Chapter. 5 Impedance Matching and Tuning

- Maximum power is delivered when the load is matched to the line (assuming the generator is matched), and power loss in the feed line is minimized.
- Impedance matching sensitive receiver components (antenna, low-noise amplifier, etc.) improves the signal-to-noise ratio of the system.
- Impedance matching in a power distribution network (such as an antenna array feed network) will reduce amplitude and phase errors.

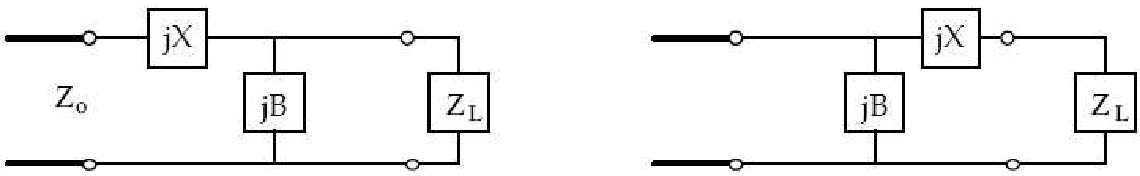
FIGURE 5.1 A lossless network matching an arbitrary load impedance to a transmission line.

- Factors that may be important in the selection of a particular matching network:

1) Complexity: simple
2) Bandwidth: restricted frequency band
3) Implementation: easy
4) Adjustability: tunable

5.1 MATCHING WITH LUMPED ELEMENTS (*L* NETWORKS)

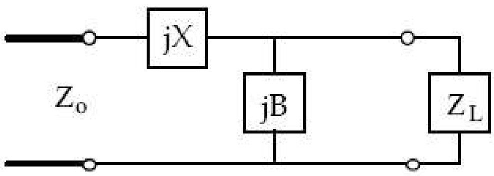
- L-section
- Simplest type including two reactive elements
- $z_L = Z_L / Z_0$
- X, B: lumped or distributed element depending on the operating frequency.



(a) **(b)**

FIGURE 5.2 *L* section matching networks. (a) Network for z_L inside the $1 + jx$ circle. (b) Network for z_L outside the $1 + jx$ circle

Analytic Solutions



- $Z_L = R_L + jX_L$ (assumption: $Z_0 < R_L$, inside of $1 + jx$ circle)
- Impedance seen looking into the matching network:

$$Z_{\text{in}} = jX + \frac{1}{jB + 1/(R_L + jX_L)} = Z_0 \qquad (5.1)$$

- Rearranging and separating into real and imaginary parts:

$$B(XR_L - X_L Z_0) = R_L - Z_0 \qquad (5.2a)$$

$$X(1 - BX_L) = BZ_0 R_L - X_L \qquad (5.2b)$$

- Solving Eq. (5.2a) for X and substituting into Eq. (5.2b)

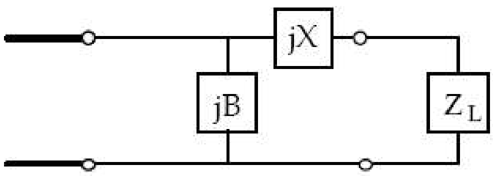
$$B = \frac{X_L \pm \sqrt{R_L / Z_0}\sqrt{R_L^2 + X_L^2 - Z_0 R_L}}{R_L^2 + X_L^2} \qquad (5.3a)$$

where $(R_L^2 + X_L^2 - Z_0 R_L) > 0 \ (\because R_L > Z_0)$

$$X = \frac{BX_L Z_0 + R_L - Z_0}{BR_L} \qquad \leftarrow (5.2a)$$

$$= \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{BR_L} \qquad (5.3b)$$

➔ Two solutions: dual valued components (B, X)



- $Z_L = R_L + jX_L$ (assumption: $Z_0 > R_L$, outside of $1 + jx$ circle)
- Admittance seen looking into the matching network:

$$Y_{\text{in}} = jB + \frac{1}{R_L + j(X + X_L)}$$

$$= \frac{1}{Z_0} \qquad (5.4)$$

- Rearranging and separating into real and imaginary parts:

$$BZ_0(X + X_L) = Z_0 - R_L \qquad (5.5a)$$

$$(X + X_L) = BZ_0R_L \qquad (5.5b)$$

- Solving for X and B:

$$X = \pm\sqrt{R_L(Z_0 - R_L)} - X_L \qquad (5.6a)$$

$$B = \pm\frac{\sqrt{(Z_0 - R_L)/R_L}}{Z_0} \qquad (5.6b)$$

➔ Two solutions: dual valued components (B, X)

Smith Chart Solutions

[Ex. 5.1] Design an L-section matching network to match a series RC load with an impedance Z_L = 200 - j100 Ω to a 100 Ω line, at a frequency of 500 MHz.

➔ Sol. 1) $z_L = 2 - j1$

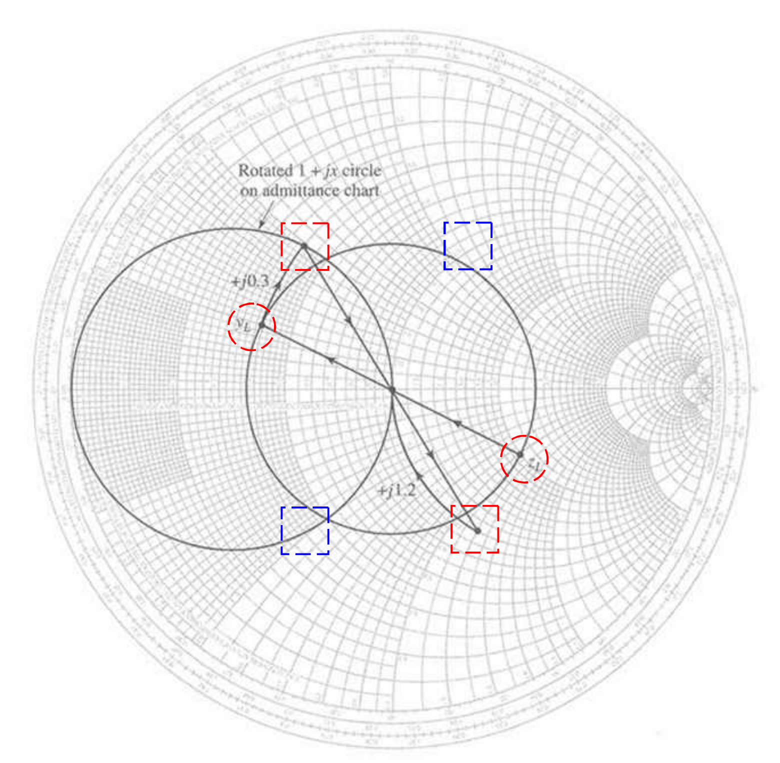
$y_L = 0.4 + j0.2$

$y = y_L + jb = 0.4 + j0.5 \leftarrow jb = j0.3$

Interconnection with admittance circle of $1 - jx$

$z = 1 - j1.2$

$x = 1.2$

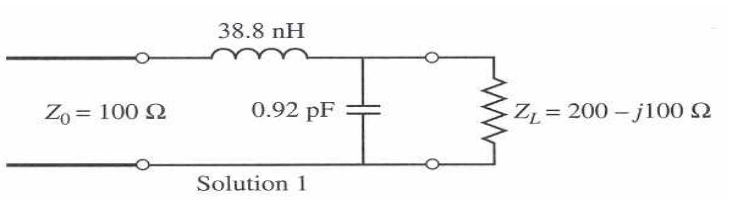


(a)

