

Engineering Electromagnetics

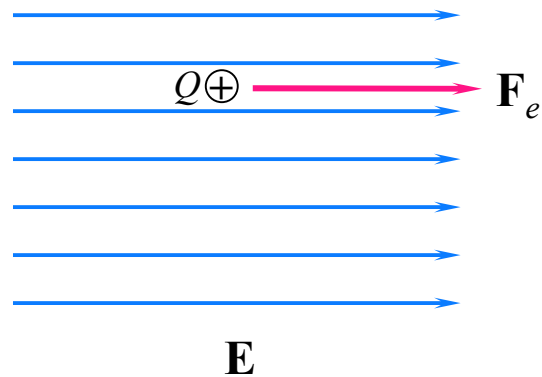
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Chapter 8: Magnetic Forces, Materials, and Inductance

8.1 Force on a Moving Charge

- Definition of electric field intensity

$$\mathbf{F}_e = Q\mathbf{E}$$



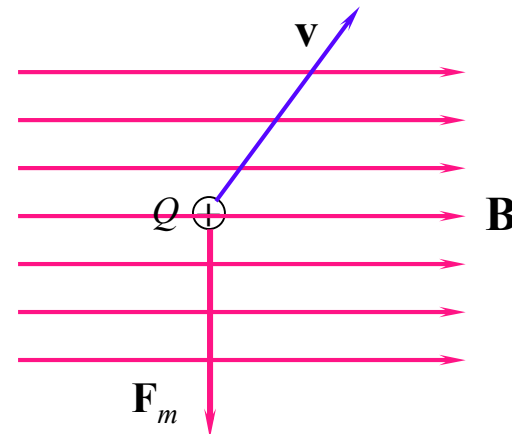
$$\vec{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2} \vec{a}_{1t}$$

$$\vec{E} = \frac{\vec{F}_t}{Q_t}$$

$$(\vec{F}_e // \vec{E}, \vec{F} = f(Q, \vec{E}))$$

- Definition of magnetic field intensity

charge is moving at velocity $\mathbf{v}$ in magnetic flux density $\vec{B}$

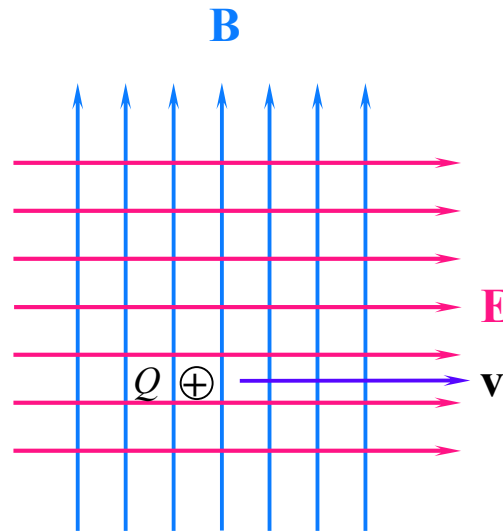


$$\mathbf{F}_m = Q (\mathbf{v} \times \mathbf{B})$$

$$(\vec{F}_m \perp \vec{v}, \vec{F}_m \perp \vec{B})$$

Lorentz Force Law

Generally, with both electric and magnetic fields present, we have both forces:



The electric field will, in this case, accelerate the charge in the direction of $\mathbf{E}$, making it cross the $\mathbf{B}$ field lines in the perpendicular sense; this gives a magnetic force component that is out of the screen

The total force on the moving charge is then the sum of the two, or

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

This is the *Lorentz Force Law* (sometimes called the “fifth Maxwell equation”)

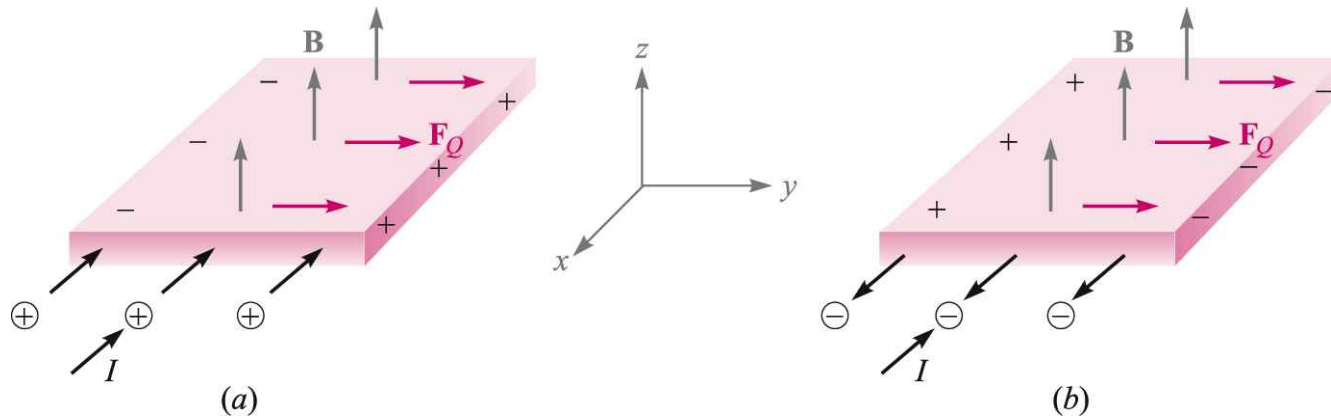
8.2 Force on a differential current element

- Definition force on a charged differential particle moving through a steady magnetic field

$$d\vec{F} = dQ\vec{v} \times \vec{B}$$

$\xrightarrow{\text{red arrow}} \rho_v dv$

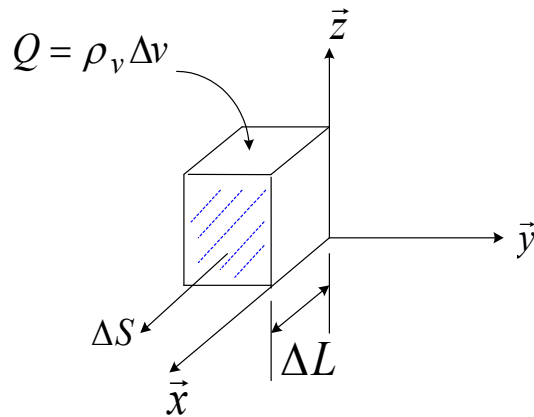
Hall Effect



- Hole injection: $\vec{v} = |\vec{v}|(-\vec{a}_x)$
 $\vec{F} = Q(\vec{v} \times \vec{B}) = Q|\vec{v}||\vec{B}|(-\vec{a}_x \times \vec{a}_z)$
 $= Q|\vec{v}||\vec{B}|\vec{a}_y$
- On hole moving, hole receive a tension to right side. ➔ **P-type semiconductor**

- Electron out-going: $\vec{v} = |\vec{v}|\vec{a}_x$
 $\vec{F} = (-Q)(\vec{v} \times \vec{B}) = (-Q)|\vec{v}||\vec{B}|(\vec{a}_x \times \vec{a}_z)$
 $= Q|\vec{v}||\vec{B}|\vec{a}_y$
- On electron moving, electron receive a tension to right side. ➔ **N-type semiconductor**

Force on a Differential Current Element



$$Q = \rho_v \Delta v = \rho_v \Delta S \Delta L$$

- In the time interval Δt , the element of charge has moved a distance Δx ,

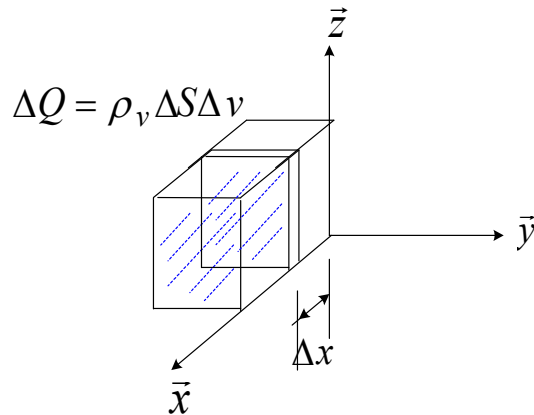
$$\Delta Q = \rho_v \Delta S \Delta x$$

- Let's assume surface ΔS moves Δx for Δt ,

$$\Delta \vec{I} = \frac{\Delta Q}{\Delta t} \vec{a}_x = \rho_v \Delta S \frac{\Delta x}{\Delta t} \vec{a}_x = \rho_v \Delta S \vec{v}_x$$

$$\therefore \vec{J} = \frac{\Delta \vec{I}}{\Delta S} = \rho_v \vec{v}_x$$

$$\boxed{\vec{J} = \rho_v \vec{v}}$$



- Since $dQ = \rho_v dv$,

$$d\vec{F} = dQ(\vec{v} \times \vec{B}) = \rho_v dv(\vec{v} \times \vec{B}) = \rho_v \vec{v} \times \vec{B} dv = \vec{J} \times \vec{B} dv$$

- For different current elements (filament current, surface current, or volume current),

$$Id\vec{L} = \vec{K}dS = \vec{J}dv$$

$$\therefore d\vec{F} = (\vec{J} \times \vec{B})dv = (\vec{K} \times \vec{B})dS = Id\vec{L} \times \vec{B}$$

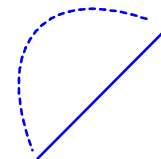
: (modified) Lorentz force equation (3, 2, 1- dimensions)

$$\begin{aligned}\vec{F} &= \int_{vol} (\vec{J} \times \vec{B})dv && \rightarrow \vec{F} = f(I(\text{or } \vec{K}, \vec{J}), \vec{B}) \\ &= \int_s (\vec{K} \times \vec{B})dS \\ &= \oint Id\vec{L} \times \vec{B} = -I \oint \vec{B} \times d\vec{L}\end{aligned}$$

- For straight line conductor in a uniform magnetic field

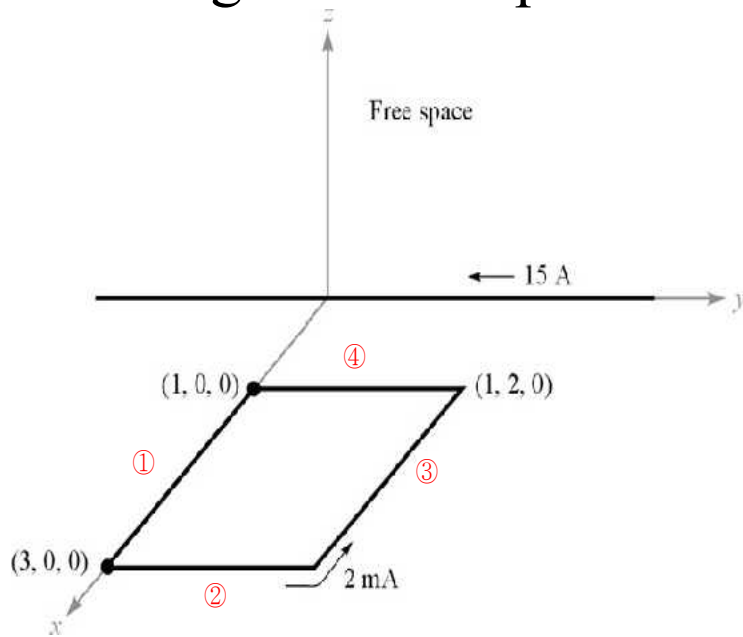
$$\vec{F} = I\vec{L} \times \vec{B} \quad \left(\vec{F} = \oint Id\vec{L} \times \vec{B} \text{ 을 이용하되 도선의 } \pm\infty \text{ 지점이 서로 연결되었다고 가정함} \right)$$

$$F = BIL \sin \theta$$



[Ex. 8.1] Force on a Square Current Loop

- Magnetic field produced in plane of loop by straight filament wire:



$$\vec{H} = \frac{I}{2\pi x} \vec{a}_z$$

$$= \frac{15}{2\pi x} \vec{a}_z \quad [\text{A/m}]$$

$$\left(\vec{H} = \oint \frac{I d\vec{L} \times \vec{a}_R}{4\pi R^2} \right) \quad \left(\begin{array}{l} d\vec{L} = dy(-\vec{a}_y) \\ \vec{a}_R = \vec{a}_x \end{array} \right)$$

$$\vec{H} = \frac{I}{2\pi\rho} \vec{a}_\rho$$

$$\vec{B} = \mu_0 \vec{H} = 4\pi \times 10^{-7} \vec{H} = \frac{3 \times 10^{-6}}{x} \vec{a}_z$$

$$\vec{F} = \oint I d\vec{L} \times \vec{B} = -I \oint \vec{B} \times d\vec{L} = -(2 \times 10^{-3}) \int \frac{3 \times 10^{-6}}{x} \vec{a}_z \times d\vec{L}$$

$$= -(2 \times 10^{-3}) \times (3 \times 10^{-6}) \left[\int_{x=1}^3 \frac{\vec{a}_z}{x} \times dx \vec{a}_x + \int_{y=0}^2 \frac{\vec{a}_z}{3} \times dy \vec{a}_y \right]$$

$$+ \int_{x=3}^1 \frac{\vec{a}_z}{x} \times dx \vec{a}_x + \int_{y=2}^0 \frac{\vec{a}_z}{1} \times dy \vec{a}_y]$$

[Ex. 8.1] Force on a Square Current Loop (continued)

$$\begin{aligned}
 &= -6 \times 10^{-9} \left[\ln x \Big|_1^3 \vec{a}_y + \frac{1}{3} y \Big|_0^2 (-\vec{a}_x) + \ln x \Big|_3^1 \vec{a}_y + y \Big|_2^0 (-\vec{a}_x) \right] \\
 &= -6 \times 10^{-9} \left[\ln 3 \vec{a}_y - \frac{2}{3} \vec{a}_x + \ln\left(\frac{1}{3}\right) \vec{a}_y + 2 \vec{a}_x \right] \\
 &= -6 \times 10^{-9} \left[\ln 3 \vec{a}_y - \frac{2}{3} \vec{a}_x - \ln 3 \vec{a}_y + 2 \vec{a}_x \right] = -8 \times 10^{-9} \vec{a}_x \text{ [N]}
 \end{aligned}$$

