Chapter. 5 Impedance Matching and Tuning

- Maximum power is delivered when the load is matched to the line (assuming the generator is matched), and power loss in the feed line is minimized.
- Impedance matching sensitive receiver components (antenna, low-noise amplifier, etc.) improves the signal-to-noise ratio of the system.
- Impedance matching in a power distribution network (such as an antenna array feed network) will reduce amplitude and phase errors.



FIGURE 5.1 A lossless network matching an arbitrary load impedance to a transmission line.

- Factors that may be important in the selection of a particular matching network:
 - 1) Complexity: simple
 - 2) Bandwidth: restricted frequency band
 - 3) Implementation: easy
 - 4) Adjustability: tunable

5.1 MATCHING WITH LUMPED ELEMENTS (L NETWORKS)

- L-section
- Simplest type including two reactive elements
- $z_L = Z_L / Z_0$
- *X*, *B*: lumped or distributed element depending on the operating frequency.



(a) (b) **FIGURE 5.2** *L* section matching networks. (a) Network for z_L inside the 1 + jx circle. (b) Network for z_L outside the 1 + jx circle

Analytic Solutions



- $Z_L = R_L + jX_L$ (assumption: $Z_0 < R_L$, inside of 1 + jx circle)
- Impedance seen looking into the matching network:

$$Z_{in} = jX + \frac{1}{jB + 1/(R_L + jX_L)}$$

= Z₀ (5.1)

- Rearranging and separating into real and imaginary parts:

$$B(XR_{L} - X_{L}Z_{0}) = R_{L} - Z_{0}$$
(5.2*a*)
$$X(1 - BX_{L}) = BZ_{0}R_{L} - X_{L}$$
(5.2*b*)

- Solving Eq. (5.2a) for X and substituting into Eq. (5.2b)

$$B = \frac{X_{L}(\pm)\sqrt{R_{L}/Z_{0}}\sqrt{R_{L}^{2} + X_{L}^{2} - Z_{0}R_{L}}}{R_{L}^{2} + X_{L}^{2}} \quad (5.3a)$$
where $(R_{L}^{2} + X_{L}^{2} - Z_{0}R_{L}) > 0 \quad (\because R_{L} > Z_{0})$

$$X = \frac{BX_{L}Z_{0} + R_{L} - Z_{0}}{BR_{L}} \quad \leftarrow (5.2a)$$

$$= \frac{1}{B} + \frac{X_{L}Z_{0}}{R_{L}} - \frac{Z_{0}}{BR_{L}} \quad (5.3b)$$

 \rightarrow Two solutions: dual valued components (*B*, *X*)



- $Z_L = R_L + jX_L$ (assumption: $Z_0 > R_L$, outside of 1 + jx circle)
- Admittance seen looking into the matching network:

$$Y_{\rm in} = jB + \frac{1}{R_L + j(X + X_L)}$$

= $\frac{1}{Z_0}$ (5.4)

- Rearranging and separating into real and imaginary parts:

$$BZ_0(X + X_L) = Z_0 - R_L (5.5a)$$

$$(X + X_L) = BZ_0 R_L \tag{5.5b}$$

- Solving for *X* and *B*:

$$X = \underbrace{\pm}_{V} \overline{R_L(Z_0 - R_L)} - X_L \qquad (5.6a)$$

$$B = \underbrace{\pm \sqrt{(Z_0 - R_L) / R_L}}_{Z_0}$$
(5.6b)

 \rightarrow Two solutions: dual valued components (*B*, *X*)

Smith Chart Solutions

[Ex. 5.1] Design an *L*-section matching network to match a series *RC* load with an impedance $Z_L = 200 - j100 \Omega$ to a 100 Ω line, at a frequency of 500 MHz.