Using Eq.(5.3*a*, *b*), $b = 0.29$ and $x = 1.22$

$$C = \frac{b}{2\pi f Z_0} = 0.92\ \text{pF} \quad \leftarrow B = bY_0 = \frac{b}{Z_0} = \omega C, \quad C = \frac{b}{\omega Z_0}$$

$$L = \frac{xZ_0}{2\pi f} = 38.8\ \text{nH} \quad \leftarrow X = xZ_0 = \omega L, \ L = \frac{xZ_0}{\omega}$$



Sol. 2) $y_L = 0.4 + j0.2$

$y = y_L + jb = 0.4 - j0.5 \leftarrow jb = -j0.7$

Interconnection with admittance circle of $1 + jx$

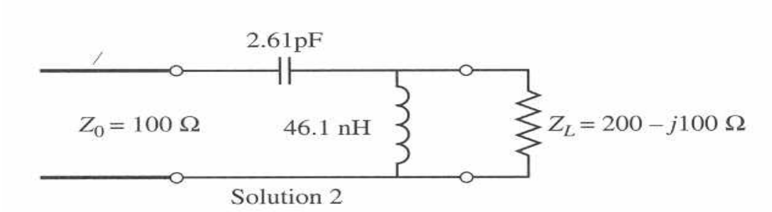
$z = 1 + j1.2$

$x = -1.2$

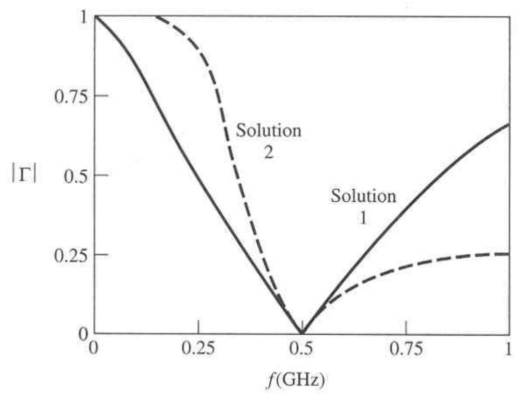
Using Eq.(5.3*a*, *b*), $b = -0.69$ and $x = -1.22$

$$C = \frac{1}{2\pi f x Z_0} = 2.61 \text{ pF} \quad \leftarrow jX = jxZ_0 = -j\frac{1}{\omega C}, \quad C = -\frac{1}{\omega x Z_0}$$

$$L = \frac{Z_0}{2\pi f b} = 46.1 \text{ nH} \quad \leftarrow jB = jbY_0 = \frac{jb}{Z_0} = -j\frac{1}{\omega L}, \ L = -\frac{Z_0}{\omega b}$$



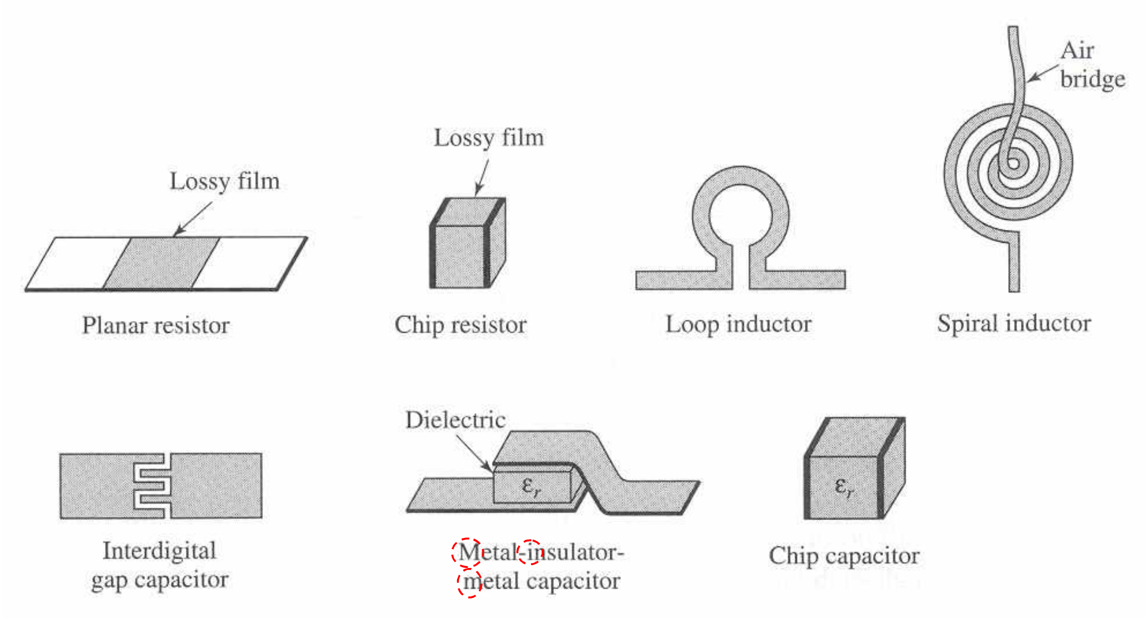
(b)



(c)

FIGURE5.3 Continued. (b) The two possible *L* section matching circuits. (c) Reflection.

coefficient magnitudes versus frequency for the matching circuits of (b).



5.2 SINGLE-STUB TUNING

- Single open-circuited or short circuited length of transmission line (a "stub"), connected either in parallel or in series with the transmission feed line at a certain distance from the load

  ➔The shunt tuning stub is especially easy to fabricate in microstrip or stripline form.

- Shunt-stub case

  $Y_L \rightarrow Y_0 + jB$ using trasmission line having length d

  $\rightarrow Y_0$ using trasmission line having length l

- Series-stub case

$Z_L \rightarrow Z_0 + jX$ using trasmission line having length d

$\rightarrow Z_0$ using trasmission line having length l

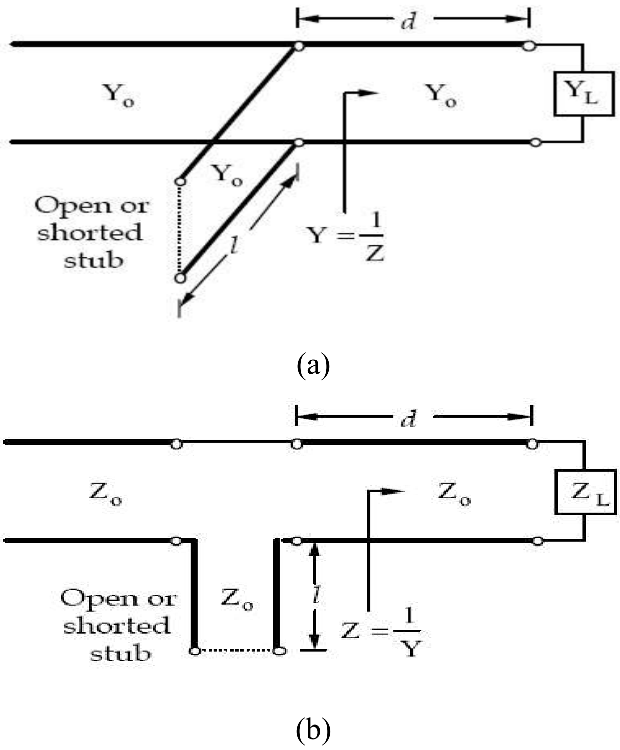


FIGURE 5.4 Single-stub tuning circuits. (a) Shunt stub. (b) Series stub.

- $\lambda/4$ transmission line with a shunt stub
- Open-circuited stubs are easier to fabricate since a via hole through the substrate to the ground plane is not needed.