8.3 Force Between Differential Current Elements

- Expression of the force on one current element directly in terms of second current element without finding the magnetic field. (직접관계식)
 $\hookrightarrow \vec{B}$ 발생 $\rightarrow$ 1st current element 와 작용하여 $\vec{F}$ 발생 (간접 관계식) (Sec. 8.2)

- Magnetic field at point 2 due to current element at point 1

$$d\vec{H}_2 = \frac{I_1 d\vec{L}_1 \times \vec{a}_{R12}}{4\pi R_{12}^2}$$

- Differential force on a differential current element

$$d\vec{F} = I d\vec{L} \times \vec{B}$$

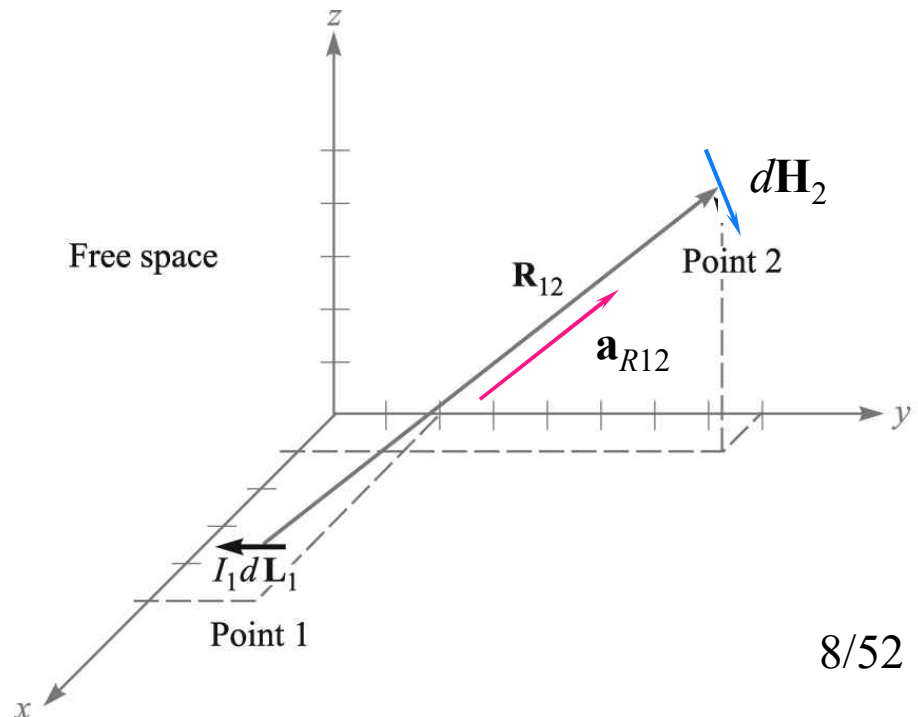
특정 point 1만이 아닌 point 1을 포함한 전체에 의한 magnetic flux density

where $d\vec{B}_2$: differential flux density at point 2 caused by current element 1

$$d(d\vec{F}_2) = I_2 d\vec{L}_2 \times d\vec{B}_2 = I_2 d\vec{L}_2 \times \mu_o \frac{I_1 d\vec{L}_1 \times \vec{a}_{R12}}{4\pi R_{12}^2} = \mu_o \frac{I_1 I_2}{4\pi R_{12}^2} d\vec{L}_2 \times (d\vec{L}_1 \times \vec{a}_{R12})$$

$$= \mu_o \frac{I_1 I_2}{4\pi R_{12}^3} d\vec{L}_2 \times (d\vec{L}_1 \times \vec{R}_{12})$$

($\because d\vec{F}_2 = I_2 d\vec{L}_2 \times \vec{B}_2$ 라고
생각할 수도 있으나 point 2에서의
 $\vec{B}$ 를 일으키는 전류가
한 점 (point 1) 이므로)

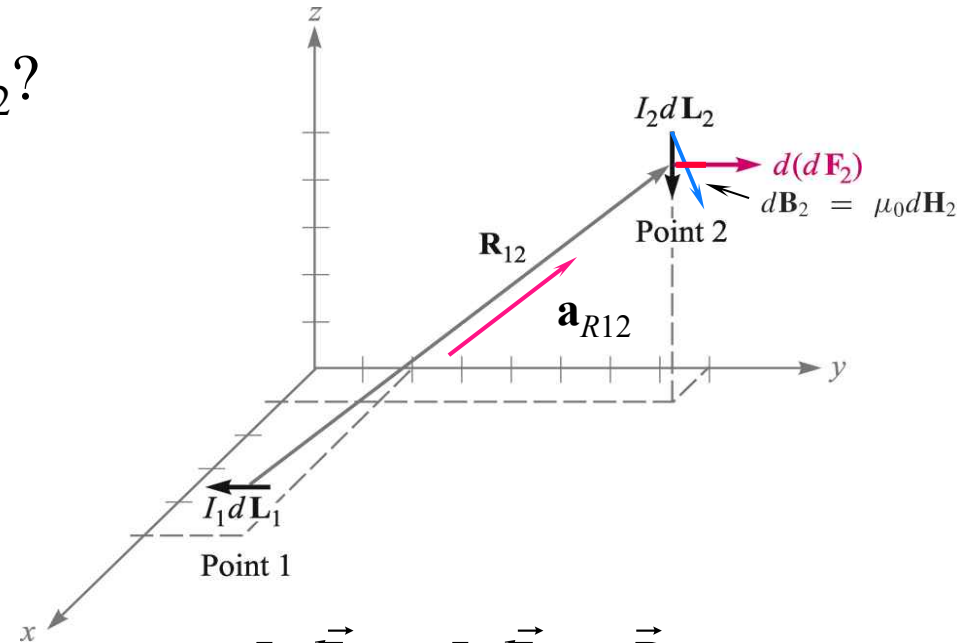


[Ex. 8.2] Differential force at P_2 ?

$$I_1 d\vec{L}_1 = -3\vec{a}_y \quad @ P_1 (5, 2, 1)$$

$$I_2 d\vec{L}_2 = -4\vec{a}_z \quad @ P_2 (1, 8, 5)$$

$$\vec{R}_{12} = -4\vec{a}_x + 6\vec{a}_y + 4\vec{a}_z$$

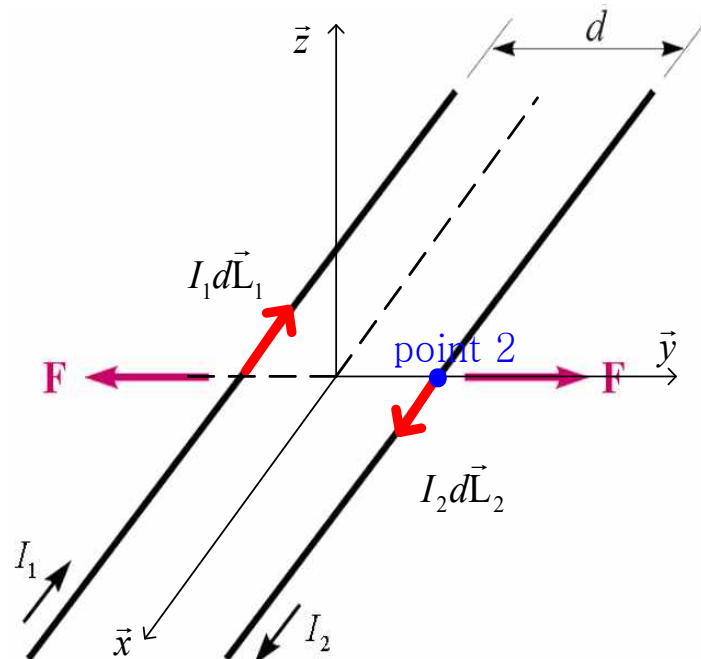


$$\begin{aligned} d(dF_2) &= \mu_o \frac{I_1 I_2}{4\pi R_{12}^3} d\vec{L}_2 \times (d\vec{L}_1 \times \vec{R}_{12}) = \frac{\mu_o}{4\pi} \frac{I_2 d\vec{L}_2 \times I_1 d\vec{L}_1 \times \vec{R}_{12}}{R_{12}^3} \\ &= \frac{4\pi \times 10^{-7}}{4\pi} \frac{(-4\vec{a}_z) \times (-3\vec{a}_y) \times (-4\vec{a}_x + 6\vec{a}_y + 4\vec{a}_z)}{(16 + 36 + 16)^{1.5}} \\ &= 10^{-7} \times \frac{(-4\vec{a}_z) \times (-12\vec{a}_z - 12\vec{a}_x)}{560.7} = 8.56 \times 10^{-9} \vec{a}_y \quad [\text{N}] \end{aligned}$$

▪ Total force between two filamentary circuits:

$$\vec{F}_2 = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[d\vec{L}_2 \times \oint \frac{d\vec{L}_1 \times \vec{a}_{R12}}{R_{12}^2} \right] = \mu_0 \frac{I_1 I_2}{4\pi} \oint \left[\oint \frac{\vec{a}_{R12} \times d\vec{L}_1}{R_{12}^2} \right] \times d\vec{L}_2$$

- Force of repulsion between two infinitely long, straight, parallel, filamentary conductors with separation d , and carrying equal but opposite current I :



$$\begin{aligned}
 d(d\vec{F}_2) &= \mu_0 \frac{I_1 I_2}{4\pi R_{12}^2} d\vec{L}_2 \times (d\vec{L}_1 \times \vec{a}_{R12}) \\
 &= k(\vec{a}_x) \times [(-\vec{a}_x) \times (\vec{a}_y)] \\
 &= k(\vec{a}_x) \times (-\vec{a}_z) = k\vec{a}_y
 \end{aligned}$$

$$\vec{H}_2 = \frac{I}{2\pi d} (-\vec{a}_z) \quad (I_1 \text{ 에 의하여 point 2 에서 느끼는 magnetic field intensity})$$

$$\begin{aligned}
 \vec{F}_2 &= \oint I_2 d\vec{L} \times \vec{B}_2 = \oint I_2 d\vec{L} \times \left(-\frac{\mu_0 I}{2\pi d} \vec{a}_z\right) \\
 &= \underline{I\vec{a}_x} \times \left(-\frac{\mu_0 I}{2\pi d} \vec{a}_z\right) = \frac{\mu_0 I^2}{2\pi d} \vec{a}_y
 \end{aligned}$$

(Straight conductor line 이므로
 $\vec{F} = \oint Id\vec{L} \times \vec{B} = -I \int \vec{B} \times d\vec{L}$
 $\rightarrow \vec{F} = I\vec{L} \times \vec{B}$)

8.4 Force and torque on a closed circuit

- Force exerted on current system:

$$\vec{F} = -I \oint \vec{B} \times d\vec{L}$$

For an **uniform** magnetic flux density,

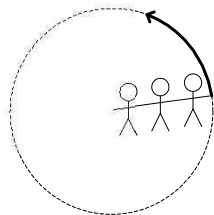
$$\begin{aligned} \vec{F} &= -I\vec{B} \times \oint d\vec{L} \quad \leftarrow \oint d\vec{L} = 0 \\ &= 0 \quad (Id\vec{L} = \vec{K}dS = \vec{J}dv \text{ are all applicable.}) \end{aligned}$$

∴ The force on a closed filamentary circuit in a uniform magnetic field is zero.
But the torque is? (Not '0')

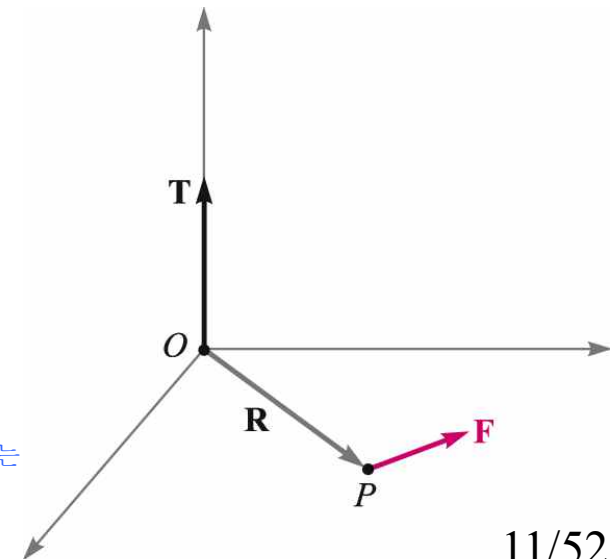
- Torque (or Moment, 회전력)

$$\vec{T} = \vec{R} \times \vec{F}$$

(원점으로부터 점 P에 이르는
rigid lever arm에 힘 $\vec{F}$ 가
주어졌을 때의 회전력)



동그라미 그릴 때 중심에서는
자리를 지키려는 힘이 작용.