→ Sol. 1)
$$z_L = 2 - j1$$

 $y_L = 0.4 + j0.2$
 $y = y_L + jb = 0.4 + j0.5 \leftarrow jb = j0.3$
Interconnection with admittance circle of $1 - jx$
 $z = 1 - j1.2$
 $x = 1.2$



(a)

Using Eq.(5.3*a*, *b*), *b* = 0.29 and *x* = 1.22

$$C = \frac{b}{2\pi f Z_0} = 0.92 \text{ pF} \quad \leftarrow B = bY_0 = \frac{b}{Z_0} = \omega C, \quad C = \frac{b}{\omega Z_0}$$
$$L = \frac{xZ_0}{2\pi f} = 38.8 \text{ nH} \quad \leftarrow X = xZ_0 = \omega L, \quad L = \frac{xZ_0}{\omega}$$

Solution 1

Sol. 2) $y_L = 0.4 + j0.2$

$$y = y_L + jb = 0.4 - j0.5 \leftarrow jb = -j0.7$$

Interconnection with admittance circle of 1 + jx

$$z = 1 + j1.2$$
$$x = -1.2$$

Using Eq.(5.3*a*, *b*), *b* = -0.69 and *x* = -1.22

$$C = \frac{1}{2\pi f x Z_0} = 2.61 \text{ pF} \quad \leftarrow jX = jxZ_0 = -j\frac{1}{\omega C}, \quad C = -\frac{1}{\omega x Z_0}$$





Solution 2



(c) **FIGURE5.3** Continued. (b) The two possible *L* section matching circuits. (c) Reflection.



coefficient magnitudes versus frequency for the matching circuits of (b).

5.2 SINGLE-STUB TUNING

- Single open-circuited or short circuited length of transmission line (a "stub"), connected either in parallel or in series with the transmission feed line at a certain distance from the load
 - → The shunt tuning stub is especially easy to fabricate in microstrip or stripline form.

- Shunt-stub case

- $Y_L \rightarrow Y_0 + jB$ using trasmission line having length *d*
 - $\rightarrow Y_0$ using trasmission line having length *l*

- Series-stub case

 $Z_L \rightarrow Z_0 + jX$ using trasmission line having length *d*

 $\rightarrow Z_0$ using trasmission line having length *l*



FIGURE 5.4 Single-stub tuning circuits. (a) Shunt stub. (b) Series stub.

- $\lambda/4$ transmission line with a shunt stub
- Open-circuited stubs are easier to fabricate since a via hole through the substrate to the ground plane is not needed.

Shunt Stubs

[Ex. 5.2] Single-Stub Shunt Tuning

- $-Z_L = 15 + j10 \Omega, f = 2 \text{ GHz}$
- Design two single-stub shunt tuning networks to match this load to a 50 Ω line.
- Plot the reflection coefficient magnitude from 1 GHz to 3 GHz for each solution.

Solution

- $z_L = 0.3 + j0.2$
- SWR circle
- Convert to the load admittance, y_L .
- SWR circle intersects the 1 + jb circle at two points, denoted as y_1 and y_2 in Figure 5.5a. $\rightarrow d_1$ or d_2
- $-d_1 = (0.328 0.284)\lambda = 0.044 \lambda$
 - $d_2 = \{(0.5 0.284) + 0.171\}\lambda = 0.387 \lambda$
- The matching stub is kept as close as possible to the load, to improve the bandwidth of the match and to reduce losses caused by a possibly large standing wave ratio on the line between the stub and the load.
- $-y_1 = 1 j1.33, \quad y_2 = 1 + j1.33$





(b)



(c)

- **FIGURE 5.5** Solution to Example 5.2. (a) Smith chart for the shunt-stub tuners. (b) The two shunt-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).
- Derivation of formulas for d and l

 $Z_L = 1/Y_L = R_L + jX_L$

Impedance Z down a length, d, of line from the load:

$$Z = Z_0 \frac{(R_L + jX_L) + jZ_0 t}{Z_0 + j(R_L + jX_L)t}$$
(5.7)
where $t = \tan \beta d$

where $t = \tan \beta d$

$$Y = G + jB = \frac{1}{Z}$$

where
$$G = \frac{R_L (1+t^2)}{R_L^2 + (X_L + Z_0 t)^2}$$
 (5.8*a*) $= \frac{1}{Z_0}$
$$B = \frac{R_L^2 t - (Z_0 - X_L t)(X_L + Z_0 t)}{Z_0 [R_L^2 + (X_L + Z_0 t)^2]}$$
 (5.8*b*)

d (which implies *t*) is chosen so that $G = Y_0 = 1 / Z_0$

$$Z_{0}(R_{L} - Z_{0})t^{2} - 2X_{L}Z_{0}t + (R_{L}Z_{0} - R_{L}^{2} - X_{L}^{2}) = 0$$

$$t = \frac{X_{L} \pm \sqrt{R_{L}[(Z_{0} - R_{L})^{2} + X_{L}^{2}]/Z_{0}}}{R_{L} - Z_{0}}, \quad \text{for } R_{L} \neq Z_{0} \qquad (5.9)$$