Shunt Stubs

[Ex. 5.2] Single-Stub Shunt Tuning

- $Z_L = 15 + j10\ \Omega$, $f = 2$ GHz
- Design two single-stub shunt tuning networks to match this load to a 50 Ω line.
- Plot the reflection coefficient magnitude from 1 GHz to 3 GHz for each solution.

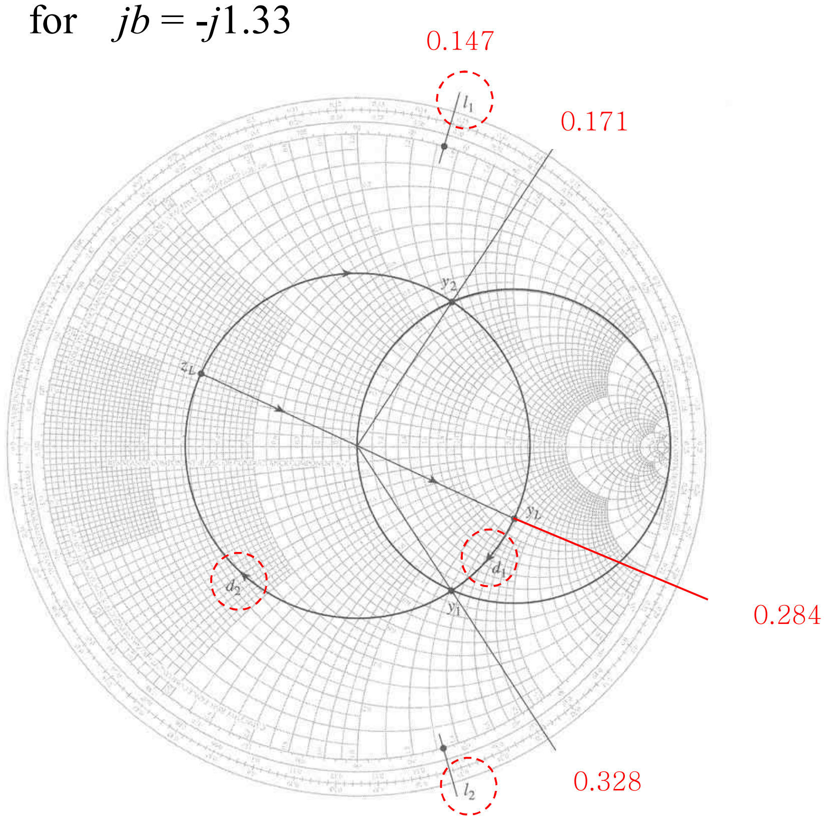
Solution

- $z_L = 0.3 + j0.2$
- SWR circle
- Convert to the load admittance, y_L.
- SWR circle intersects the $1 + jb$ circle at two points, denoted as y_1 and y_2 in Figure 5.5a. → d_1 or d_2
- $d_1 = (0.328 - 0.284)\lambda = 0.044\ \lambda$

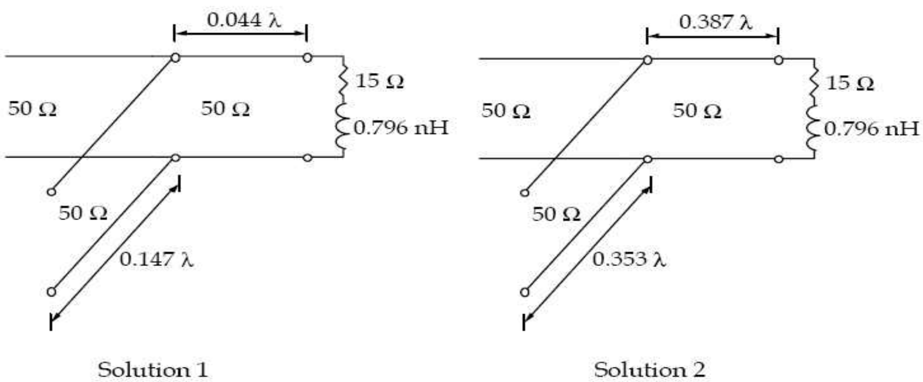
  $d_2 = \{(0.5 - 0.284) + 0.171\}\lambda = 0.387\ \lambda$
- The matching stub is kept as close as possible to the load, to improve the bandwidth of the match and to reduce losses caused by a possibly large standing wave ratio on the line between the stub and the load.
- $y_1 = 1 - j1.33$, $y_2 = 1 + j1.33$

- $l_1 = 0.147\,\lambda$ for $jb = j1.33$

$l_2 = 0.353\,\lambda$ for $jb = -j1.33$



(a)



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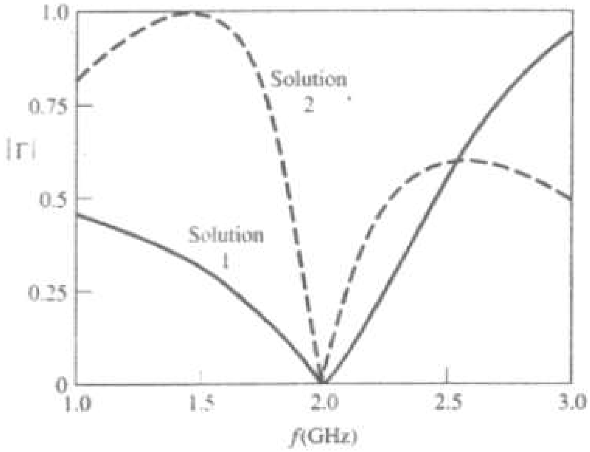


(c)

FIGURE 5.5 Solution to Example 5.2. (a) Smith chart for the shunt-stub tuners. (b) The two shunt-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

- Derivation of formulas for d and l

$Z_L = 1/Y_L = R_L + jX_L$

Impedance Z down a length, d, of line from the load:

$$Z = Z_0 \frac{(R_L + jX_L) + jZ_0 t}{Z_0 + j(R_L + jX_L)t} \tag{5.7}$$

where $t = \tan \beta d$

$$Y = G + jB = \frac{1}{Z}$$

where $G = \dfrac{R_L(1+t^2)}{R_L^2 + (X_L + Z_0 t)^2}$ (5.8a) $= \dfrac{1}{Z_0}$

$$B = \frac{R_L^2 t - (Z_0 - X_L t)(X_L + Z_0 t)}{Z_0[R_L^2 + (X_L + Z_0 t)^2]} \quad (5.8b)$$

d (which implies t) is chosen so that $G = Y_0 = 1 / Z_0$

$$Z_0(R_L - Z_0)t^2 - 2X_L Z_0 t + (R_L Z_0 - R_L^2 - X_L^2) = 0$$

$$t = \frac{X_L \pm \sqrt{R_L[(Z_0 - R_L)^2 + X_L^2]/Z_0}}{R_L - Z_0}, \quad \text{for } R_L \neq Z_0 \qquad (5.9)$$

If $R_L = Z_0$, then $t = -X_L / 2Z_0$.

If $R_L \neq Z_0$,

$$t = \tan\beta d = \tan\frac{2\pi d}{\lambda}$$

$$\frac{d}{\lambda} = \begin{cases} \dfrac{1}{2\pi}\tan^{-1} t, & \text{for } t \geq 0 \\ \dfrac{1}{2\pi}(\pi + \tan^{-1} t), & \text{for } t < 0. \end{cases} \qquad (5.10)$$

To find the required stub lengths, $B_S = -B$.