- $\vec{F}_1 @ P_1 \rightarrow$ lever arm $\vec{R}_1$
 $\vec{F}_2 @ P_2 \rightarrow$ lever arm $\vec{R}_2$

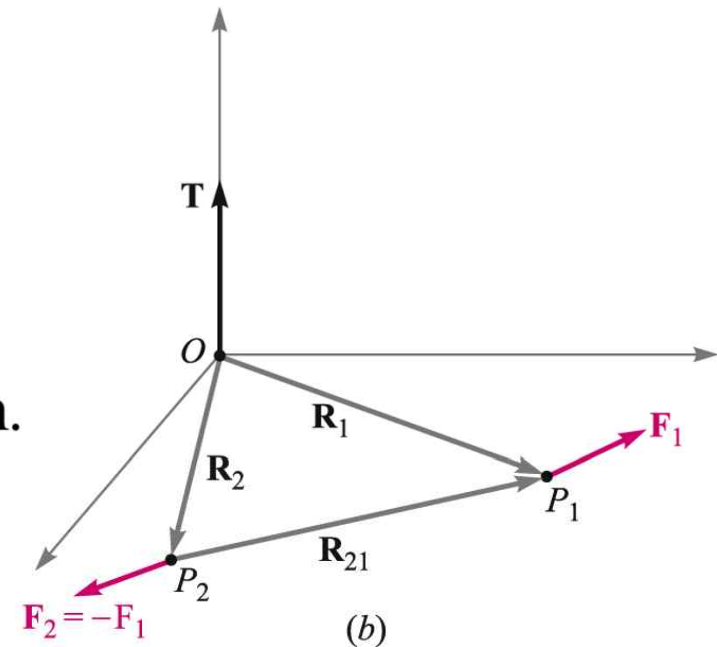
Common origin: O

Object does not undergo any translation.

$$\mathbf{T} = \mathbf{R}_1 \times \mathbf{F}_1 + \mathbf{R}_2 \times \mathbf{F}_2$$

$$\text{where } \mathbf{F}_1 + \mathbf{F}_2 = 0$$

$$\mathbf{T} = (\mathbf{R}_1 - \mathbf{R}_2) \times \mathbf{F}_1 = \mathbf{R}_{21} \times \mathbf{F}_1$$



➔ *Independent of the choice of origin for $\vec{R}_1$ and $\vec{R}_2$.*

($\vec{R}_{21}$ is a relative distance and not relative on an origin.)

➔ *The torque is independent of the choice of origin if total force is zero.*

Torque on a Differential Current Loop

- Consider the torque on a differential current loop in a magnetic field $\vec{B}$.
- Magnetic flux density at center: $\vec{B}_0$
- Since the loop is of differential size, the value of $\vec{B}$ at any points of the loop can be assumed as $\vec{B}_0$.
- Vector force on side 1:

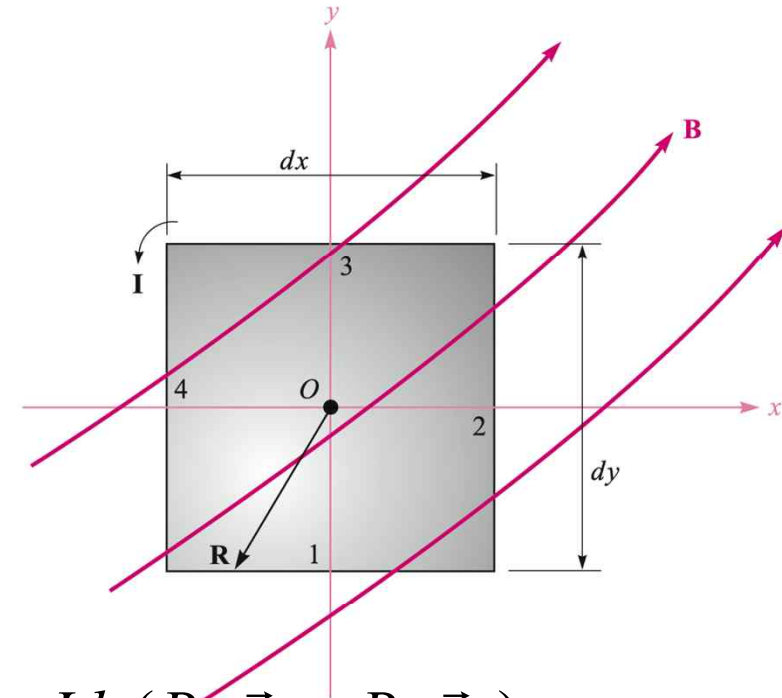
$$d\vec{F}_1 = Idx\vec{a}_x \times \vec{B}_0 \quad \leftarrow \quad \boxed{\vec{F} = I\vec{L} \times \vec{B}}$$

$$= Idx\vec{a}_x \times (B_{0x}\vec{a}_x + B_{0y}\vec{a}_y + B_{0z}\vec{a}_z) = Idx(B_{0y}\vec{a}_z - B_{0z}\vec{a}_y)$$

Since $\vec{R}_1 = -\frac{1}{2}dy\vec{a}_y$,

$$d\vec{T}_1 = \vec{R}_1 \times d\vec{F}_1 = \left(-\frac{1}{2}dy\vec{a}_y\right) \times Idx(B_{0y}\vec{a}_z - B_{0z}\vec{a}_y)$$

$$= -\frac{1}{2}dxdyIB_{0y}\vec{a}_x$$



- By the same procedure on side 3,

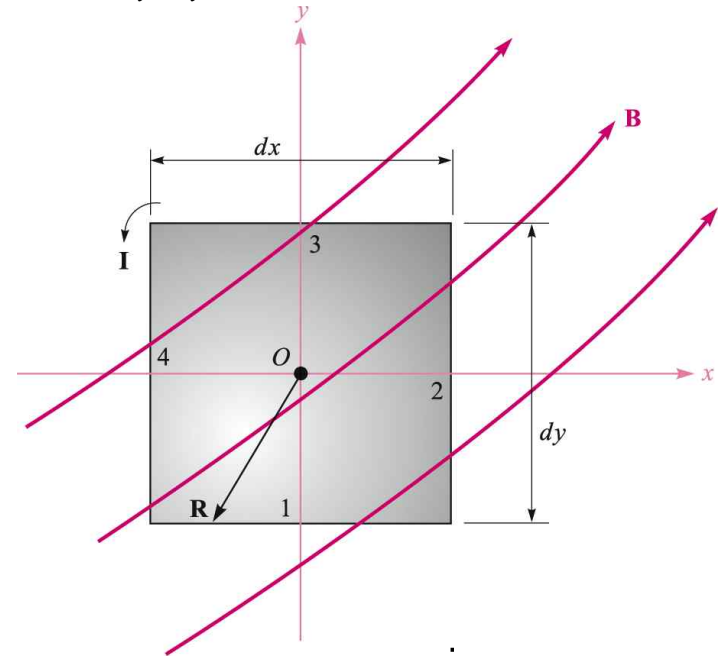
$$\begin{aligned}
 d\vec{T}_3 &= \vec{R}_3 \times d\vec{F}_3 = \left(\frac{1}{2} dy \vec{a}_y\right) \times \left[(-I dx \vec{a}_x) \times (B_{0x} \vec{a}_x + B_{0y} \vec{a}_y + B_{0z} \vec{a}_z)\right] \\
 &= \left(\frac{1}{2} dy \vec{a}_y\right) \times \left[-I dx (B_{0y} \vec{a}_z - B_{0z} \vec{a}_y)\right] \\
 &= -\frac{1}{2} dx dy I B_{0y} \vec{a}_x = d\vec{T}_1
 \end{aligned}$$

$$\therefore d\vec{T}_1 + d\vec{T}_3 = -dx dy I B_{0y} \vec{a}_x$$

- By the same procedure on side 2,

$$\begin{aligned}
 d\vec{T}_2 &= \vec{R}_2 \times d\vec{F}_2 = \left(\frac{1}{2} dx \vec{a}_x\right) \times \left[(I dy \vec{a}_y) \times (B_{0x} \vec{a}_x + B_{0y} \vec{a}_y + B_{0z} \vec{a}_z)\right] \\
 &= \left(\frac{1}{2} dx \vec{a}_x\right) \times \left[I dy (-B_{0x} \vec{a}_z + B_{0z} \vec{a}_x)\right] = \frac{1}{2} dx dy I B_{0x} \vec{a}_y = d\vec{T}_4
 \end{aligned}$$

$$\therefore d\vec{T}_2 + d\vec{T}_4 = dx dy I B_{0x} \vec{a}_y$$



- Total torque:

$$\begin{aligned}
 d\vec{T} &= d\vec{T}_1 + d\vec{T}_2 + d\vec{T}_3 + d\vec{T}_4 \\
 &= I dx dy (-B_{0y} \vec{a}_x + B_{0x} \vec{a}_y) = I dx dy \vec{a}_z \times (B_{0x} \vec{a}_x + B_{0y} \vec{a}_y + B_{0z} \vec{a}_z) \\
 &= I dx dy (\vec{a}_z \times \vec{B}_0) \\
 &= I \left[(dx dy \vec{a}_z) \times \vec{B}_0 \right] \\
 &= I (d\vec{S} \times \vec{B}_0) = Id\vec{S} \times \vec{B}_0
 \end{aligned}$$

where $d\vec{S}$: vector area of differential current loop

- Differential magnetic dipole moment: $d\vec{m}$ [A· m²]

$$d\vec{m} = Id\vec{S}$$

$$\therefore d\vec{T} = d\vec{m} \times \vec{B}$$

- Similar procedure in $\vec{E}$ -field,

$$d\vec{T} = d\vec{p} \times \vec{E} : \text{Torque produced on differential electric dipole by an electric field}$$

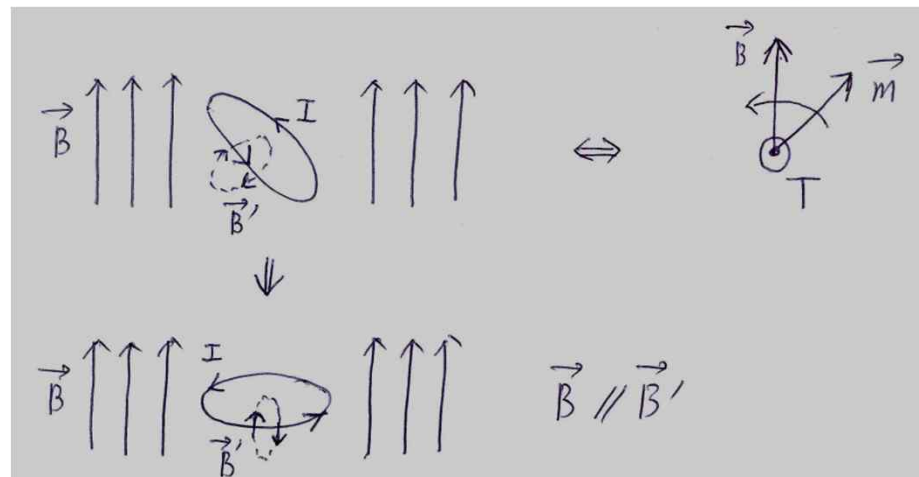
- Torque on a planar loop of any size or shape in an uniform magnetic field:

$$\vec{T} = I\vec{S} \times \vec{B} = \vec{m} \times \vec{B}$$

↗ magnetic dipole moment

(loop 전류에 의해 생기는 자기방향을 외부의 자기방향과 일치시키도록 (전류 loop를 회전시키려는) 회전력이 가해짐.

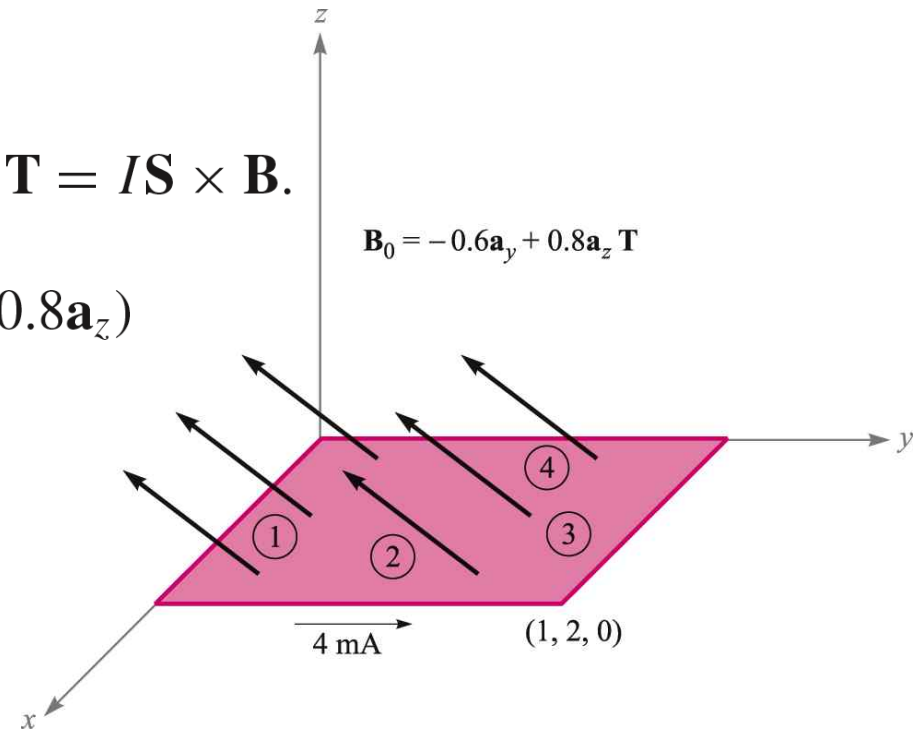
→ **The easiest way to determine the direction of the torque**



[Ex. 8.3] Calculate the torque by using $\mathbf{T} = I\mathbf{S} \times \mathbf{B}$.

$$\begin{aligned}\mathbf{T} &= 4 \times 10^{-3} [(1)(2)\mathbf{a}_z] \times (-0.6\mathbf{a}_y + 0.8\mathbf{a}_z) \\ &= 4.8\mathbf{a}_x \text{ mN} \cdot \text{m}\end{aligned}$$

(Thus, the loop tends to rotate about an axis parallel to the positive x -axis. The small magnetic field produced by the 4 mA loop current tends to line up with $\vec{B}$.)



