and $J_z = 0$ in $a < \rho < b$, we can consider only $\vec{A} = A \vec{a}_z$ component.)

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_z}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 A_z}{\partial \phi^2} + \frac{\partial^2 A_z}{\partial z^2} = 0 \quad (\because \text{The current is varying for } \rho, \text{ symmetrical for } \phi, \text{ and constant for } z.)$$

$$\Rightarrow \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_z}{\partial \rho} \right) = 0$$

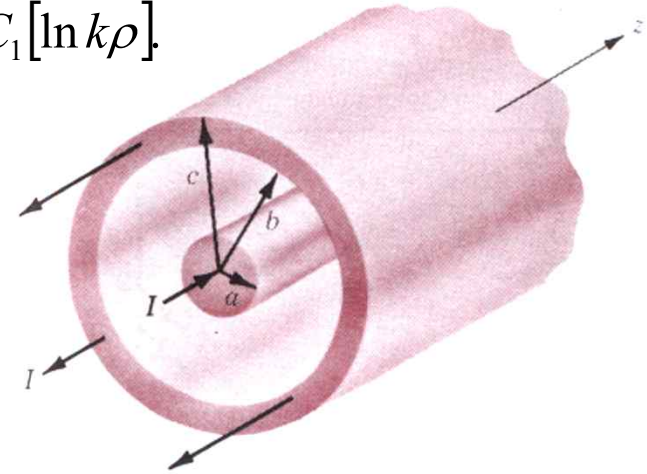
- General solution:

$$A_z = C_1 \ln \rho + C_2 = C_1 \left[\ln \rho + \frac{C_2}{C_1} \right] = C_1 [\ln \rho + \ln k] = C_1 [\ln k\rho].$$

- Since $A_z = 0$ @ $\rho = b$,

$$kb = 1 \quad \therefore k = \frac{1}{b}$$

$$\therefore A_z = C_1 \ln \frac{\rho}{b}$$



$$\begin{aligned} \nabla \times \vec{A} &= \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) \vec{a}_\rho + \left(\frac{1}{\rho} \frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \vec{a}_\phi + \left[\frac{1}{\rho} \frac{\partial (\rho A_\phi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial A_\rho}{\partial \phi} \right] \vec{a}_z \\ &= -\frac{\partial A_z}{\partial \rho} \vec{a}_\phi = -C_1 \cdot \frac{1}{\rho} \cdot \frac{1}{b} \vec{a}_\phi = -C_1 \cdot \frac{1}{\rho} \vec{a}_\phi \quad (\because \text{비례상수이므로}) \\ &= \vec{B} = \mu_0 \vec{H} \end{aligned}$$

$$\therefore \vec{H} = -\frac{C_1}{\mu_0 \rho} \vec{a}_\phi$$

$$\oint \vec{H} \cdot d\vec{L} = \int_0^{2\pi} \left(-\frac{C_1}{\mu_0 \rho} \right) \vec{a}_\phi \cdot \rho d\phi \vec{a}_\phi = -\frac{C_1}{\mu_0} \cdot 2\pi = I$$

$$C_1 = -\frac{\mu_0 I}{2\pi}$$

$$A_z = C_1 \ln \frac{\rho}{b} = \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho}$$

$$\vec{H} = -\frac{C_1}{\mu_0 \rho} \vec{a}_\phi = \frac{I}{2\pi \rho} \vec{a}_\phi$$

[Ex.] A plot of A_z versus ρ for $b = 5a$

$$\begin{aligned} A_z &= \frac{\mu_0 I}{2\pi} \ln \frac{b}{\rho} = \frac{\mu_0 I}{2\pi} \ln \frac{5a}{\rho} \\ &= \frac{\mu_0 I}{2\pi} \left[\ln 5 + \ln \frac{a}{\rho} \right] = \frac{\mu_0 I}{2\pi} \left[\ln 5 - \ln \frac{\rho}{a} \right] \end{aligned}$$