1) For an open-circuited stub,

$jB_S = jY_0 \tan\beta l_o = jY_0 \tan(2\pi l_o/\lambda) =- jB$

$$\frac{l_o}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{B_s}{Y_0}\right) = \frac{-1}{2\pi}\tan^{-1}\left(\frac{B}{Y_0}\right) \qquad (5.11a)$$

2) For a short-circuited stub,

$jB_S = -jY_0\cot\beta l_s = -jY_0\cot(2\pi l_s/\lambda) = -jB$

$$\frac{l_s}{\lambda} = \frac{-1}{2\pi}\tan^{-1}\left(\frac{Y_0}{B_s}\right) = \frac{1}{2\pi}\tan^{-1}\left(\frac{Y_0}{B}\right) \qquad (5.11b)$$

➔If the lengths (l_o, l_s) are negative, $\lambda/2$ can be added to give a positive result.

Shunt Stubs

[Ex. 5.2] Single-Stub Series Tuning

- $Z_L = 100 + j80\ \Omega$, $f = 2$ GHz
- Match a load to a 50 Ω using a single series open-circuited stub.
- Plot the reflection coefficient magnitude from 1 GHz to 3 GHz for each solution.

Solution

- $z_L = 2 + j1.6$
- SWR circle
- The SWR circle intersects the $1 + jx$ circle at two points, denoted as z_1 and z_2.
- The shortest distance, d_1, from the load to the stub:

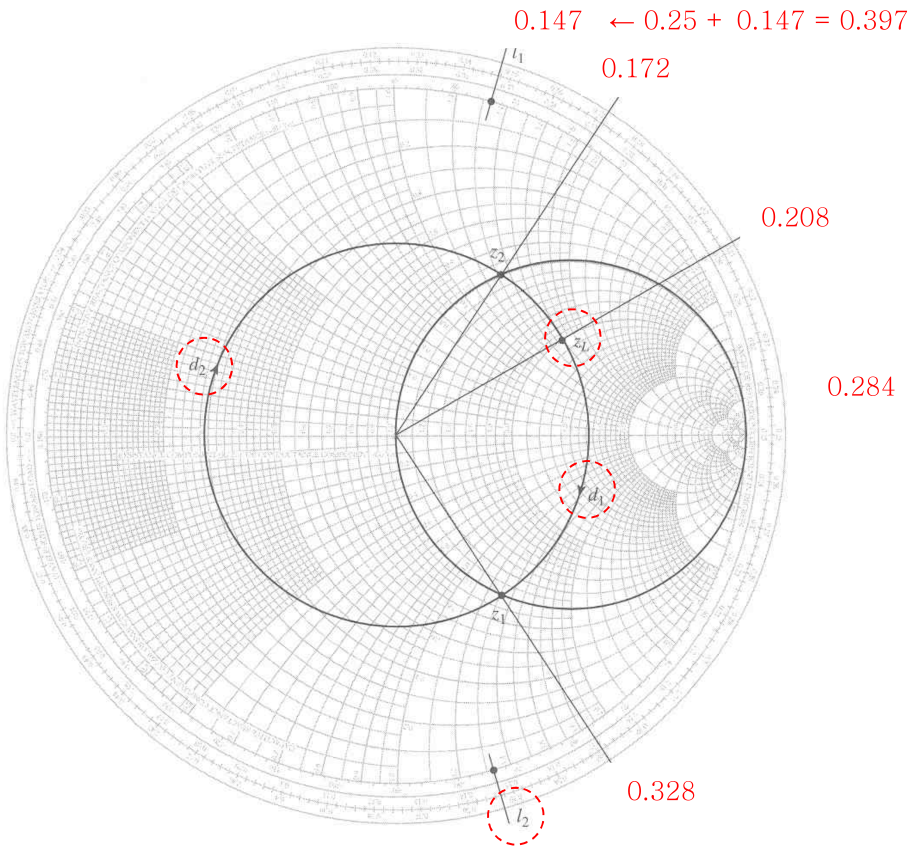
$d_1 = (0.328 - 0.208)\ \lambda = 0.120\ \lambda$

The second distance:

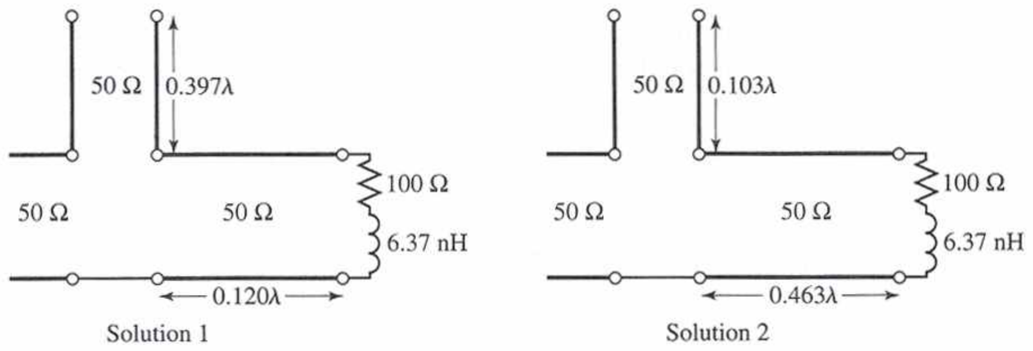
$d_2 = \{(0.5 - 0.208) + 0.172\}\ \lambda = 0.463\ \lambda$

- $z_1 = 1 - j1.33$, $z_2 = 1 + j1.33$
- $l_1 = 0.397\,\lambda$ for $jx = j1.33$ (open stub)

 $l_2 = 0.103\,\lambda$ for $jx = -j1.33$ (open stub)



(a)



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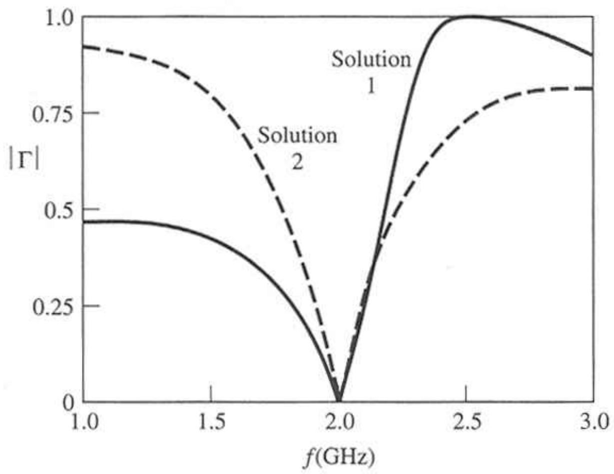


(c)

FIGURE 5.6 Solution to Example 5.3. (a) Smith chart for the series-stub tuners. (b) The two series-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

- Derivation of formulas for *d* and *l* for series-stub tuner

$Y_L = 1 / Z_L = G_L + jB_L$

Impedance *Y* down a length, *d*, of line from the load:

$$Y = Y_0 \frac{(G_L + jB_L) + jtY_0}{Y_0 + jt(G_L + jB_L)} \qquad (5.12)$$

where $t = \tan \beta d$, $Y_0 = 1/Z_0$

$$Z = R + jX = \frac{1}{Y}$$

where $R = \dfrac{G_L(1+t^2)}{G_L^2 + (B_L + Y_0 t)^2} \qquad (5.13a) = \dfrac{1}{Y_0}$

$$X = \frac{G_L^2 t - (Y_0 - tB_L)(B_L + tY_0)}{Y_0[G_L^2 + (B_L + Y_0 t)^2]} \qquad (5.13b)$$

Now d (which implies t) is chosen so that $R = Z_0 = 1 / Y_0$.

$$Y_0(G_L - Y_0)t^2 - 2B_L Y_0 t + (G_L Y_0 - G_L^2 - B_L^2) = 0$$
$$\leftarrow \text{Eq.}(5.13a) = 1/Y_0$$

$$t = \frac{B_L \pm \sqrt{G_L[(Y_0 - G_L)^2 + B_L^2]/Y_0}}{G_L - Y_0}, \qquad \text{for } G_L \neq Y_0$$

If $G_L = Y_0$, then $t = -B_L / 2Y_0$.