[Ex. 8.4] Decomposite solution

- On side ①, $\vec{F}_1 = I\vec{L}_1 \times \vec{B}_0 = (4 \times 10^{-3})\vec{a}_x \times (-0.6\vec{a}_y + 0.8\vec{a}_z)$
 $= (-2.4\vec{a}_z - 3.2\vec{a}_y) \times 10^{-3} \text{ [N]}$
- On side ③, $\vec{F}_3 = I\vec{L}_3 \times \vec{B}_0 = 4 \times 10^{-3}(-\vec{a}_x) \times (-0.6\vec{a}_y + 0.8\vec{a}_z)$
 $= (2.4\vec{a}_z + 3.2\vec{a}_y) \times 10^{-3} \text{ [N]}$

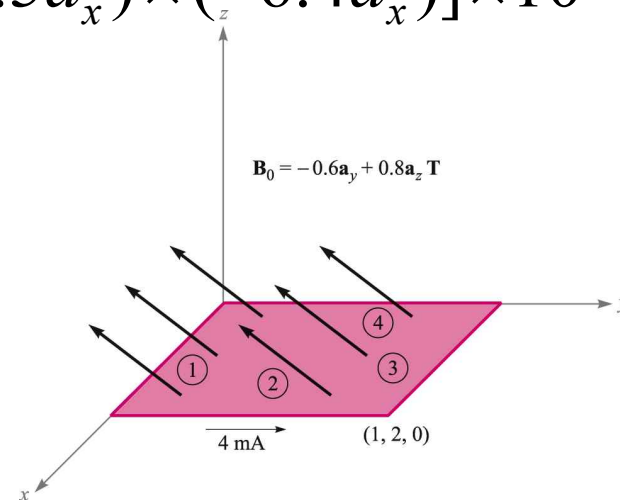
- On side ②, $\vec{F}_2 = I\vec{L}_2 \times \vec{B}_0 = (4 \times 10^{-3})(2\vec{a}_y) \times (-0.6\vec{a}_y + 0.8\vec{a}_z)$
 $= 6.4\vec{a}_x \times 10^{-3} \text{ [N]}$

- On side ④, $\vec{F}_4 = I\vec{L}_4 \times \vec{B}_0 = -6.4\vec{a}_x \times 10^{-3} \text{ [N]}$

- Since these forces are distributed uniformly along each of the sides, each forces were applied at the center (1/2, 1).

$$\begin{aligned}\vec{T} &= \vec{T}_1 + \vec{T}_2 + \vec{T}_3 + \vec{T}_4 = \vec{R}_1 \times \vec{F}_1 + \vec{R}_2 \times \vec{F}_2 + \vec{R}_3 \times \vec{F}_3 + \vec{R}_4 \times \vec{F}_4 \\ &= [(-1\vec{a}_y) \times (-3.2\vec{a}_y - 2.4\vec{a}_z) + (0.5\vec{a}_x) \times (6.4\vec{a}_x) \\ &\quad + (1\vec{a}_y) \times (3.2\vec{a}_y + 2.4\vec{a}_z) + (-0.5\vec{a}_x) \times (-6.4\vec{a}_x)] \times 10^{-3} \\ &= (2.4\vec{a}_x + 2.4\vec{a}_x) \times 10^{-3} \\ &= 4.8 \times 10^{-3} \vec{a}_x \text{ [N/m]}\end{aligned}$$

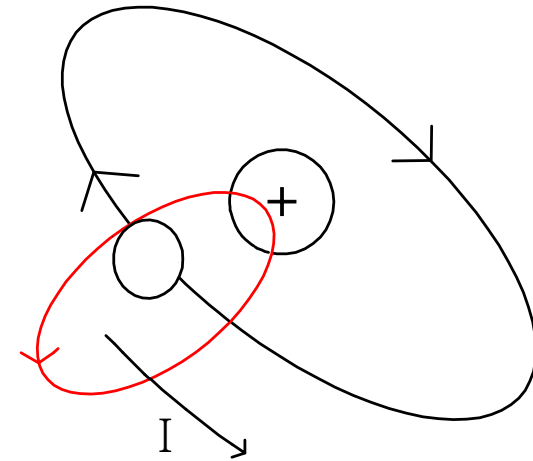
➔ Same result with Ex. 8.3



8.5 The Nature of Magnetic Materials

1) The 1st moment in magnetic fields

- An electron in an orbit is analogous to a small current loop.
- $\vec{T}$ occurs due to $\vec{T} = I\vec{S} \times \vec{B}$.
- The torque tends to align the magnetic field produced by the orbiting electron with the external magnetic fields.

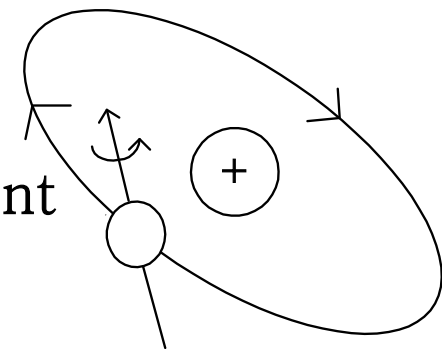


2) The 2nd moment is attributed to electron spin

$$d\vec{m} = \pm 9 \times 10^{-24} A \cdot m^2 : \text{spin magnetic moment}$$



alignment aiding or opposing an external magnetic field.



3) The 3rd moment is caused by nuclear spin. ➔ Negligible effect

Magnetic Material Summary

Classification	Magnetic Moments	B Values	Comments
Diamagnetic	$\mathbf{m}_{\text{orb}} + \mathbf{m}_{\text{spin}} = 0$	$B_{\text{int}} < B_{\text{appl}}$	$B_{\text{int}} \doteq B_{\text{appl}}$
Paramagnetic	$\mathbf{m}_{\text{orb}} + \mathbf{m}_{\text{spin}} = \text{small}$	$B_{\text{int}} > B_{\text{appl}}$	$B_{\text{int}} \doteq B_{\text{appl}}$
Ferromagnetic	$ \mathbf{m}_{\text{spin}} \gg \mathbf{m}_{\text{orb}} $	$B_{\text{int}} \gg B_{\text{appl}}$	Domains
Antiferromagnetic	$ \mathbf{m}_{\text{spin}} \gg \mathbf{m}_{\text{orb}} $	$B_{\text{int}} \doteq B_{\text{appl}}$	Adjacent moments oppose
Ferrimagnetic	$ \mathbf{m}_{\text{spin}} \gg \mathbf{m}_{\text{orb}} $	$B_{\text{int}} > B_{\text{appl}}$	Unequal adjacent moments oppose; low σ
Superparamagnetic	$ \mathbf{m}_{\text{spin}} \gg \mathbf{m}_{\text{orb}} $	$B_{\text{int}} > B_{\text{appl}}$	Nonmagnetic matrix; recording tapes

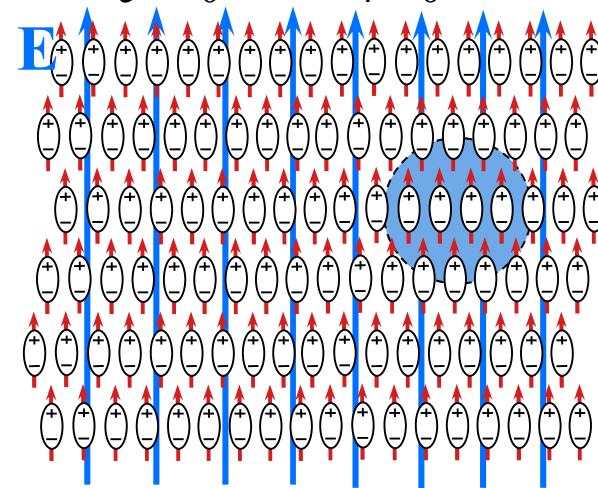
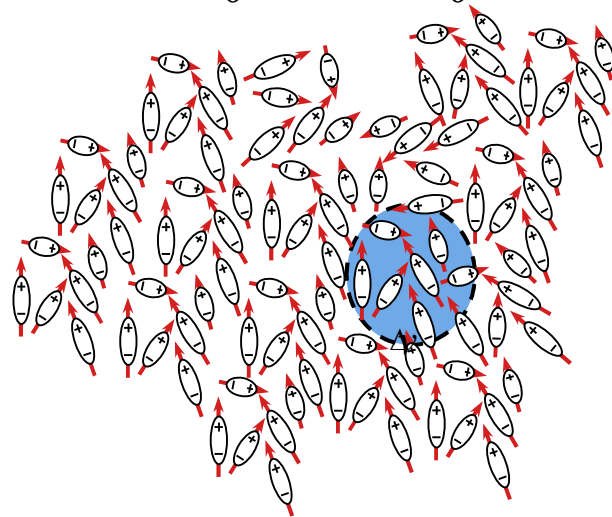
8.6 Magnetization and Permeability

- The magnetic dipoles act as a distributed source for the magnetic field.

cf.) Electric dipole: $\vec{P} = Q\vec{d} = \chi_e \epsilon_0 \vec{E}$

Electric flux density in dielectric materials

$$\vec{D} = \vec{D}_0 + \vec{P} = \epsilon_0 \vec{E} + \chi_e \epsilon_0 \vec{E} = (1 + \chi_e) \epsilon_0 \vec{E} = \epsilon_r \epsilon_0 \vec{E}$$



- Movement of bound charges (orbital electrons, electron spin, nuclear spin): bound current or Amperian current
 → A new field caused by the bound charged currents has the same dimension of $\vec{H}$ and is called the magnetization $\vec{M}$.

Magnetic Dipole Ensembles

- Consider a bound current, I_b , surrounding a differential area, dS .
- Magnetic dipole moment:

$$\mathbf{m} = I_b d\mathbf{S}$$

- Consider n magnetic dipoles per unit volume and volume Δv

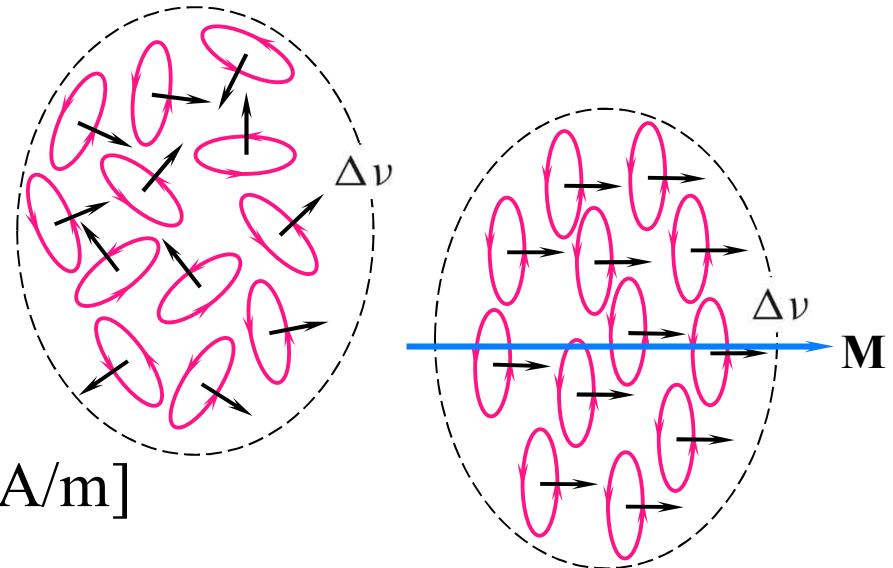
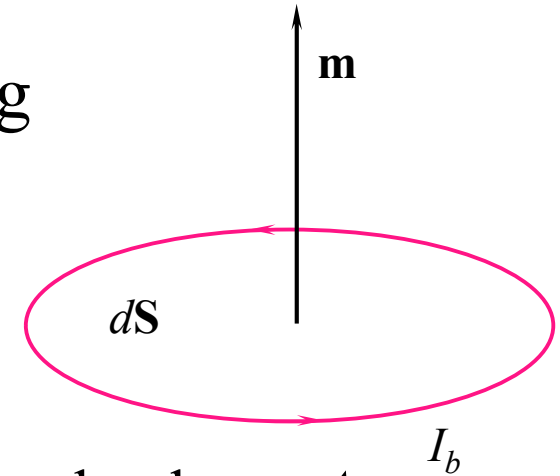
$$\mathbf{m}_{\text{total}} = \sum_{i=1}^{n\Delta v} \mathbf{m}_i$$

- Magnetization* $\vec{M}$: magnetic dipole moment per unit volume as the volume shrinks to a point

$$\vec{M} = \lim_{\Delta v \rightarrow 0} \frac{\vec{m}_{\text{total}}}{\Delta v} = \lim_{\Delta v \rightarrow 0} \frac{1}{\Delta v} \sum_{i=1}^{n\Delta v} \vec{m}_i \quad [\text{A/m}]$$

- If all have the same orientation, the magnetization is simplified as

$$\mathbf{M} = n\mathbf{m} = nI_b d\mathbf{S}$$



Bound Current Formulation

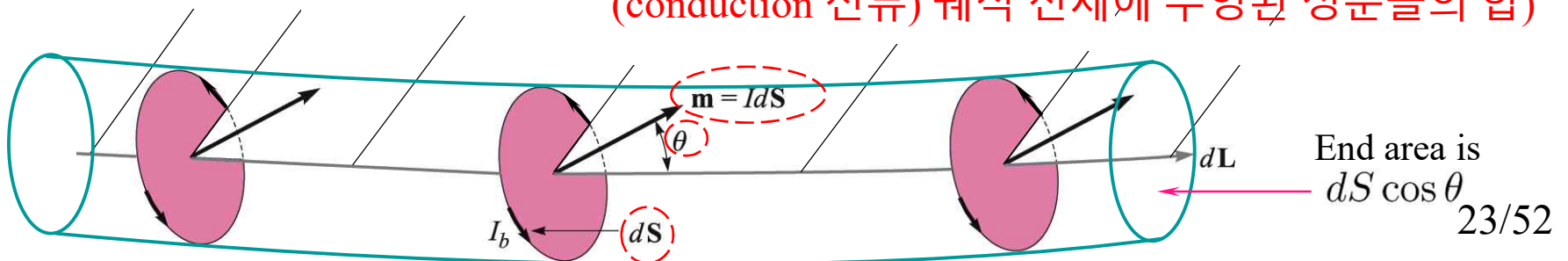
- Identical magnetic moments ($\vec{m}$, n per unit volume) makes an angle θ with the element of path $d\mathbf{L}$.
- $\vec{m}$ consists of a bound current I_b circulation about an area $d\vec{S}$.
- Differential *bound current*, crossing the surface along length dL :

$$dI_B = \underbrace{nI_b}_{\text{Dipole current per unit volume}} \underbrace{d\mathbf{S} \cdot d\mathbf{L}}_{\text{Differential volume, } dv} = \mathbf{M} \cdot d\mathbf{L}$$