If
$$R_L = Z_0$$
, then $t = -X_L / 2Z_0$.
If $R_L \neq Z_0$,

$$t = \tan \beta d = \tan \frac{2\pi d}{\lambda}$$

$$\frac{d}{\lambda} = \begin{cases} \frac{1}{2\pi} \tan^{-1} t, & \text{for } t \ge 0\\ \frac{1}{2\pi} (\pi + \tan^{-1} t), & \text{for } t < 0. \end{cases}$$
(5.10)

To find the required stub lengths, $B_S = -B$.

1) For an open-circuited stub,

$$jB_{S} = jY_{0}\tan\beta l_{o} = jY_{0}\tan(2\pi l_{o}/\lambda) = -jB$$
$$\frac{l_{o}}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{B_{s}}{Y_{0}}\right) = \frac{-1}{2\pi}\tan^{-1}\left(\frac{B}{Y_{0}}\right)$$
(5.11a)

2) For a short-circuited stub,

$$jB_{s} = -jY_{0}\cot\beta l_{s} = -jY_{0}\cot(2\pi l_{s}/\lambda) = -jB$$
$$\frac{l_{s}}{\lambda} = \frac{-1}{2\pi}\tan^{-1}\left(\frac{Y_{0}}{B_{s}}\right) = \frac{1}{2\pi}\tan^{-1}\left(\frac{Y_{0}}{B}\right)$$
(5.11b)

→ If the lengths (l_o, l_s) are negative, $\lambda/2$ can be added to give a positive result.

Shunt Stubs

[Ex. 5.2] Single-Stub Series Tuning

- $Z_L = 100 + j80 \Omega, f = 2 \text{ GHz}$
- Match a load to a 50 Ω using a single series open-circuited stub.
- Plot the reflection coefficient magnitude from 1 GHz to 3 GHz for each solution.

Solution

- $z_L = 2 + j1.6$
- SWR circle
- The SWR circle intersects the 1 + jx circle at two points, denoted as z_1 and z_2 .
- The shortest distance, d_1 , from the load to the stub:

 $d_1 = (0.328 - 0.208) \lambda = 0.120 \lambda$

The second distance:

 $d_2 = \{(0.5 - 0.208) + 0.172\} \ \lambda = 0.463 \ \lambda$







(c)

FIGURE 5.6 Solution to Example 5.3. (a) Smith chart for the series-stub tuners. (b) The two series-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

- Derivation of formulas for d and l for series-stub tuner

 $Y_L = 1 / Z_L = G_L + jB_L$

Impedance *Y* down a length, *d*, of line from the load:

$$Y = Y_0 \frac{(G_L + jB_L) + jtY_0}{Y_0 + jt(G_L + jB_L)}$$
(5.12)
where $t = \tan \beta d$, $Y_0 = 1/Z_0$

$$Z = R + jX = \frac{1}{Y}$$

where $R = \frac{G_L(1+t^2)}{G_L^2 + (B_L + Y_0 t)^2}$ (5.13a) $= \frac{1}{Y_0}$

$$X = \frac{G_L^2 t - (Y_0 - tB_L)(B_L + tY_0)}{Y_0 [G_L^2 + (B_L + Y_0 t)^2]}$$
(5.13b)

Now *d* (which implies *t*) is chosen so that $R = Z_0 = 1 / Y_0$.

$$Y_{0}(G_{L} - Y_{0})t^{2} - 2B_{L}Y_{0}t + (G_{L}Y_{0} - G_{L}^{2} - B_{L}^{2}) = 0$$

$$\leftarrow \text{Eq.}(5.13a) = 1/Y_{0}$$

$$t = \frac{B_{L} \pm \sqrt{G_{L}[(Y_{0} - G_{L})^{2} + B_{L}^{2}]/Y_{0}}}{G_{L} - Y_{0}}, \quad \text{for } G_{L} \neq Y_{0}$$