If $G_L \neq Y_0$,

$$t = \tan\beta d = \tan\frac{2\pi d}{\lambda}$$

$$\frac{d}{\lambda} = \begin{cases} \dfrac{1}{2\pi}\tan^{-1} t, & \text{for } t \geq 0 \\ \dfrac{1}{2\pi}(\pi + \tan^{-1} t), & \text{for } t < 0. \end{cases} \qquad (5.15)$$

To find the required stub lengths, $X_S = -X$.

1) For a short-circuited stub,

$jX_S = jZ_0\tan\beta l_s = jZ_0\tan(2\pi l_s/\lambda) = -jX$

$$\frac{l_s}{\lambda} = \frac{1}{2\pi}\tan^{-1}\left(\frac{X_s}{Z_0}\right) = \frac{-1}{2\pi}\tan^{-1}\left(\frac{X}{Z_0}\right) \qquad (5.16a)$$

2) For a open-circuited stub,

$jX_s = -jZ_0\cot\beta l_o = -jZ_0\cot(2\pi l_o/\lambda) = -jX$

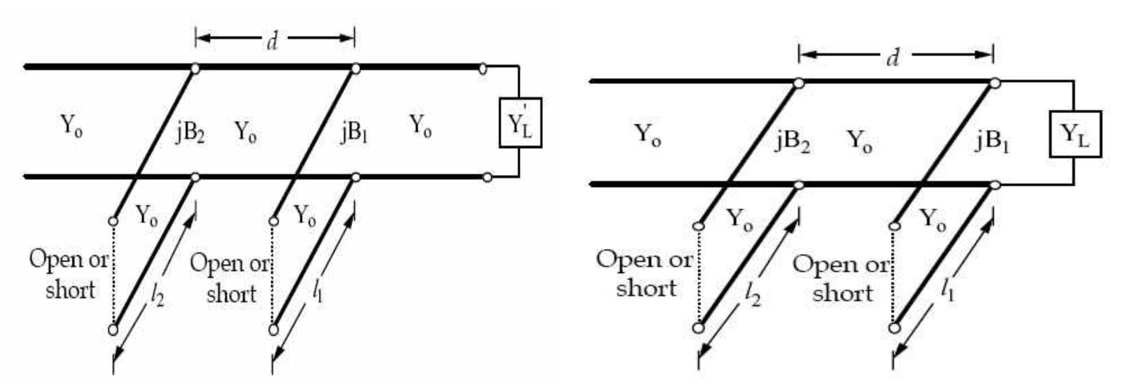
$$\frac{l_o}{\lambda} = \frac{-1}{2\pi}\tan^{1}\left(\frac{Z_0}{X_s}\right) = \frac{1}{2\pi}\tan^{-1}\left(\frac{Z_0}{X}\right) \qquad (5.16b)$$

➔If the lengths (l_o, l_s) are negative, $\lambda/2$ can be added to give a positive result.

5.3 DOUBLE-STUB TUNING

- Disadvantage of single-stub tuning: requiring a variable length of line between the load and the stub.

  ➔ Difficult if an adjustable tuner was desired



(a) (b)

FIGURE 5.7 Double-stub tuning. (a) Original circuit with the load an arbitrary distance from the first stub. (b) Equivalent circuit with load at the first stub.

Smith Chart Solution

- The susceptance of the first stub, b_1 (or b_1', for the second solution), moves the load admittance ($y_L = g_0 + jb$) to y_1 (or y_1').
- The amount of rotation is d wavelengths toward the load
  → Transforming y_1 (or y_1') to y_2 (or y_2'): $1 + jb$ circle
- The second stub then adds a susceptance b_2 (or b_2'), which brings us to the center of the chart, and completes the match.

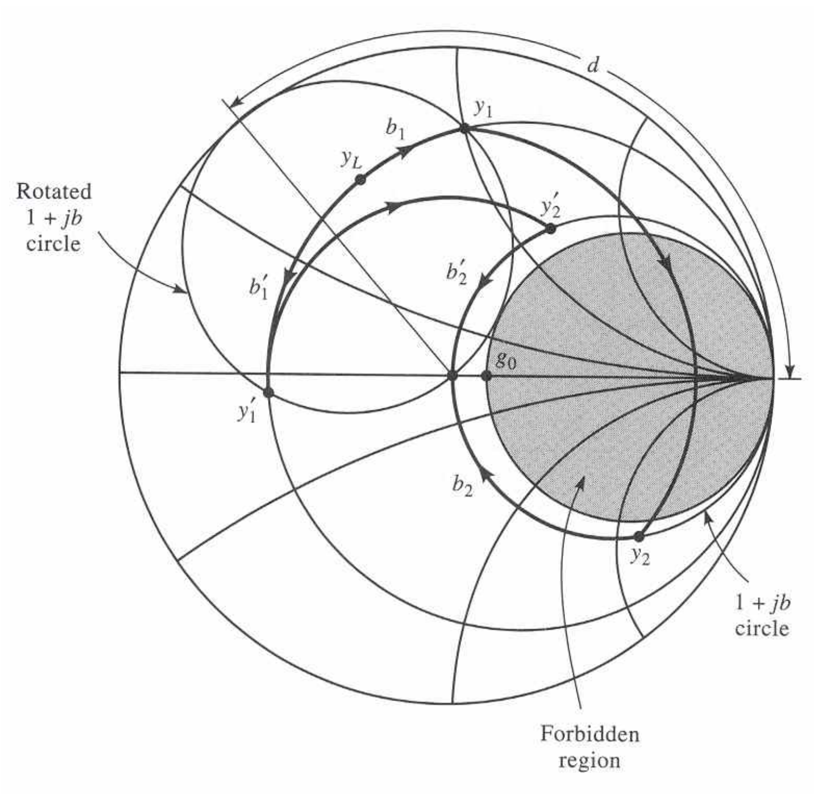


FIGURE 5.8 Smith chart diagram for the operation of a double-stub tuner.

[Ex. 5.4] Double-Stub Tuning

- Design a double-stub shunt tuner to match a load impedance Z_L = 60 - j80 Ω to a 50 Ω line at 2 GHz.
- The stubs are to be open-circuited stubs, and are spaced $\lambda/8$ apart.
- Plot the reflection coefficient magnitude versus frequency from 1 ~ 3 GHz.