(Magnetic moment를 일으키는 전체 전류 (nI_b) 중에 $d\mathbf{L}$ 방향으로 투영된 n 개의 bounded current를 고려)

$$I_B = \oint \mathbf{M} \cdot d\mathbf{L} \quad (21) \quad : \text{within and entire closed contour}$$

(bounded 전류가 실제로 움직이는 (conduction 전류) 궤적 전체에 투영된 성분들의 합)



General Relation Between **B** and **H** in specific media except free space

- The *total current* (I_T) in a general medium consist of the sum of bound and free currents
- Defining **B** as the fundamental magnetic field quantity, Ampere's Circuital Law for the total current becomes:

$$\oint \frac{\mathbf{B}}{\mu_0} \cdot d\mathbf{L} = I_T = I_{\mathbf{B}} + I$$

- *Total free current* (I):

$$\oint \mathbf{H} \cdot d\mathbf{L} = I = I_T - I_{\mathbf{B}} = \oint \left(\frac{\mathbf{B}}{\mu_0} - \mathbf{M} \right) \cdot d\mathbf{L}$$

Thus: $\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}$... and finally: $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$

- Current and Related Current Densities with Stoke's theorem

$$I_T = I + I_B \quad \leftarrow \quad \begin{aligned} I &= \oint \vec{H} \cdot d\vec{L} = \int (\nabla \times \vec{H}) \cdot d\vec{S} \\ &= \int \vec{J} \cdot d\vec{S} \end{aligned} \quad (21)$$

Bound Current: $I_B = \int_s \mathbf{J}_B \cdot d\mathbf{S}$

$$= \oint \vec{M} \cdot d\vec{L} = \int (\nabla \times \vec{M}) \cdot d\vec{S} \quad \rightarrow \quad \nabla \times \vec{M} = \vec{J}_B$$

Total Current: $I_T = \int_s \mathbf{J}_T \cdot d\mathbf{S}$ (22)

$$= \oint \frac{\vec{B}}{\mu_0} \cdot d\vec{L} = \int (\nabla \times \frac{\vec{B}}{\mu_0}) \cdot d\vec{S} \quad \rightarrow \quad \nabla \times \frac{\vec{B}}{\mu_0} = \vec{J}_T$$

Conduction Current: $I = \int_s \mathbf{J} \cdot d\mathbf{S}$ (26)

$$= \oint \vec{H} \cdot d\vec{L} = \int (\nabla \times \vec{H}) \cdot d\vec{S} \quad \rightarrow \quad \nabla \times \vec{H} = \vec{J}$$

- Relation between $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{M}$

$$\mathbf{M} = \chi_m \mathbf{H} \quad \text{for linear isotropic media}$$

↑
magnetic susceptibility function
(frequency domain)

$$\vec{B} = \mu_0 (\vec{H} + \chi_m \vec{H}) = \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_r \vec{H}$$

where χ_m : magnetic susceptibility

$\mu_r = 1 + \chi_m$: relative permeability

- *Permeability*: $\mu = \mu_0 \mu_r$

- Consequently: $\mathbf{B} = \mu \mathbf{H}$

[Ex. 8.5] Ferrite material ($\mu_r = 50$). $|B| = 0.05$ [T]

$$\chi_m = \mu_r - 1 = 49 \quad \text{(isotropic material 이므로 vector로 표기하지 않음)}$$

$\vec{H} // \vec{B} // \vec{M}$)

$$H = \frac{B}{\mu_0 \mu_r} = \frac{0.05}{4\pi \times 10^{-7} \times 50} = 796 \quad [\text{A/m}]$$

$$M = \chi_m H = 49H = 39000 \quad [\text{A/m}] \quad (>> \vec{H})$$

➔ Magnetic flux density depends on the motion of the bound charges mainly

▪ Permeability of an anisotropic magnetic material

$$B_x = \mu_{xx}H_x + \mu_{xy}H_y + \mu_{xz}H_z$$

$$B_y = \mu_{yx}H_x + \mu_{yy}H_y + \mu_{yz}H_z \quad \Rightarrow [B] = [\mu][H]$$

$$B_z = \mu_{zx}H_x + \mu_{zy}H_y + \mu_{zz}H_z$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) : \text{valid} \quad (\vec{H} \nparallel \vec{B}, \vec{H} \nparallel \vec{M}, \vec{B} \nparallel \vec{M})$$

Permeability
tensor

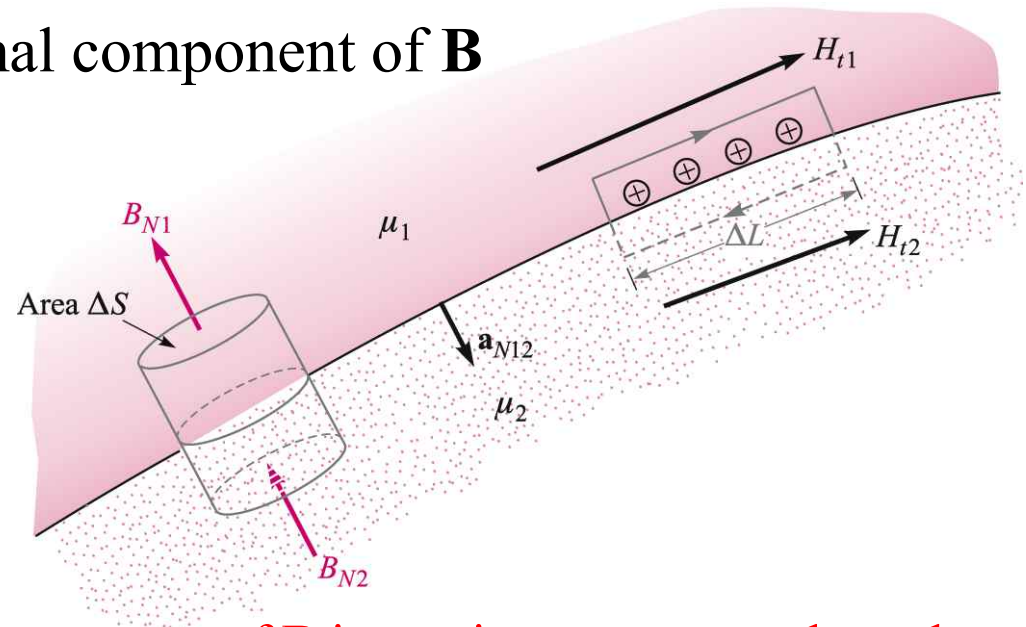
8.7 Magnetic (Field) Boundary Conditions

- Two isotropic homogeneous linear material with permeabilities μ_1 and μ_2 .
- Boundary condition for normal component of $\mathbf{B}$ (with Gauss' Law)

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

$$B_{N1} \Delta S - B_{N2} \Delta S = 0$$

$$@ h \rightarrow 0$$



$B_{N2} = B_{N1}$ → The normal component of $\mathbf{B}$ is continuous across a boundary.

$$\Rightarrow \mu_2 H_{N2} = \mu_1 H_{N1}, \quad H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \quad \leftarrow M = \chi_m H, \quad H = \frac{M}{\chi_m}$$

$$\frac{M_{N2}}{\chi_{m2}} = \frac{\mu_1}{\mu_2} \frac{M_{N1}}{\chi_{m1}} \Rightarrow M_{N2} = \frac{\chi_{m2} \mu_1}{\chi_{m1} \mu_2} M_{N1}$$

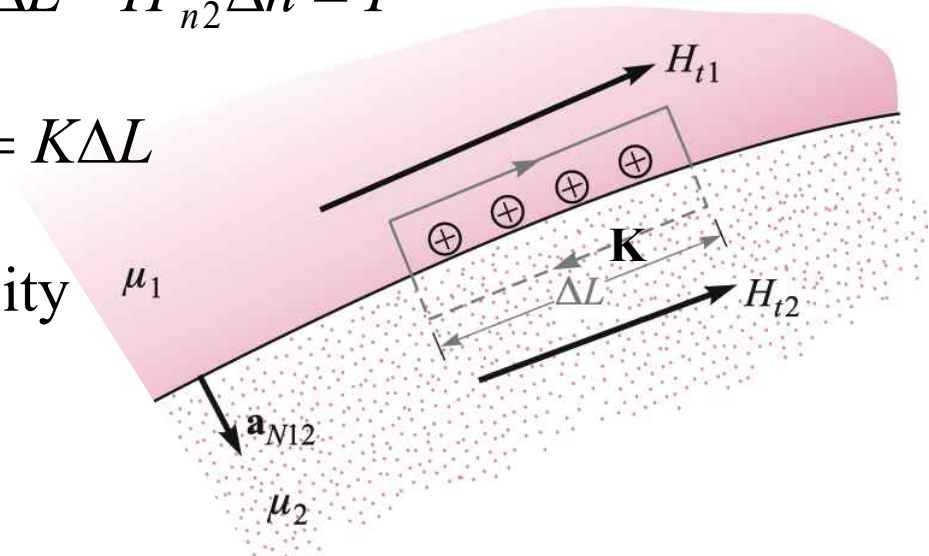
- Boundary condition for tangential component of $\mathbf{H}$ with Ampere's circuital law

$$\oint \vec{H} \cdot d\vec{L} = H_{t1}\Delta L + H_{n1}\Delta h - H_{t2}\Delta L - H_{n2}\Delta h = I$$

$$\lim_{\Delta h \rightarrow 0} \oint \vec{H} \cdot d\vec{L} = H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

where $\mathbf{K}$: surface current density

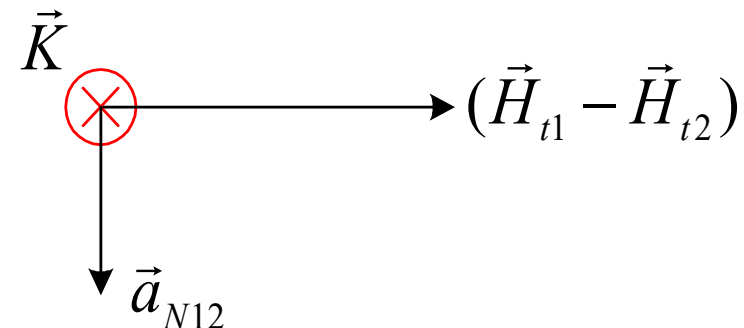
$$\therefore \boxed{H_{t1} - H_{t2} = K}$$



- More generally:

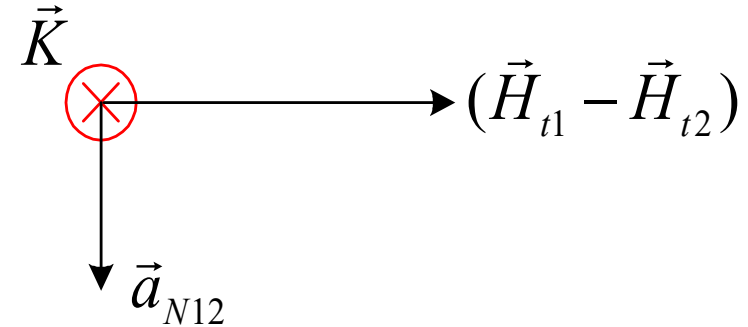
$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{N12} = \mathbf{K}$$

$$\text{or } \mathbf{H}_{t1} - \mathbf{H}_{t2} = \mathbf{a}_{N12} \times \mathbf{K}$$



- An equivalent formulation

$$\vec{H}_{t1} - \vec{H}_{t2} = \vec{a}_{N12} \times \vec{K}$$



$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \quad \leftarrow \quad B = \mu_0 \mu_r H = \mu \frac{M}{\chi_m}$$

$$\frac{\cancel{\mu_1} \frac{M_{t1}}{\chi_{m1}}}{\cancel{\mu_1}} - \frac{\cancel{\mu_2} \frac{M_{t2}}{\chi_{m2}}}{\cancel{\mu_2}} = K$$

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K$$

[Ex. 8.6] $\mu_1 = 4 [\mu\text{H/m}]$ @ $z > 0$ (region 1)

$\mu_2 = 7 [\mu\text{H/m}]$ @ $z < 0$ (region 2)

$$\vec{K} = 80\vec{a}_x [\text{A/m}] \text{ @ } z = 0, \quad \vec{B}_1 = 2\vec{a}_x - 3\vec{a}_y + \vec{a}_z [\text{mT}] \quad \vec{B}_2 = ?$$

$$\rightarrow \mathbf{B}_{N1} = (\mathbf{B}_1 \cdot \mathbf{a}_{N12})\mathbf{a}_{N12} = [(2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z) \cdot (-\mathbf{a}_z)](-\mathbf{a}_z) = \mathbf{a}_z \text{ mT}$$

$$\therefore \underline{\mathbf{B}_{N2} = \mathbf{B}_{N1} = \mathbf{a}_z \text{ mT}}$$

$$\mathbf{B}_{t1} = \mathbf{B}_1 - \mathbf{B}_{N1} = 2\mathbf{a}_x - 3\mathbf{a}_y \text{ mT}$$

$$\mathbf{H}_{t1} = \frac{\mathbf{B}_{t1}}{\mu_1} = \frac{(2\mathbf{a}_x - 3\mathbf{a}_y)10^{-3}}{4 \times 10^{-6}} = 500\mathbf{a}_x - 750\mathbf{a}_y \text{ A/m}$$

$$\begin{aligned} \mathbf{H}_{t2} &= \mathbf{H}_{t1} - \mathbf{a}_{N12} \times \mathbf{K} = 500\mathbf{a}_x - 750\mathbf{a}_y - (-\mathbf{a}_z) \times 80\mathbf{a}_x \\ &= 500\mathbf{a}_x - 750\mathbf{a}_y + 80\mathbf{a}_y = 500\mathbf{a}_x - 670\mathbf{a}_y \text{ A/m} \end{aligned}$$

$$\mathbf{B}_{t2} = \mu_2 \mathbf{H}_{t2} = 7 \times 10^{-6}(500\mathbf{a}_x - 670\mathbf{a}_y) = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y \text{ mT}$$

$$\mathbf{B}_2 = \mathbf{B}_{N2} + \mathbf{B}_{t2} = \underline{3.5\mathbf{a}_x - 4.69\mathbf{a}_y + \mathbf{a}_z \text{ mT}}$$