If
$$G_L = Y_0$$
, then $t = -B_L / 2Y_0$.
If $G_L \neq Y_0$,

$$t = \tan \beta d = \tan \frac{2\pi d}{\lambda}$$

$$\frac{d}{\lambda} = \begin{cases} \frac{1}{2\pi} \tan^{-1} t, & \text{for } t \ge 0\\ \frac{1}{2\pi} (\pi + \tan^{-1} t), & \text{for } t < 0. \end{cases}$$
(5.15)

To find the required stub lengths, $X_S = -X$.

1) For a short-circuited stub,

$$jX_{s} = jZ_{0}\tan\beta l_{s} = jZ_{0}\tan(2\pi l_{s}/\lambda) = -jX$$

$$\frac{l_{s}}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{X_{s}}{Z_{0}}\right) = \frac{-1}{2\pi}\tan^{-1}\left(\frac{X}{Z_{0}}\right)$$
(5.16a)

2) For a open-circuited stub,

$$jX_s = -jZ_0 \cot\beta l_o = -jZ_0 \cot(2\pi l_o/\lambda) = -jX$$
$$\frac{l_o}{\lambda} = \frac{-1}{2\pi} \tan^{-1}\left(\frac{Z_0}{X_s}\right) = \frac{1}{2\pi} \tan^{-1}\left(\frac{Z_0}{X}\right)$$
(5.16b)

→ If the lengths (l_o , l_s) are negative, $\lambda/2$ can be added to give a positive result.

5.3 DOUBLE-STUB TUNING

- Disadvantage of single-stub tuning: requiring a variable length of line between the load and the stub.

→ Difficult if an adjustable tuner was desired



FIGURE 5.7 Double-stub tuning. (a) Original circuit with the load an arbitrary distance from the first stub. (b) Equivalent circuit with load at the first stub.

Smith Chart Solution

- The susceptance of the first stub, b_1 (or b_1 ', for the second solution), moves the load admittance ($y_L = g_0 + jb$) to y_1 (or y_1 ').
- The amount of rotation is d wavelengths toward the load

 \rightarrow Transforming y_1 (or y_1 ') to y_2 (or y_2 '): 1 + jb circle

- The second stub then adds a susceptance b_2 (or b_2 '), which brings us to the center of the chart, and completes the match.



FIGURE 5.8 Smith chart diagram for the operation of a double-stub tuner.

[Ex. 5.4] Double-Stub Tuning

- Design a double-stub shunt tuner to match a load impedance $Z_L = 60 j80 \Omega$ to a 50 Ω line at 2 GHz.
- The stubs are to be open-circuited stubs, and are spaced $\lambda/8$ apart.
- Plot the reflection coefficient magnitude versus frequency from 1 ~
 3 GHz.

→
$$y_L = 0.3 + j0.4$$

By moving every point on the g = 1 circle $\lambda/8$ toward the load, we then find the susceptance of the first stub, which can be one of two possible values: $b_1 = 1.314$ or b_1 ' = -0.114

 $\rightarrow y_2 = 1 - j3.38$ or $y_2 = 1 + j1.38$

Susceptance of the second stub: $b_2 = 3.38$ or $b_2' = -1.38$

Lengths of the short-circuited stubs are then found as,

 $l_1 = 0.146 \lambda, \quad l_2 = 0.482 \lambda,$ or $l_1' = 0.204 \lambda, \quad l_2' = 0.350 \lambda$



(a)



(b)



(c)

FIGURE 5.9 Solution to Example 5.4. (a) Smith chart for the double-stub tuners. (b) The two double-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

Analytic Solution

- Load admittance with the first stub:

$$Y_1 = Y_L + jB_1 = G_L + j(B_L + B_1)$$
(5.17)

- Admittance just to the right of the second stub:

$$Y_{2} = Y_{0} \frac{G_{L} + j(B_{L} + B_{1} + Y_{0}t)}{Y_{0} + jt(G_{L} + jB_{L} + jB_{1})}$$
(5.18)
where $t = \tan \beta d$, $Y_{0} = 1/Z_{0}$