➔ $y_L = 0.3 + j0.4$

By moving every point on the $g = 1$ circle $\lambda/8$ toward the load, we then find the susceptance of the first stub, which can be one of two possible values: $b_1 = 1.314$ or $b_1' = -0.114$

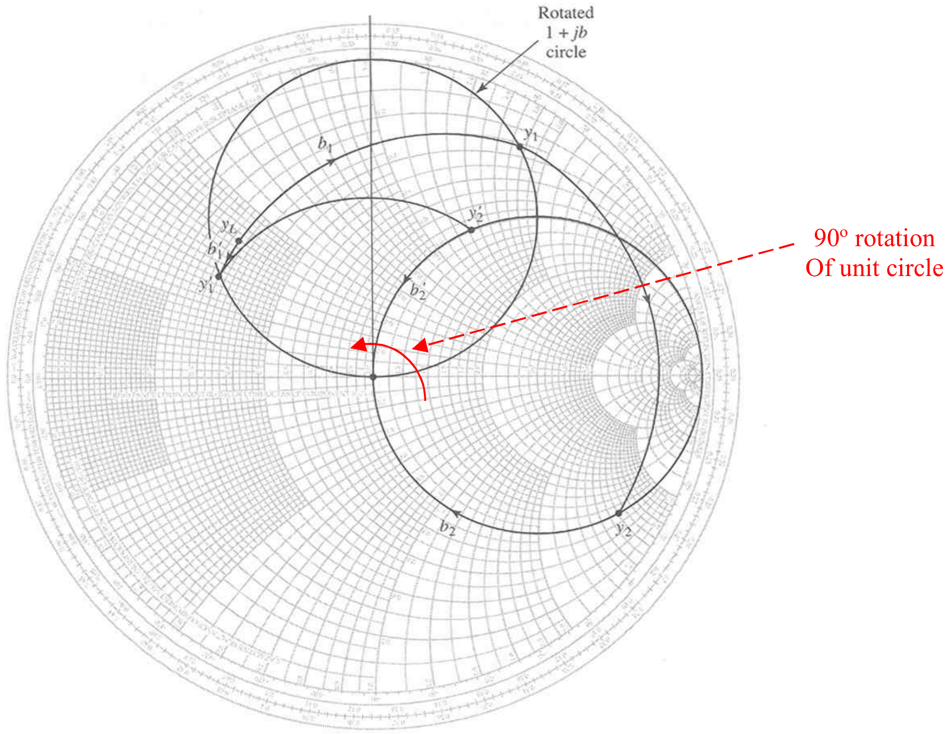
→ $y_2 = 1 - j3.38$ or $y_2' = 1 + j1.38$

Susceptance of the second stub: $b_2 = 3.38$ or $b_2' = -1.38$

Lengths of the short-circuited stubs are then found as,

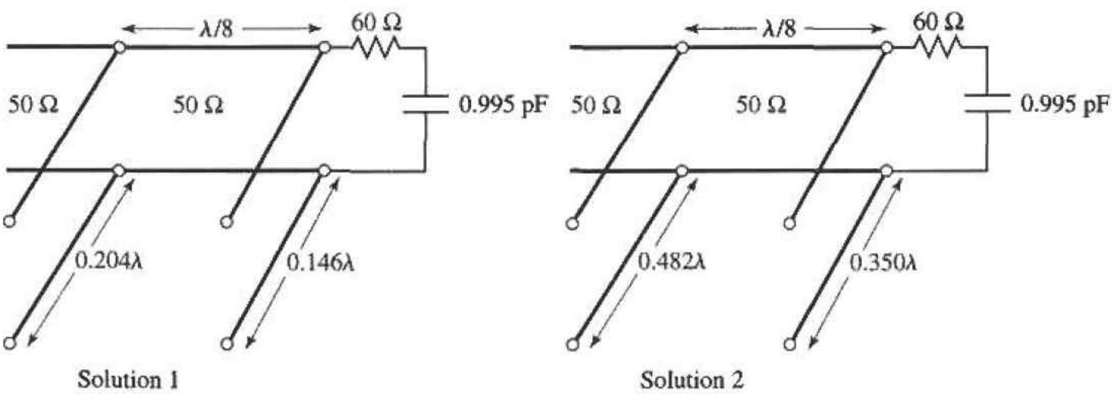
$l_1 = 0.146\lambda$, $l_2 = 0.204\lambda$,

or $l_1' = 0.482\lambda$, $l_2' = 0.350\lambda$



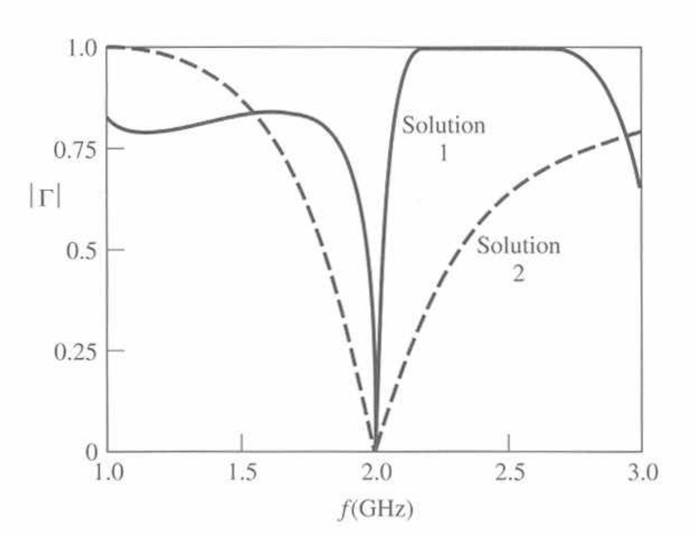
Rotated
1 + jb
circle
y1
b1
y2'
yL
b1'
y1'
b2'
90° rotation
Of unit circle
b2
y2

(a)



λ/8
60 Ω
50 Ω
50 Ω
0.995 pF
0.204λ
0.146λ
Solution 1
λ/8
60 Ω
50 Ω
50 Ω
0.995 pF
0.482λ
0.350λ
Solution 2

(b)



(c)

FIGURE 5.9 Solution to Example 5.4. (a) Smith chart for the double-stub tuners. (b) The two double-stub tuning solutions. (c) Reflection coefficient magnitudes versus frequency for the tuning circuits of (b).

Analytic Solution

- Load admittance with the first stub:

$$Y_1 = Y_L + jB_1 = G_L + j(B_L + B_1) \qquad (5.17)$$

- Admittance just to the right of the second stub:

$$Y_2 = Y_0 \frac{G_L + j(B_L + B_1 + Y_0 t)}{Y_0 + jt(G_L + jB_L + jB_1)} \qquad (5.18)$$

where $t = \tan \beta d,\ \ Y_0 = 1/Z_0$

- $\mathrm{Re}(Y_2) = Y_0$:

$$G_L^2 - G_L Y_0 \frac{1+t^2}{t^2} + \frac{(Y_0 - B_L t - B_1 t)^2}{t^2} = 0 \qquad (5.19)$$

$$G_L = Y_0 \frac{1+t^2}{2t^2}\left[1 \pm \sqrt{1 - \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y_0^2(1+t^2)^2}}\right] \qquad (5.20)$$

- Since G_L is real,

$$0 \le \frac{4t^2(Y_0 - B_L t - B_1 t)^2}{Y_0^2(1+t^2)^2} \le 1$$

$$\Rightarrow \quad 0 \le G_L \le Y_0 \frac{1+t^2}{t^2} = \frac{Y_0}{\sin^2 \beta d} \qquad (5.21)$$

- After d has been fixed, the first stub susceptance can be determined from (5.19) as:

$$B_1 = -B_L + \frac{Y_0 \pm \sqrt{(1+t^2)G_L Y_0 - G_L^2 t^2}}{t} \qquad (5.22)$$

- $jB_2 = -j\mathrm{Im}(Y_2)$:

$$B_2 = \frac{\pm Y_0 \sqrt{Y_0 G_L (1+t^2) - G_L^2 t^2} + G_L Y_0}{G_L t} \qquad (5.23)$$

- Open-circuited stub length: $\dfrac{l_o}{\lambda} = \dfrac{1}{2\pi}\tan^{-1}\left(\dfrac{B}{Y_0}\right)$ $\qquad (5.24a)$

- Short-circuited stub length: $\frac{l_s}{\lambda} = \frac{-1}{2\pi}\tan^{-1}\left(\frac{Y_0}{B}\right)$ (5.24*b*)

Where $B = B_1, B_2$

5.4 THE QUARTER-WAVE TRANSFROMER

- A single-section $\lambda/4$ transformer: narrow band impedance match
  A multi-section $\lambda/4$ transformer: broad band impedance match

- One drawback of the quarter-wave transformer: match a real load impedance.