8.8 The Magnetic Circuit

- Relationship between the electrostatic potential and electric field intensity:

$$\mathbf{E} = -\nabla V$$

- Its analogous relation to the magnetic field intensity.

$$\mathbf{H} = -\nabla V_m \quad @ \vec{J} = 0$$

where V_m : scalar magnetic potential or magnetomotive force (mmf)

- Electric potential difference between A and B

$$V_{AB} = \int_A^B \mathbf{E} \cdot d\mathbf{L} \quad (\text{이동 경로에 무관})$$

By the corresponding relation between the mmf and $\vec{H}$,

$$V_{mAB} = \int_A^B \mathbf{H} \cdot d\mathbf{L} \quad (\text{특정 이동 경로를 지정하면 specific value 발생})$$

- Ohm's law for electric field: $\vec{J} = \sigma \vec{E}$
- The magnetic flux density will be the analog of the current density.

$$\vec{B} = \mu \vec{H}$$

- Total (electric) current: $I = \int_S \mathbf{J} \cdot d\mathbf{S}$

↔ Total magnetic flux flowing through the cross section of a magnetic circuit

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

- Resistance: $V = IR$

↔ Reluctance: $V_m = \Phi \underline{\underline{\mathfrak{R}}}$

└─→ [A · turn/Wb]

$$\mathfrak{R} = \frac{V_m}{\Phi} : \text{a ratio of magnetomotive force (mmf) to total flux}$$

Reluctance as an Analogy to Resistance

$$\cdot R = \frac{d}{\sigma S} \quad \Leftrightarrow \quad \boxed{\Re = \frac{d}{\mu S}}$$

where d : isotropic homogeneous material length

S : uniform cross section

$$\cdot \text{Source voltage in electric circuit: } \oint \mathbf{E} \cdot d\mathbf{L} = 0$$

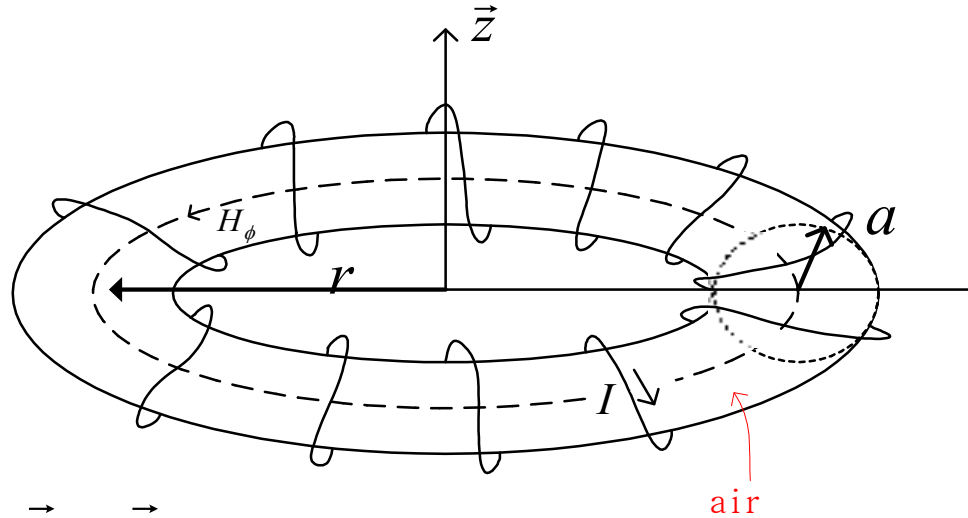
➔ KVL: The rise in potential (electromotive force, emf) through the source is exactly equal to the fall in potential through the load.

$$\Leftrightarrow \oint \mathbf{H} \cdot d\mathbf{L} = I_{\text{total}}$$

(= NI) (in case of a current I flown through an N -turn coil)

$$V_{mAB} = \int_A^B \mathbf{H} \cdot d\mathbf{L} : \text{This quantity is the **mmf** around a closed path, which we use as } V_m \text{ in our magnetic circuit equation.}$$

[Ex.] $N = 500$ turns, $S = 6 \text{ cm}^2$, $r = 15 \text{ cm}$, $I = 4 \text{ A}$, Toroid



$$V_{mAB} = \int_A^B \vec{H} \cdot d\vec{L}$$

$$V_m = \oint \vec{H} \cdot d\vec{L} = NI = 500 \times 4 = 2000 \quad [\text{A} \cdot \text{t}] \quad \leftarrow \text{mmf}$$

$$\mathfrak{R} = \frac{d}{\mu S} = \frac{2\pi \times 0.15}{4\pi \times 10^{-7} \times 6 \times 10^{-4}} = 1.25 \times 10^9 \quad [\text{A} \cdot \text{t/Wb}] \quad \leftarrow \text{reluctance}$$

$$\Phi = \frac{V_m}{\mathfrak{R}} = \frac{2000}{1.25 \times 10^9} = 1.6 \times 10^{-6} \quad [\text{Wb}] \quad \leftarrow \text{total flux}$$

$$B = \frac{\Phi}{S} = \frac{1.6 \times 10^{-6}}{6 \times 10^{-4}} = 2.67 \times 10^{-3} \quad [\text{T}] \quad \leftarrow \text{magnetic flux density}$$

[Ex.] (Continued)

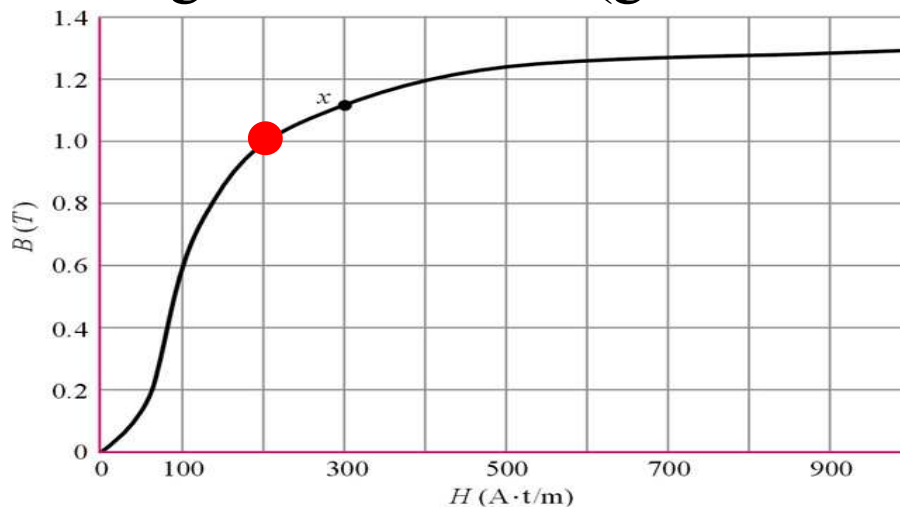
$$H = \frac{B}{\mu} = \frac{2.67 \times 10^{-3}}{4\pi \times 10^{-7}} = 2125 \quad [\text{A} \cdot \text{t/m}] \leftarrow \text{magnetic field intensity in free space}$$

By Ampere's circuital law,

$$H_{\phi} 2\pi r = NI$$

$$H_{\phi} = \frac{NI}{2\pi r} = \frac{500 \times 4}{2\pi \times 0.15} = 2122 \quad [\text{A} \cdot \text{t/m}] : \text{similar results}$$

- Ferromagnetic materials (general material): $|\vec{m}_{spin}| \gg |\vec{m}_{orb}|$



In ferromagnetic materials, **B** increases with increasing **H**, but in a nonlinear manner as shown in the typical curve

FIGURE 8.11 Magnetization curve of a sample of silicon sheet steel.

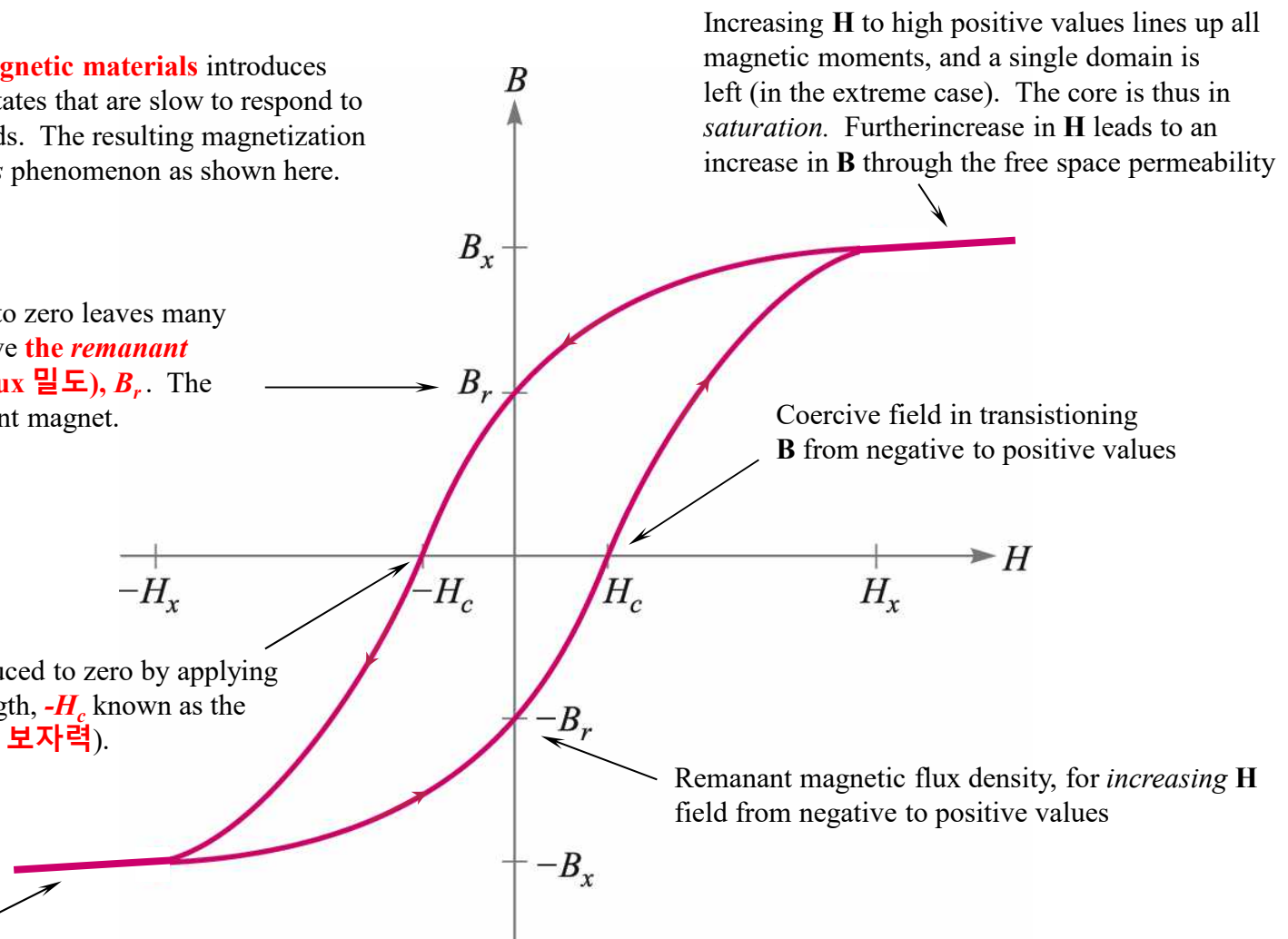
A Further Complication: Hysteresis

Domain wall shifting in **ferromagnetic materials** introduces semi-permanent magnetization states that are slow to respond to changes in applied magnetic fields. The resulting magnetization curve demonstrates the *hysteresis* phenomenon as shown here.

Decreasing the applied $\mathbf{H}$ field to zero leaves many dipoles still aligned, and we have **the remanent magnetic flux density (잔여 flux 밀도), B_r** . The material has become a permanent magnet.

The remanent flux density is reduced to zero by applying an opposing magnetic field strength, $-\mathbf{H}_c$ known as the **coercive field (or coercive force, 보자력)**.