- $\operatorname{Re}(Y_2) = Y_0$:

$$G_L^2 - G_L Y_0 \frac{1 + t^2}{t^2} + \frac{(Y_0 - B_L t - B_1 t)^2}{t^2} = 0$$
 (5.19)

$$G_{L} = Y_{0} \frac{1+t^{2}}{2t^{2}} \left[1 \pm \sqrt{1 - \frac{4t^{2} (Y_{0} - B_{L}t - B_{1}t)^{2}}{Y_{0}^{2} (1+t^{2})^{2}}} \right]$$
(5.20)

- Since G_L is real,

$$0 \leq \frac{4t^{2}(Y_{0} - B_{L}t - B_{1}t)^{2}}{Y_{0}^{2}(1 + t^{2})^{2}} \leq 1$$

$$\Rightarrow 0 \leq G_{L} \leq Y_{0} \frac{1 + t^{2}}{t^{2}} = \frac{Y_{0}}{\sin^{2}\beta d}$$
(5.21)

- After *d* has been fixed, the first stub susceptance can be determined from (5.19) as:

$$B_1 = -B_L + \frac{Y_0 \pm \sqrt{(1+t^2)G_L Y_0 - G_L^2 t^2}}{t}$$
(5.22)

$$-jB_{2} = -j\text{Im}(Y_{2}):$$

$$B_{2} = \frac{\pm Y_{0}\sqrt{Y_{0}G_{L}(1+t^{2}) - G_{L}^{2}t^{2}} + G_{L}Y_{0}}{G_{L}t} \qquad (5.23)$$

- Open-circuited stub length:
$$\frac{l_o}{\lambda} = \frac{1}{2\pi} \tan^{-1} \left(\frac{B}{Y_0}\right)$$
 (5.24*a*)

- Short-circuited stub length:
$$\frac{l_s}{\lambda} = \frac{-1}{2\pi} \tan^{-1} \left(\frac{Y_0}{B}\right)$$
 (5.24*b*)

Where $B = B_1, B_2$

5.4 THE QUARTER-WAVE TRANSFROMER

- A single-section $\lambda/4$ transformer: narrow band impedance match A multi-section $\lambda/4$ transformer: broad band impedance match
- One drawback of the quarter-wave transformer: match a real load impedance.



FIGURE 5.10 A single-section quarter-wave matching transformer. $l=\lambda_0/4$ at the design frequency f_0 .

$$Z_1 = \sqrt{Z_0 Z_L} \tag{5.25}$$

where $l = \lambda_0/4$: electrical length at operating frequency, f_0 .

- Input impedance seen looking into the matching section:

$$Z_{\rm in} = Z_1 \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t}$$
(5.26)

where $t = \tan \beta l$ and $\beta l = \theta = \pi / 2$

- Reflection coefficient:

$$\Gamma = \frac{Z_{\rm in} - Z_0}{Z_{\rm in} + Z_0} = \frac{Z_1(Z_L - Z_0) + jt(Z_1^2 - Z_0 Z_L)}{Z_1(Z_L + Z_0) + jt(Z_1^2 + Z_0 Z_L)}$$
(5.27)

$$\leftarrow Z_1^2 = Z_0 Z_L$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0 + j2t\sqrt{Z_0 Z_L}}$$
(5.28)

- Reflection coefficient magnitude:

$$\Gamma = \frac{|Z_{L} - Z_{0}|}{[(Z_{L} + Z_{0})^{2} + 4t^{2}Z_{0}Z_{L}]^{1/2}}$$

$$= \frac{1}{\{[(Z_{L} + Z_{0})/(Z_{L} - Z_{0})]^{2} + [4t^{2}Z_{0}Z_{L}/(Z_{L} - Z_{0})^{2}]\}^{1/2}}$$

$$= \frac{1}{\{1 + [4Z_{0}Z_{L}/(Z_{L} - Z_{0})^{2}] + [4Z_{0}Z_{L}t^{2}/(Z_{L} - Z_{0})^{2}]\}^{1/2}}$$

$$= \frac{1}{\{1 + [4Z_{0}Z_{L}/(Z_{L} - Z_{0})^{2}] \sec^{2}\theta\}^{1/2}}$$
(5.29)
$$(\because 1 + t^{2} = 1 + \tan^{2}\theta = \sec^{2}\theta)$$