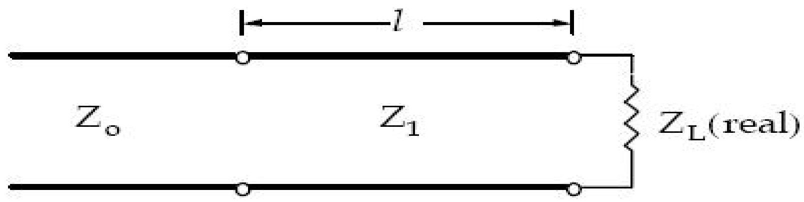


FIGURE 5.10 A single-section quarter-wave matching transformer. $l=\lambda_0/4$ at the design frequency f_0.

$$Z_1 = \sqrt{Z_0 Z_L} \qquad (5.25)$$

where $l = \lambda_0/4$: electrical length at operating frequency, f_0.

- Input impedance seen looking into the matching section:

$$Z_{\text{in}} = Z_1 \frac{Z_L + jZ_1 t}{Z_1 + jZ_L t} \qquad (5.26)$$

where $t = \tan \beta l$ and $\beta l = \theta = \pi / 2$

- Reflection coefficient:

$$\Gamma = \frac{Z_{\text{in}} - Z_0}{Z_{\text{in}} + Z_0} = \frac{Z_1(Z_L - Z_0) + jt(Z_1^2 - Z_0 Z_L)}{Z_1(Z_L + Z_0) + jt(Z_1^2 + Z_0 Z_L)} \qquad (5.27)$$

$$\leftarrow Z_1^2 = Z_0 Z_L$$

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0 + j2t\sqrt{Z_0 Z_L}} \qquad (5.28)$$

- Reflection coefficient magnitude:

$$|\Gamma| = \frac{|Z_L - Z_0|}{[(Z_L + Z_0)^2 + 4t^2 Z_0 Z_L]^{1/2}}$$

$$= \frac{1}{\{[(Z_L + Z_0)/(Z_L - Z_0)]^2 + [4t^2 Z_0 Z_L /(Z_L - Z_0)^2]\}^{1/2}}$$

$$= \frac{1}{\{1 + [4Z_0 Z_L /(Z_L - Z_0)^2] + [4Z_0 Z_L t^2 /(Z_L - Z_0)^2]\}^{1/2}}$$

$$= \frac{1}{\{1 + [4Z_0 Z_L /(Z_L - Z_0)^2] \sec^2 \theta\}^{1/2}} \qquad (5.29)$$

$(\because\ 1 + t^2 = 1 + \tan^2 \theta = \sec^2 \theta)$

- If the frequency is near the design frequency, f_0, then $l \approx \lambda_0/4$ and $\theta \approx \pi/2$. Then $\sec^2\theta >> 1$.

$$|\Gamma| \cong \frac{|Z_L - Z_0|}{2\sqrt{Z_0 Z_L}} |\cos\theta|, \qquad \text{for } \theta \text{ near } \pi/2 \qquad (5.30)$$

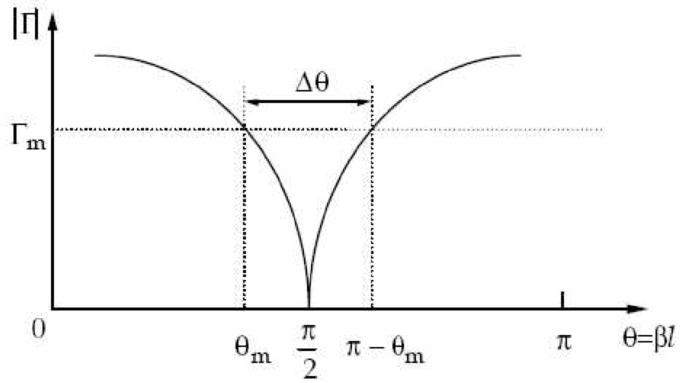


FIGURE 5.11 Approximate behavior of the reflection coefficient magnitude for a single-section quarter-wave transformer operation near its design frequency.

- If we set a maximum value, Γ_m, of the reflection coefficient magnitude that can be tolerated, then we can define the bandwidth of the matching transformer as

$$\Delta\theta = 2\left(\frac{\pi}{2} - \theta_m\right) \qquad (5.31)$$

$$\frac{1}{\Gamma_m^2} = 1 + \left(\frac{2\sqrt{Z_0 Z_L}}{Z_L - Z_0} \sec\theta_m\right)^2$$

$$\cos\theta_m = \frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|} \qquad (5.32)$$

- If we assume TEM line,

$$\theta = \beta l = \frac{2\pi f}{v_p}\frac{v_p}{4 f_0} = \frac{\pi f}{2 f_0}$$

$$f_m = \frac{2\theta_m f_0}{\pi}$$

- Fractional bandwidth:

$$\frac{\Delta f}{f_0} = \frac{2(f_0 - f_m)}{f_0} = 2 - \frac{2 f_m}{f_0} = 2 - \frac{4\theta_m}{\pi}$$

$$= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}} \frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right] \qquad (5.33)$$

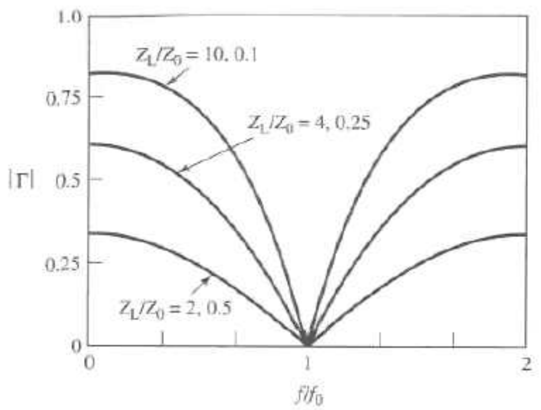


FIGURE 5.12 Reflection coefficient magnitude versus frequency for a single-section quarter-wave matching transformer with various load mismatches.

- When non-TEM lines (such as waveguides) are used, the propagation constant is no longer a linear function of frequency, and the wave impedance will be frequency dependent.

- The effect of reactances associated with discontinuities of transmission line must be considered.

[Ex. 5.5] Quarter-Wave Transformer Bandwidth

Design a single-section quarter-wave matching transformer to match a 10 Ω load to a 50 Ω line, at f_0 = 3 GHz. Determine the percent bandwidth for which the SWR $\leq$ 1.5.