Increasing $\mathbf{H}$ to high negative values again leads to saturation



[Ex. 8.7] Toroid, $d(\text{air gap}) = 2 \text{ mm}$, $N = 500 \text{ turns}$, $S = 6 \text{ cm}^2$, $r = 15 \text{ cm}$, $I = ?$ (To establish a magnetic flux density of 1 [T])

$$\rightarrow \mathfrak{R}_{air} = \frac{d_{air}}{\mu S} = \frac{2 \times 10^{-3}}{4\pi \times 10^{-7} \times 6 \times 10^{-4}} = 2.653 \times 10^6 \text{ [A} \cdot \text{t/Wb]} \leftarrow \text{Reluctance}$$

$$\Phi = BS = 1 \times (6 \times 10^{-4}) = 6 \times 10^{-4} \text{ [Wb]}$$

$$V_{m.air} = \Phi \mathfrak{R}_{air} = (6 \times 10^{-4}) \times (2.653 \times 10^6) = 1592 \text{ [A} \cdot \text{t]}$$

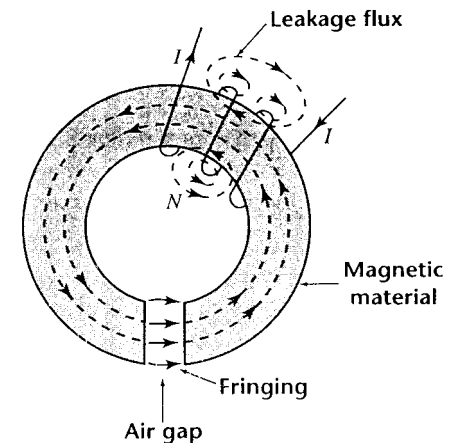
From Fig. 8.11, $H = 200 \text{ [A} \cdot \text{t/m]}$ at $B = 1 \text{ [T]}$

$$H_{steel} = 200 \text{ [A} \cdot \text{t/m]}$$

$$\begin{aligned} V_{m.steel} &= \oint \vec{H} \cdot d\vec{L} \cong H \cdot (\pi \times 2r) = 200 \times (\pi \times 2 \times 0.15) \\ &= 188 \text{ [A} \cdot \text{t]} \end{aligned}$$

$$\begin{aligned} \therefore V_m &= V_{m.air} + V_{m.steel} = 1592 + 188 = 1780 \text{ [A} \cdot \text{t]} \\ &= NI \end{aligned}$$

$$I = V_m / N = 1780 / 500 = 3.56 \text{ [A]}$$



[Ex. 8.8] Given a coil current 4 A in previous example, $B=?$

$$\rightarrow V_m = NI = 500 \times 4 = 2000 \text{ [A}\cdot\text{t]}$$

$$\text{In the previous example, } V_{m,steel} = 188 = \Phi \cdot \mathfrak{R}_{steel} = 6 \times 10^{-4} \mathfrak{R}_{steel}$$

$$\mathfrak{R}_{steel} = 0.313 \times 10^6$$

$$\mathfrak{R}_{total} = \mathfrak{R}_{steel} + \mathfrak{R}_{air} = 2.96 \times 10^6 \text{ [A}\cdot\text{t]}$$

$$V_m = \Phi \cdot \mathfrak{R}_{total} = 2000$$

$$\Phi = 2000 / \mathfrak{R}_{total} = 6.76 \times 10^{-4} \text{ [Wb]} = B \cdot S$$

$$B = 6.76 \times 10^{-4} / S = 1.13 \text{ [T]}$$

8.9 Potential Energy and Force on Magnetic Materials

- Work necessary to bring the prerequisite point charge from infinity to their final resting places:

$$W_E = \int_v \frac{1}{2} \mathbf{D} \cdot \mathbf{E} dv \quad [\text{J}]$$

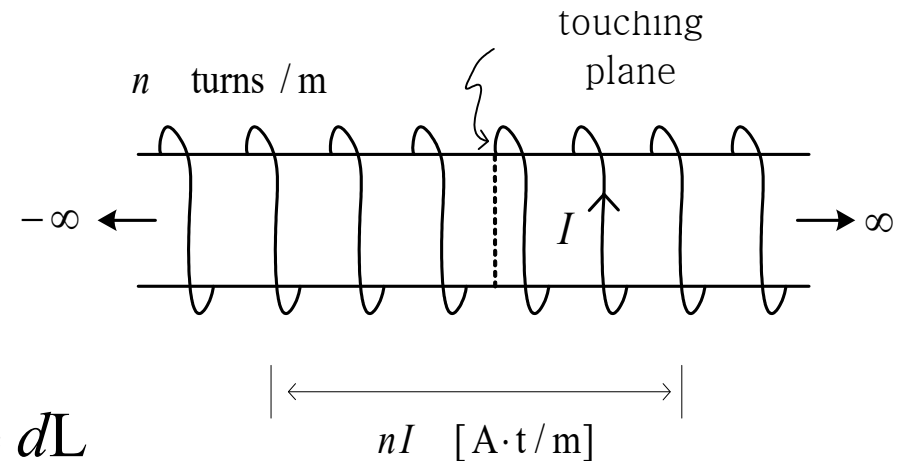
- Total energy stored in a steady magnetic field in which $\vec{B}$ is linear to $\vec{H}$ (in linear media):

$$W_H = \frac{1}{2} \int_{\text{vol}} \vec{B} \cdot \vec{H} dv = \frac{1}{2} \int_{\text{vol}} \mu H^2 dv = \frac{1}{2} \int_{\text{vol}} \frac{B^2}{\mu} dv$$

It is convenient to think that of this energy as being distributed throughout the volume with an energy of $\frac{1}{2} \vec{B} \cdot \vec{H}$ [J/m³].

[Ex.]

- Apply a mechanical force to separate these two sections of the core while keeping the flux density constant: F over distance dL



- The magnetic flux density can be obtained from magnetization curve for silicon steel.: B_{ST}
- Work moving one core appears as stored energy in the air gap

$$dW_H = F dL = \frac{1}{2} \frac{B_{st}^2}{\mu_0} dv = \frac{1}{2} \frac{B_{st}^2}{\mu_0} S dL \quad \text{where } S: \text{core-sectional area}$$

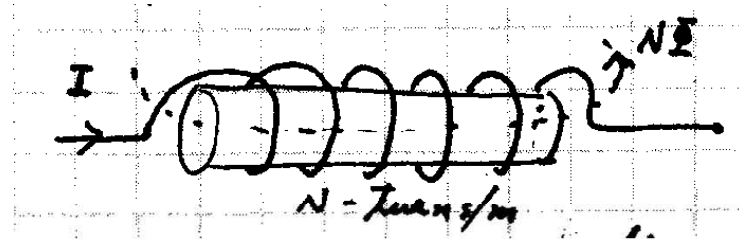
$$F = \frac{B_{st}^2 S}{2\mu_0}$$

[Ex.] If $B = 1.4$ [T], $F = \frac{1.4^2 S}{2 \times 4\pi \times 10^{-7}} = 7.8 \times 10^5 S$ [N]

8.10 Inductance and Mutual Inductance

- Resistance ($R = V / I$) : A ratio of potential difference between two equipotential surfaces of conducting material to total current crossing either equipotential surfaces. ($R = d / (\sigma S)$): geometric function)
- Capacitance ($C = Q / V$) : A ratio of total charge on either of two equipotential conducting surfaces to potential difference between surfaces. ($C = \epsilon \frac{S}{d}$)
- Consider a toroid of N -turns in which a current I produces a total flux Φ . $\rightarrow$ (Total) Flux linkage (λ)

$$\lambda = N\Phi$$



- (Self-) Inductance (L):

$$L = \frac{\lambda}{I} = \frac{N\Phi}{I} \quad \text{for linear magnetic material only.}$$

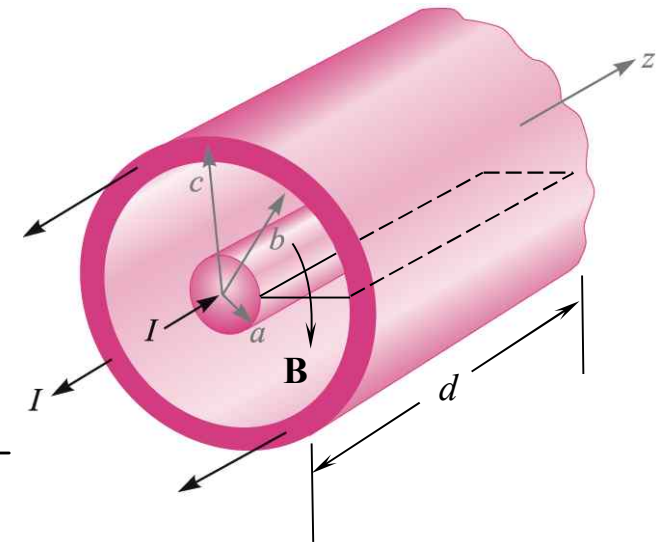
[Ex.] $H_\phi = \frac{I}{2\pi\rho}$ @ $a < \rho < b$ in coaxial cable

$$\vec{B} = \mu_0 \vec{H} = \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi$$

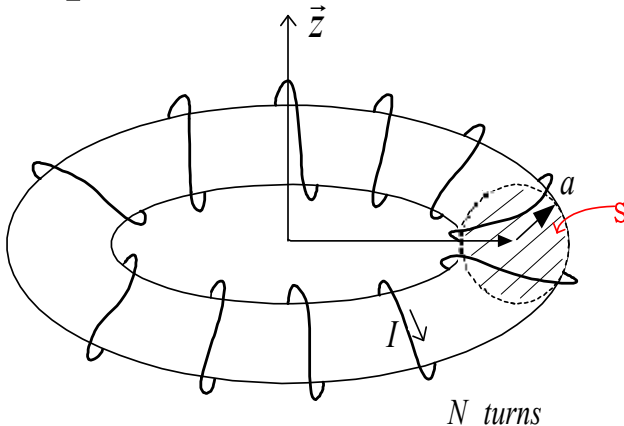
$$\Phi = \int_S \vec{B} \cdot d\vec{S} = \int_0^d \int_a^b \frac{\mu_0 I}{2\pi\rho} \vec{a}_\phi \cdot d\rho dz \vec{a}_\phi = \frac{\mu_0 I d}{2\pi} \ln \frac{b}{a}$$

$$L = \frac{\lambda}{I} = \frac{\Phi}{I} = \frac{\mu_0 d}{2\pi} \ln \frac{b}{a} \quad [\text{H}] \quad \leftarrow \text{assumption : } N=1 \text{ due to coaxial cable}$$

$$= \frac{\mu_0}{2\pi} \ln \frac{b}{a} \quad [\text{H/m}] \quad \leftarrow \text{inductance per unit length}$$



[Ex.] Toroid coil of N -turns, current: I , mean toroid radius: ρ_0



$$B_\phi = \frac{\mu_0 N I}{2\pi\rho_0} \quad \Phi = \frac{\mu_0 N I S}{2\pi\rho_0} \quad \leftarrow \Phi = \int \vec{B} \cdot d\vec{S}$$

$$L = \frac{N\Phi}{I} = \frac{\mu_0 N^2 S}{2\pi\rho_0} \quad \rightarrow \text{page 47}$$

Departure from the Ideal

- In reality, flux density generated by each turn may not link the entire coil. Such *fringing fields* may link only one or two turns.

- Total flux linkages:

$$(N\Phi)_{\text{total}} = \Phi_1 + \Phi_2 + \Phi_3 + \cdots + \Phi_N$$
$$= \sum_{i=1}^N \Phi_i$$

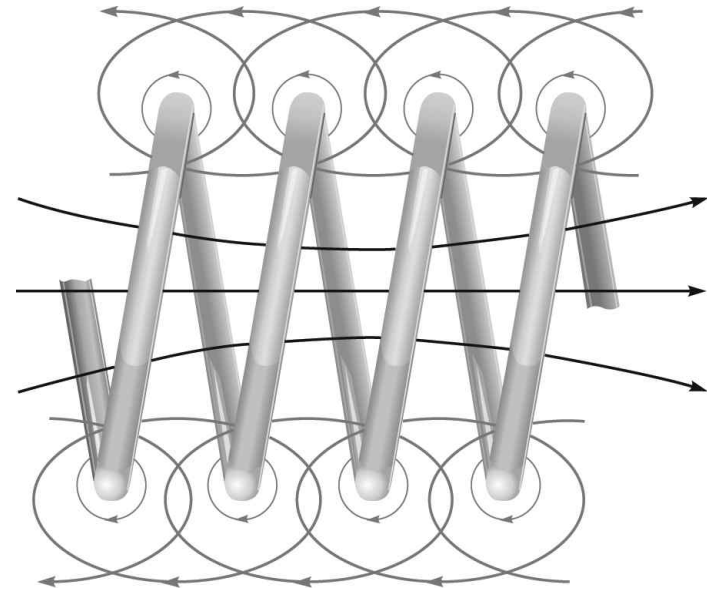
where Φ_i : flux linking the i^{th} turn

- An equivalent (or another) definition for inductance

$$L = \frac{2W_H}{I^2} \quad (\leftarrow W_H = \frac{1}{2} LI^2)$$

where I : total current flowing in closed path

W_H : energy in magnetic field produced by current



$$L = \frac{2W_H}{I^2} \quad (\leftarrow W_H = \frac{1}{2} LI^2)$$

[Proof]

$$\begin{aligned} L &= \frac{2W_H}{I^2} = \frac{2 \times \frac{1}{2} \int_{vol} \vec{B} \cdot \vec{H} dv}{I^2} = \frac{\int_{vol} \vec{B} \cdot \vec{H} dv}{I^2} && \leftarrow \vec{B} = \nabla \times \vec{A} \\ &= \frac{1}{I^2} \int_{vol} \vec{H} \cdot (\nabla \times \vec{A}) dv && \leftarrow \nabla \cdot (\vec{A} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\nabla \times \vec{H}) \\ &= \frac{1}{I^2} \left[\int_{vol} \nabla \cdot (\vec{A} \times \vec{H}) dv + \int_{vol} \vec{A} \cdot (\nabla \times \vec{H}) dv \right] \\ &= \frac{1}{I^2} \left[\oint_S (\vec{A} \times \vec{H}) \cdot d\vec{S} + \int_{vol} \vec{A} \cdot \vec{J} dv \right] = \frac{1}{I^2} \int_{vol} \vec{A} \cdot \vec{J} dv \end{aligned}$$

(모든 magnetic energy를 갖고 있는 체적 element들을 포함하게끔 임의의 폐곡면을 설정하면, 폐곡면 표면에는 magnetic flux density가 '0'이므로 ($\vec{B} = \nabla \times \vec{A} = \mu_0 \vec{H} = 0$), 폐곡면에서 $\vec{A} = 0 = \vec{H}$)

[Proof (continued)]

$$\text{Since } \vec{A} = \int_{vol} \frac{\mu_0 \vec{J} dv}{4\pi R} \quad (7.51),$$

$$\begin{aligned} L &= \frac{1}{I^2} \int_{vol} \left(\int_{vol} \frac{\mu \vec{J} dv}{4\pi R} \right) \cdot \vec{J} dv \quad \leftarrow \vec{J} dv = I d\vec{L} \\ &= \frac{1}{I^2} \oint \left(\oint \frac{\mu I d\vec{L}}{4\pi R} \right) \cdot I d\vec{L} = \frac{\mu}{4\pi} \oint \left(\oint \frac{d\vec{L}}{R} \right) \cdot d\vec{L} \quad (= f(\text{distribution of current in space})) \end{aligned}$$

Let hypothesize a uniform current distribution for simplicity.