- If the frequency is near the design frequency, f_0 , then $l \approx \lambda_0/4$ and $\theta \approx \pi/2$. Then $\sec^2\theta >> 1$.

$$|\Gamma| \simeq \frac{|Z_L - Z_0|}{2\sqrt{Z_0 Z_L}} |\cos\theta|, \quad \text{for } \theta \text{ near } \pi/2$$
 (5.30)



FIGURE 5.11 Approximate behavior of the reflection coefficient magnitude for a singlesection quarter-wave transformer operation near its design frequency.

- If we set a maximum value, Γ_m , of the reflection coefficient magnitude that can be tolerated, then we can define the bandwidth of the matching transformer as

$$\Delta \theta = 2 \left(\frac{\pi}{2} - \theta_m \right) \tag{5.31}$$

$$\frac{1}{\Gamma_m^2} = 1 + \left(\frac{2\sqrt{Z_0 Z_L}}{Z_L - Z_0} \sec \theta_m\right)^2$$

$$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}$$
(5.32)

- If we assume TEM line,

$$\theta = \beta l = \frac{2\pi f}{v_p} \frac{v_p}{4f_0} = \frac{\pi f}{2f_0}$$
$$f_m = \frac{2\theta_m f_0}{\pi}$$

- Fractional bandwidth:

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{2f_m}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$= 2 - \frac{4}{\pi} \cos^{-1} \left[\frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|} \right] \quad (5.33)$$

0



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- When non-TEM lines (such as waveguides) are used, the propagation constant is no longer a linear function of frequency, and the wave impedance will be frequency dependent.
- The effect of reactances associated with discontinuities of transmission line must be considered.

[Ex. 5.5] Quarter-Wave Transformer Bandwidth

Design a single-section quarter-wave matching transformer to match a 10 Ω load to a 50 Ω line, at $f_0 = 3$ GHz. Determine the percent bandwidth for which the SWR ≤ 1.5 .

→
$$Z_1 = \sqrt{Z_0 Z_L} = \sqrt{(50)(10)} = 22.36 \,\Omega$$

The length of the matching section is $\lambda/4$ at 3 GHz.

$$\Gamma_m = \frac{\text{SWR} - 1}{\text{SWR} + 1} = \frac{1.5 - 1}{1.5 + 1} = 0.2$$

Fractional bandwidth:

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi} \cos^{-1} \left[\frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|} \right]$$
$$= 2 - \frac{4}{\pi} \cos^{-1} \left[\frac{0.2}{\sqrt{1 - (0.2)^2}} \frac{2\sqrt{(50)(10)}}{|10 - 50|} \right]$$
$$= 0.29 \text{ or } 29\%.$$

5.5 THE THEORY OF SMALL REFLECTIONS

- Theory of small reflections: total reflection coefficient caused by the partial reflections from several small discontinuities

Single-Section Transformer

- Derivation an approximate expression for the overall reflection coefficient $\boldsymbol{\Gamma}$



FIGURE 5.13 Partial reflections and transmissions on a single-section matching transformer.

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} \tag{5.34}$$

$$\Gamma_{2} = -\Gamma_{1} \qquad (5.35)$$

$$\Gamma_{3} = \frac{Z_{L} - Z_{2}}{Z_{L} + Z_{2}} \qquad (5.36)$$

$$T_{21} = 1 + \Gamma_{1} = \frac{2Z_{2}}{Z_{1} + Z_{2}} \qquad (5.37)$$

$$T_{12} = 1 + \Gamma_{2} = \frac{2Z_{1}}{Z_{1} + Z_{2}} \qquad (5.38)$$

- Expression of the total reflection as an infinite sum of partial reflections and transmissions:

$$\Gamma = \Gamma_{1} + T_{12}T_{21}\Gamma_{3}e^{-2j\theta} + T_{12}T_{21}\Gamma_{3}^{2}\Gamma_{2}e^{-4j\theta} + \dots$$
$$= \Gamma_{1} + T_{12}T_{21}\Gamma_{3}e^{-2j\theta}\sum_{n=0}^{\infty}\Gamma_{2}^{n}\Gamma_{3}^{n}e^{-2jn\theta}$$
(5.39)