➔ $Z_1 = \sqrt{Z_0 Z_L} = \sqrt{(50)(10)} = 22.36\ \Omega$

The length of the matching section is $\lambda/4$ at 3 GHz.

$$\Gamma_m = \frac{\mathrm{SWR}-1}{\mathrm{SWR}+1} = \frac{1.5-1}{1.5+1} = 0.2$$

Fractional bandwidth:

$$\frac{\Delta f}{f_0} = 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{\Gamma_m}{\sqrt{1-\Gamma_m^2}}\frac{2\sqrt{Z_0 Z_L}}{|Z_L - Z_0|}\right]$$

$$= 2 - \frac{4}{\pi}\cos^{-1}\left[\frac{0.2}{\sqrt{1-(0.2)^2}}\frac{2\sqrt{(50)(10)}}{|10-50|}\right]$$

$$= 0.29 \quad \text{or} \quad 29\%.$$

5.5 THE THEORY OF SMALL REFLECTIONS

- Theory of small reflections: total reflection coefficient caused by the partial reflections from several small discontinuities

Single-Section Transformer

- Derivation an approximate expression for the overall reflection coefficient Γ

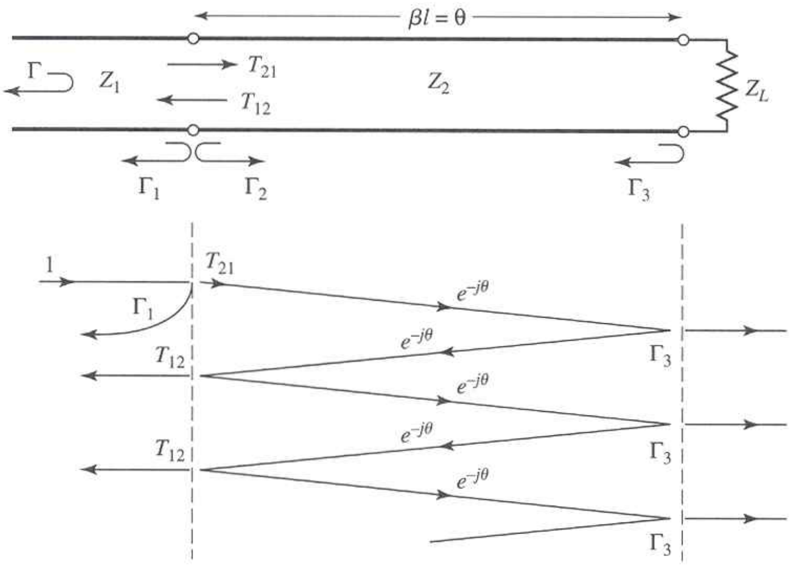


FIGURE 5.13 Partial reflections and transmissions on a single-section matching transformer.

$$\Gamma_1 = \frac{Z_2 - Z_1}{Z_2 + Z_1} \qquad (5.34)$$

$$\Gamma_2 = -\Gamma_1 \qquad (5.35)$$

$$\Gamma_3 = \frac{Z_L - Z_2}{Z_L + Z_2} \qquad (5.36)$$

$$T_{21} = 1 + \Gamma_1 = \frac{2Z_2}{Z_1 + Z_2} \qquad (5.37)$$

$$T_{12} = 1 + \Gamma_2 = \frac{2Z_1}{Z_1 + Z_2} \qquad (5.38)$$

- Expression of the total reflection as an infinite sum of partial reflections and transmissions:

$$\Gamma = \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta} + T_{12}T_{21}\Gamma_3^2\Gamma_2 e^{-4j\theta} + \ldots$$

$$= \Gamma_1 + T_{12}T_{21}\Gamma_3 e^{-2j\theta}\sum_{n=0}^{\infty}\Gamma_2^n\Gamma_3^n e^{-2jn\theta} \qquad (5.39)$$

- Using the geometric series and small reflection condition,

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \qquad \text{for} \quad |x| < 1$$

$$\Gamma = \Gamma_1 + \frac{T_{12}T_{21}\Gamma_3 e^{-2j\theta}}{1 - \Gamma_2\Gamma_3 e^{-2j\theta}} \qquad (5.40)$$

$$\leftarrow \Gamma_2 = -\Gamma_1,\ T_{21} = 1 + \Gamma_1,\ T_{12} = 1 + \Gamma_2 = 1 - \Gamma_1$$

$$= \frac{\Gamma_1 - \Gamma_1\Gamma_2\Gamma_3 e^{-2j\theta} + (1-\Gamma_1)(1+\Gamma_1)\Gamma_3 e^{-2j\theta}}{1 + \Gamma_1\Gamma_3 e^{-2j\theta}}$$

$$\cong \frac{\Gamma_1 + \Gamma_3 e^{-2j\theta}}{1 + \Gamma_1\Gamma_3 e^{-2j\theta}} \qquad (5.41)$$

$$\Gamma \cong \Gamma_1 + \Gamma_3 e^{-2j\theta} \qquad (5.42)$$

- Intuitive ideas:
  1) The total reflection is dominated by the reflection from the initial discontinuity between Z_1 and Z_2 (Γ_1), and the first reflection from the discontinuity between Z_2 and Z_L ($\Gamma_3 e^{-2j\theta}$).
  2) The $e^{-2j\theta}$ term accounts for the phase delay when the incident wave travels up and down the line.

Multi-Section Transformer

- N equal-length (commensurate) sections of transmission lines.

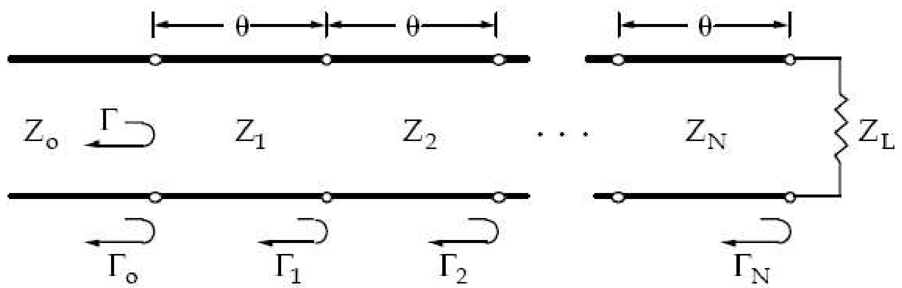


FIGURE 5.14 Partial reflection coefficients for a multisection matching transformer.

- Derivation an approximate expression for the total reflection coefficient Γ.

$$\Gamma_0 = \frac{Z_1 - Z_0}{Z_1 + Z_0}, \qquad (5.43a)$$

$$\Gamma_n = \frac{Z_{n+1} - Z_n}{Z_{n+1} + Z_n}, \qquad (5.43b)$$

$$\Gamma_N = \frac{Z_L - Z_N}{Z_L + Z_N}. \qquad (5.43c)$$

- Assumptions:
  1) All Z_n increase or decrease monotonically across the transformer.
  2) Z_L is real.

$$\Rightarrow \; \Gamma_n > 0 \text{ if } Z_L > Z_0$$
$$\Gamma_n < 0 \text{ if } Z_L < Z_0$$

- Overall reflection coefficient:

$$\Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-2j\theta} + \Gamma_2 e^{-4j\theta} + \cdots + \Gamma_N e^{-2jN\theta} \qquad (5.44) \leftarrow (5.42)$$

- If the transformer be made symmetrical, so that $\Gamma_0 = \Gamma_N$, $\Gamma_1 = \Gamma_{N-1}$, $\Gamma_2 = \Gamma_{N-2}$, etc. Then (5.44) can be written as;

$$\Gamma(\theta) = e^{-jN\theta}\{\Gamma_0[e^{jN\theta} + e^{-jN\theta}] + \Gamma_1[e^{j(N-2)\theta} + e^{-j(N-2)\theta}] + \cdots\} \quad (5.45)$$

Where if N is odd, the last term is $\Gamma_{(N-1)/2}(e^{j\theta}+e^{-j\theta})$, while if N is even the last term is $\Gamma_{N/2}$.

$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + ... + \Gamma_n \cos(N-2n)\theta$$
$$+ \cdots + \frac{1}{2}\Gamma_{N/2}] \qquad \text{for } N \text{ even} \qquad (5.46a)$$

$$\Gamma(\theta) = 2e^{-jN\theta}[\Gamma_0 \cos N\theta + \Gamma_1 \cos(N-2)\theta + ... + \Gamma_n \cos(N-2n)\theta$$
$$+ \cdots + \Gamma_{(N-1)/2} \cos\theta] \qquad \text{for } N \text{ odd} \qquad (5.46b)$$