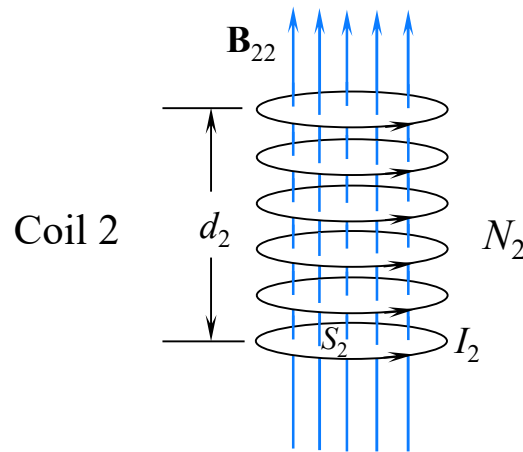
$$\begin{aligned} L &= \frac{1}{I^2} \int_{vol} \vec{A} \cdot \vec{J} dv = \frac{1}{I^2} \oint \vec{A} \cdot I d\vec{L} = \frac{1}{I} \oint \vec{A} \cdot d\vec{L} \quad \leftarrow \text{Stoke's theorem} \\ &= \frac{1}{I} \int_S (\nabla \times \vec{A}) \cdot d\vec{S} = \frac{1}{I} \int_S \vec{B} \cdot d\vec{S} = \frac{\Phi}{I} \quad [\text{H/turn}] \end{aligned}$$

When the closed line integral consist of N laps about this common path.

$$L = \frac{N\Phi}{I} \quad [\text{H}] \quad (\text{Proven})$$

Two Inductors

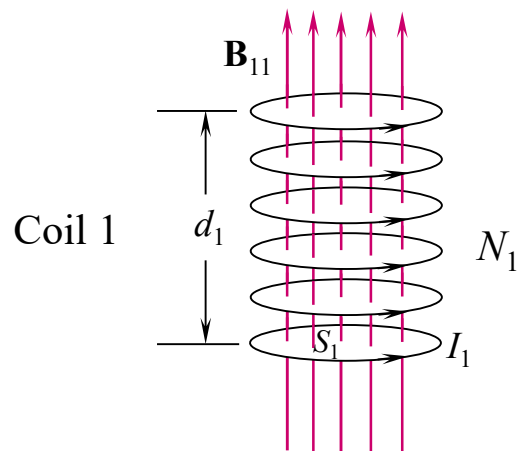
Suppose we have two solenoids, having different specifications as indicated:



The *self linkage* and *self inductance* of each coil are determined in the manner that we used before, assuming identical fluxes through each turn.

$$\longrightarrow \lambda_{22} = N_2 \Phi_{22} = N_2 \int_{S_2} \mathbf{B}_{22} \cdot d\mathbf{S}_2$$

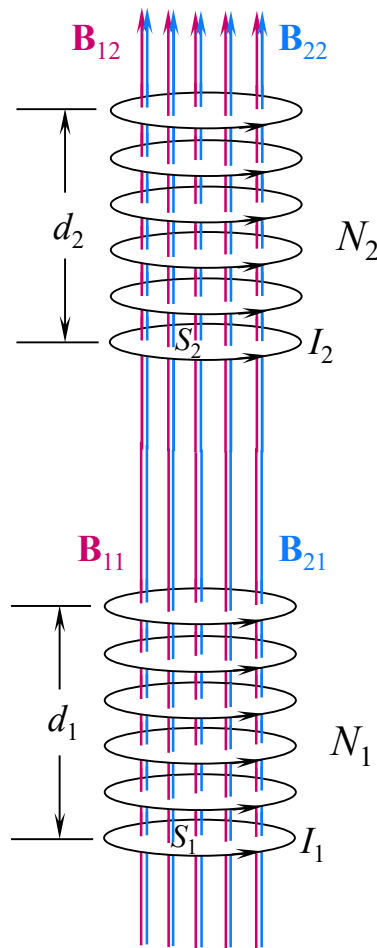
$$\text{and } L_{22} = \frac{\lambda_{22}}{I_2} = N_2^2 \frac{\mu_2 S_2}{d_2}$$



$$\longrightarrow \lambda_{11} = N_1 \Phi_{11} = N_1 \int_{S_1} \mathbf{B}_{11} \cdot d\mathbf{S}_1$$

$$\text{and } L_{11} = \frac{\lambda_{11}}{I_1} = N_1^2 \frac{\mu_1 S_1}{d_1}$$

Interaction Between Inductors



Actually, the magnetic fields generated by each coil will link the other, as shown here. This flux overlap is the basis of *mutual inductance*.

Throughout this discussion, the field in **red** is that generated by Coil 1, while the **blue** field is generated by Coil 2

With both currents on, all the fields indicated here will be present. The fields and other quantities are kept track of by the subscripts, the meaning of which is:

$$\begin{array}{ccc} \mathbf{B}_{ij} & \longrightarrow & \mathbf{M}_{ij} \\ \swarrow \quad \searrow & & \swarrow \quad \searrow \\ \text{arising} & & \text{evaluated} \\ \text{from coil } i & & \text{within coil } j \end{array}$$

$i, j = 1, 2$

Note that the diagrams shown here are oversimplified, because there will be significant spreading of the crossover fields, $\mathbf{B}_{12}$ and $\mathbf{B}_{21}$.

- Mutual inductance between circuits 1 and 2: M_{21}

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1}$$

where Φ_{12} : flux produced by I_1 which links the path of the
filamentary current I_2

N_2 : # of turns in circuit 2

$$M_{12} = \frac{1}{I_1 I_2} \int_{vol} (\vec{B}_1 \cdot \vec{H}_2) dv \quad \leftarrow L = \frac{2W_H}{I^2} \cdot I_1 \text{에 의해 magnetic flux}$$

$$= \frac{1}{I_1 I_2} \int_{vol} (\mu \vec{H}_1 \cdot \vec{H}_2) dv \quad \vec{B}_1 \text{가 발생하여 전류 } I_2 \text{가 흐르는 곳에 영향을 미치는 것.}$$

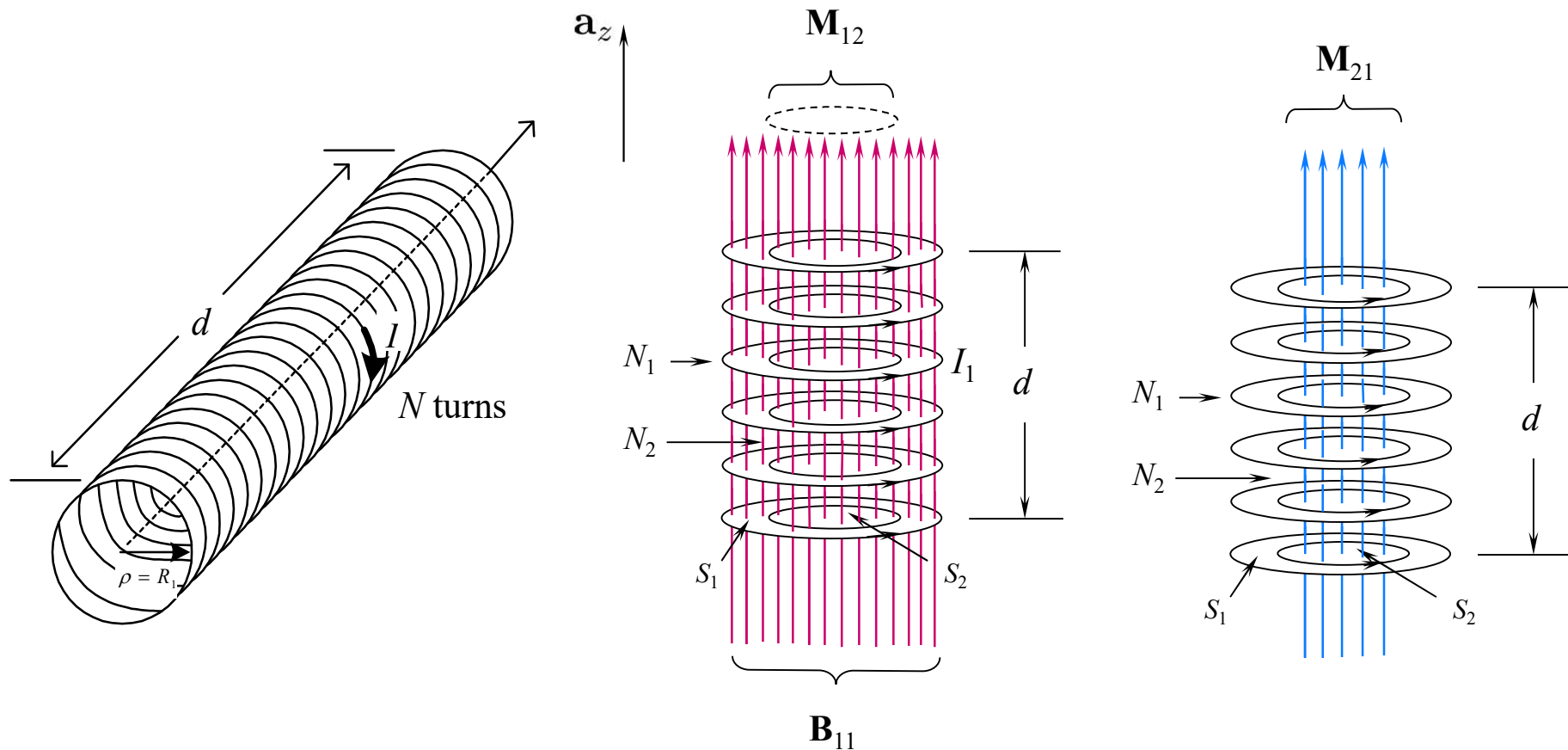
where $\vec{B}_1$: field resulting from I_1 with $I_2 = 0$

$\vec{B}_2$: field resulting from I_2 with $I_1 = 0$

$$\therefore M_{12} = \frac{1}{I_1 I_2} \int_{vol} (\mu \vec{H}_1 \cdot \vec{H}_2) = \frac{1}{I_2 I_1} \int_{vol} (\mu \vec{H}_2 \cdot \vec{H}_1) = \frac{1}{I_2 I_1} \int_{vol} (\vec{B}_2 \cdot \vec{H}_1)$$

$$= M_{21}$$

[Ex. 8.9] Two coaxial solenoid of radius R_1, R_2 ($R_2 > R_1$) carrying currents I_1 and I_2 with n_1, n_2 [turns/m]



$$\begin{aligned} \vec{H}_1 &= \frac{NI_1}{d} \vec{a}_z = n_1 I_1 \vec{a}_z & (0 < \rho < R_1) & \quad \vec{H}_2 = n_2 I_2 \vec{a}_z & (0 < \rho < R_2) \\ &= 0 & (\rho > R_1) & \quad & = 0 & (\rho > R_2) \end{aligned}$$

For uniform field,

$$\Phi_{12} = \mu_0 n_1 I_1 \pi R_1^2 \quad \leftarrow \Phi = \int \vec{B} \cdot d\vec{S}, M_{12} = \frac{\int_v (\vec{B}_1 \cdot \vec{H}_2) dv}{I_1 I_2}$$

B_1 (H_1 - field 발생 가능 영역이 $0 < \rho < R_1$ 이므로, coupling 영역도 $0 < \rho < R_1$ 임.)

$$M_{12} = \frac{(\mu_0 n_1 I_1 \pi R_1^2)(n_2 I_2)}{I_1 I_2} = \mu_0 n_1 n_2 \pi R_1^2$$

Similarly,

$$\Phi_{21} = \mu_0 n_2 I_2 \pi R_1^2$$

B_1 (비록 I_2 에 의해 flux가 πR_2^2 만큼 발생해도 2차측의 면적 $\pi R_1^2 = S_1$ 만큼만 flux가 쇄교 가능)

$$M_{21} = \frac{(\mu_0 n_2 I_2 \pi R_1^2)(n_1 I_1)}{I_2 I_1} = \mu_0 n_1 n_2 \pi R_1^2 = M_{12}$$

If $N_1 = 50$ [turns/cm], $N_2 = 80$ [turns/cm], $R_1 = 2$ [cm], and $R_2 = 3$ [cm],

$$M_{12} = M_{21} = 4\pi \times 10^{-7} \times 5000 \times 8000 \times \pi \times 0.02^2 = 63.17 \times 10^{-3} \text{ [H/m]}$$

$$\begin{aligned} L_1 &= \frac{n_1 \Phi_1}{I_1} = \frac{n_1 B_1 S_1}{I_1} = \frac{n_1 \mu_0 H_1 S_1}{I_1} = \frac{n_1 \mu_0 n_1 I_1 \pi R_1^2}{I_1} \quad (\leftarrow \frac{N_1}{d} = n_1) \\ &= n_1^2 \mu_0 \pi R_1^2 = 5000^2 \times 4\pi \times 10^{-7} \times \pi \times 0.02^2 = 39.48 \times 10^{-3} \text{ [H/m]} \end{aligned}$$

$$\begin{aligned} L_2 &= n_2^2 \mu_0 S_2 \\ &= n_2^2 \mu_0 \pi R_2^2 \\ &= 8000^2 \times 4\pi \times 10^{-7} \times \pi \times 0.03^2 = 227.4 \times 10^{-3} \text{ [H/m]} \end{aligned}$$