- Using the geometric series and small reflection condition,

$$\sum_{n=0}^{\infty} x^{n} = \frac{1}{1-x}, \quad \text{for } |x| < 1$$

$$\Gamma = \Gamma_{1} + \frac{T_{12}T_{21}\Gamma_{3}e^{-2j\theta}}{1-\Gamma_{2}\Gamma_{3}e^{-2j\theta}} \qquad (5.40)$$

$$\leftarrow \Gamma_{2} = -\Gamma_{1}, T_{21} = 1 + \Gamma_{1}, T_{12} = 1 + \Gamma_{2} = 1 - \Gamma_{1}$$

$$= \frac{\Gamma_{1} - \Gamma_{1}\Gamma_{2}\Gamma_{3}e^{-2j\theta} + (1 - \Gamma_{1})(1 + \Gamma_{1})\Gamma_{3}e^{-2j\theta}}{1+\Gamma_{1}\Gamma_{3}e^{-2j\theta}}$$

$$\cong \frac{\Gamma_{1} + \Gamma_{3}e^{-2j\theta}}{1+\Gamma_{1}\Gamma_{3}e^{-2j\theta}} \qquad (5.41)$$

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-2j\theta} \tag{5.42}$$

- Intuitive ideas:

- 1) The total reflection is dominated by the reflection from the initial discontinuity between Z_1 and Z_2 (Γ_1), and the first reflection from the discontinuity between Z_2 and Z_L ($\Gamma_3 e^{-2j\theta}$).
- 2) The $e^{-2j\theta}$ term accounts for the phase delay when the incident wave travels up and down the line.

Multi-Section Transformer

- N equal-length (commensurate) sections of transmission lines.



FIGURE 5.14 Partial reflection coefficients for a multisection matching transformer.

- Derivation an approximate expression for the total reflection coefficient Γ .

$$\Gamma_{0} = \frac{Z_{1} - Z_{0}}{Z_{1} + Z_{0}}, \qquad (5.43a)$$

$$\Gamma_{n} = \frac{Z_{n+1} - Z_{n}}{Z_{n+1} + Z_{n}}, \qquad (5.43b)$$

$$\Gamma_{N} = \frac{Z_{L} - Z_{N}}{Z_{L} + Z_{N}}. \qquad (5.43c)$$

- Assumptions:

All Z_n increase or decrease monotonically across the transformer.
 Z_L is real.

$$\Rightarrow \Gamma_n > 0 \text{ if } Z_L > Z_0$$

$$\Gamma_n < 0 \text{ if } Z_L < Z_0$$

- Overall reflection coefficient:

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \dots + \Gamma_N e^{-2jN\theta} \qquad (5.44) \leftarrow (5.42)$$

- If the transformer be made symmetrical, so that $\Gamma_0 = \Gamma_N$, $\Gamma_1 = \Gamma_{N-1}$, $\Gamma_2 = \Gamma_{N-2}$, etc. Then (5.44) can be written as;

$$\Gamma(\theta) = e^{-jN\theta} \{ \Gamma_0[e^{jN\theta} + e^{-jN\theta}] + \Gamma_1[e^{j(N-2)\theta} + e^{-j(N-2)\theta}] + \cdots \}$$
(5.45)

Where if *N* is odd, the last term is $\Gamma_{(N-1)/2}(e^{j\theta}+e^{-j\theta})$, while if *N* is even the last term is $\Gamma_{N/2}$.

$$\Gamma(\theta) = 2e^{-jN\theta} [\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \dots + \Gamma_n \cos(N-2n)\theta + \dots + \frac{1}{2}\Gamma_{N/2}] \qquad \text{for } N \text{ even} \qquad (5.46a)$$

$$\Gamma(\theta) = 2e^{-jN\theta} [\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + \dots + \Gamma_n \cos(N-2n)\theta + \dots + \Gamma_{(N-1)/2} \cos \theta] \quad \text{for } N \text{ odd} \quad (5.46b)$$